Autonomous multispecies reaction-diffusion systems with more-than-two-site interactions

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Autonomous multispecies systems with more-than-two-neighbor interactions are studied. Conditions necessary and sufficient for the closedness of the evolution equations of the n-point functions are obtained. The average numbers of the particles at each site for one species and three-site interactions, and its generalization to the more-than-three-site interactions, are explicitly obtained. Generalizations of the Glauber model in different directions, using generalized rates, generalized numbers of states at each site, and generalized numbers of interacting sites, are also investigated.

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I. INTRODUCTION

The principles of equilibrium statistical mechanics are well established. However, thermal equilibrium is a special case, and little is known about the properties of systems not in equilibrium, for example about relaxation toward the stationary state. Some interesting problems in nonequilibrium systems include nonequilibrium phase transitions described by phenomenological rate equations, and the way the system relaxes to its steady state. As mean-field techniques, generally, do not give correct results for low-dimensional systems, people are motivated to study exactly solvable stochastic models in low dimensions. Moreover, solving one-dimensional systems should in principle be easier. Exact results for some models on a one-dimensional lattice were obtained, for example, in Refs. [1–12]. Different methods have been used to study these models, including analytical and asymptotic methods, mean field methods, and large-scale numerical methods. Systems with more than one species were also studied [13–27]. Many of the arguments were based on simulation results. There are, however, some exact results as well. For most of the models studied, the interaction is between nearest neighbors. However, there exist studies on models with more-than-two-site interactions (see Ref. [28], for example).

In Ref. [29], a ten-parameter family of stochastic models with interactions between nearest neighbors was studied. In these models, the k-point equal time correlation functions $\langle n_i n_j \cdots n_k \rangle$ satisfy linear differential equations involving no higher-order correlations. We call these models *autonomous*, in the sense that the evolution equations of n-point functions are closed (contain only n- or less-point functions). These linear equations for the average number of the particles $\langle n_i \rangle$ have been solved. The same models were studied on lattices with boundaries in Ref. [30]. It was shown that these models may exhibit dynamic and static phase transitions. The same idea was generalized to multi-species models [22] in one dimension with two-site interactions. There, conditions were

obtained that the Hamiltonian should satisfy in order that the evolution equation for correlation functions be closed. The set of equations for average densities can be written in terms of four matrices. These matrices are not determined uniquely from the Hamiltonian: there is a kind of gauge transformation one can apply on them which of course, does not change the evolution equation. A formal solution for the average densities of different species was found. The large-time behavior of the average densities of different species was also studied. The time evolution equations for more-point functions, generally contain not only these four matrices, but also elements of the Hamiltonian, and to obtain a closed form for their solution is generally not easy.

The Glauber dynamics was originally proposed to study the relaxation of the Ising model near equilibrium states. It was also shown that there is a relation between the kinetic Ising model at zero temperature and the diffusion annihilation model in one dimension. There is an equivalence between domain walls in the Ising model and particles in the diffusion annihilation model. Kinetic generalizations of the Ising model, for example the Glauber model or the Kawasaki model, are phenomenological models and were studied extensively [31–42].

In this paper, autonomous multispecies systems with more-than-two-neighbor interactions are studied. Necessary and sufficient conditions for the closedness of the evolution equations of the n-point functions are obtained. As an example, we explicitly obtain the average number of particles at each site for one species and three-site interactions. This is then generalized to the case where more than three sites interact. As another example, a generalization of the Glauber model is presented. In this generalization, the processes are the same as those of the ordinary Glauber model, but the rates depend on three free parameters, rather than one free parameter in the ordinary Glauber model. Finally, this model is further generalized to the case where the number of interacting sites is more than three and the number of states at each site is more than two.

II. MODELS LEADING TO A CLOSED SET OF EVOLUTION EQUATIONS

The models addressed are multispecies exclusion reaction-diffusion models. That is, each site is a vacancy or

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contains one particle. There are several kinds of particles, but at any time at most one kind can be present at each site. Throughout the paper, the dynamics is assumed to be translationally invariant. First consider a case where the interaction is between three neighboring sites. Then the Hamiltonian describing the system can be written as

$$\mathcal{H} = \sum_{i=1}^{L} H_{i,i+1,i+2}.$$ (1)

The number of sites is L and the number of possible states in a site is q (one of these states, for example the qth one, may be the vacancy); different states of each site are denoted by α, $\alpha = 1, \ldots, q$. Introducing n_i^{α} as the number operator of the particles of type α in the site i, we have

$$\sum_{\alpha=1}^{q} n_i^{\alpha} = 1.$$ (2)

The average number of the particles of the type α in the site i at the time t is

$$\langle n_i^{\alpha} \rangle = \langle S | n_i^{\alpha} | P(t) \rangle,$$ (3)

where $|P(t)\rangle := \exp(t\mathcal{H})|P(0)\rangle$ represents the state of the system at the time t,

$$\langle S | := \underbrace{\langle s | \otimes \cdots \otimes \langle s |}_{L},$$ (4)

and

$$\langle s | := \underbrace{(1 \; 1 \; \cdots 1)}_{q}.$$ (5)

Thus the time evolution of $\langle n_i^{\alpha} \rangle$ is given by

$$\frac{d}{dt} \langle n_i^{\alpha} \rangle = \langle S | n_i^{\alpha} \mathcal{H} | P(t) \rangle.$$ (6)

The only terms of the Hamiltonian $\mathcal{H}$ entering the above equation are $H_{i,i+1,i+2}$, $H_{i-1,i,i+1}$, and, $H_{i-2,i-1,i}$. The result of acting with any matrix Q on the bra $\langle s |$ is equal to that of acting the diagonal matrix $\tilde{Q}$ on the same bra, provided each diagonal element of the matrix $\tilde{Q}$ is equal to the sum of all elements of the corresponding column in the matrix Q. Thus the actions of $(1 \otimes 1 \otimes n^{\alpha})H$, $(1 \otimes n^{\alpha} \otimes 1)H$, and $(n^{\alpha} \otimes 1 \otimes 1)H$ on $\langle s | \otimes \langle s | \otimes \langle s |$ are equal to the actions of three diagonal matrices on $\langle s | \otimes \langle s | \otimes \langle s |$. We use the symbol $\sim$ to denote the equality of the action on $\langle s | \otimes \langle s | \otimes \langle s |$. We have

$$(n^{\alpha} \otimes 1 \otimes 1)H \sim \sum_{\beta\gamma\lambda} {}^{1}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} n^{\beta} \otimes n^{\gamma} \otimes n^{\lambda},$$

$$(1 \otimes n^{\alpha} \otimes 1)H \sim \sum_{\beta\gamma\lambda} {}^{2}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} n^{\beta} \otimes n^{\gamma} \otimes n^{\lambda},$$ (7)

$$(1 \otimes 1 \otimes n^{\alpha})H \sim \sum_{\beta\gamma\lambda} {}^{3}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} n^{\beta} \otimes n^{\gamma} \otimes n^{\lambda},$$

where ${}^{i}\mathcal{A}^{\alpha}_{\beta\gamma\lambda}$'s are defined as

$${}^{1}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} := s_{\tau} s_{\omega} H^{\alpha\tau\omega}_{\beta\gamma\lambda},$$

$${}^{2}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} := s_{\tau} s_{\omega} H^{\tau\alpha\omega}_{\beta\gamma\lambda},$$ (8)

$${}^{3}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} := s_{\tau} s_{\omega} H^{\tau\omega\alpha}_{\beta\gamma\lambda}.$$

An implicit summation (from 1 to q) over the same subscript and superscript is always assumed. From these, Eq. (6) takes the form

$$\langle \dot{n}_i^{\alpha} \rangle = {}^{1}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} \langle n_i^{\beta} n_{i+1}^{\gamma} n_{i+2}^{\lambda} \rangle + {}^{2}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} \langle n_{i-1}^{\beta} n_i^{\gamma} n_{i+1}^{\lambda} \rangle$$
$$+ {}^{3}\mathcal{A}^{\alpha}_{\beta\gamma\lambda} \langle n_{i-2}^{\beta} n_{i-1}^{\gamma} n_i^{\lambda} \rangle.$$ (9)

Generally, the right-hand side of Eq. (9) contains one-, two-, and three-point functions. (Note that n^{α}'s are not independent.) We want to obtain a condition that only one-point functions appear in the right-hand side. To do this, we consider the expression

$$u = f_{\alpha\beta\gamma} n_i^{\alpha} n_j^{\beta} n_k^{\gamma},$$ (10)

and ask for the condition that the right-hand side is expressible in terms of linear combinations of n's, provided

$$s_{\alpha} n_l^{\alpha} = 1.$$ (11)

It is obvious that if

$$f_{\alpha\beta\gamma} = {}_{1}F_{\alpha} + {}_{2}F_{\beta} + {}_{3}F_{\gamma},$$ (12)

then the right-hand side of Eq. (10) is expressible in terms of linear combinations of n's. To prove that this form for f is necessary as well, we just count the number of independent variables in f's satisfying the desired property. One can write n_l^q in terms one 1 and other n_l^{α}'s. Then it is seen that a general cubic form of n's is expressible in terms of q^3 independent forms of n's, each containing no more than three n's. Of these, $1 + 3(q-1)$ expressions [the monomials of degree zero and one of $3(q-1)$ independent variables] are desirable. The coefficients of other monomials should be zero. Thus from q^3 independent variables in f, there remain only $3(q-1) + 1$ independent variables in f's satisfying the desired condition. It seems that the right-hand side of Eq. (12) contains more independent variables, namely $3q$. But we note that the transformation

$${}_{i}F_{\alpha} \rightarrow {}_{i}F_{\alpha} + {}_{i}B,$$ (13)

does not change the right-hand side of Eq. (12), provided

$$\sum_{i} {}_{i}B = 0.$$ (14)

This means that there are $3 - 1$ redundant variables in expression (12). Thus Eq. (12) actually contains the correct number of independent variables, and hence is the most general form of f with the desired property.

So, in order that Eq. (9) be expressible in terms of only one-point functions, one must have

$$^i\!\mathcal{A}^{\,\alpha}_{\,\beta\gamma\lambda}={}^i_1\!\mathcal{A}^{\,\alpha}_{\,\beta}+{}^i_2\!\mathcal{A}^{\,\alpha}_{\,\gamma}+{}^i_3\!\mathcal{A}^{\,\alpha}_{\,\lambda}. \tag{15}$$

Note that ${}^i_j\mathcal{A}$'s are not determined uniquely. Applying the *gauge* transformation

$$^i_j\!\mathcal{A}^{\,\alpha}_{\,\beta}\rightarrow{}^i_j\!\mathcal{A}^{\,\alpha}_{\,\beta}+{}^i_j\!B^{\alpha}={}^i_j\!\mathcal{A}^{\,\alpha}_{\,\beta}+{}^i_j\!B^{\alpha}s_{\beta},\quad\text{with}\quad\sum_j{}^i_j\!B^{\alpha}=0 \tag{16}$$

does not change the right-hand side of Eq. (15).

If Eq. (15) is satisfied, Eq. (6) takes the form

$$\langle\dot{n}_i^{\alpha}\rangle=({}^1_1\!\mathcal{A}^{\,\alpha}_{\,\beta}+{}^2_2\!\mathcal{A}^{\,\alpha}_{\,\beta}+{}^3_3\!\mathcal{A}^{\,\alpha}_{\,\beta})\langle n_i^{\beta}\rangle+({}^1_2\!\mathcal{A}^{\,\alpha}_{\,\beta}+{}^3_3\!\mathcal{A}^{\,\alpha}_{\,\beta})\langle n_{i+1}^{\beta}\rangle$$
$$+({}^2_1\!\mathcal{A}^{\,\alpha}_{\,\beta}+{}^3_2\!\mathcal{A}^{\,\alpha}_{\,\beta})\langle n_{i-1}^{\beta}\rangle+{}^1_3\!\mathcal{A}^{\,\alpha}_{\,\beta}\langle n_{i+2}^{\beta}\rangle+{}^3_1\!\mathcal{A}^{\,\alpha}_{\,\beta}\langle n_{i-2}^{\beta}\rangle. \tag{17}$$

Equation (15) in fact guarantees that the time-evolution equations of n-point functions contain only n- and less-point functions. In the simplest case, the one-species case, each site is vacant or occupied by only one kind of particles. Then, the matrices ${}^i_j\mathcal{A}$ are two by two.

One can do the same arguments for the case where more than three neighboring sites interact. Suppose the number of interacting sites is N. We define

$$^i\!\mathcal{A}^{\,\alpha_i}_{\,\beta_1\cdots\beta_N}:=\left(\prod_{l\neq i}s_{\alpha_l}\right)H^{\alpha_1\cdots\alpha_N}_{\beta_1\cdots\beta_N}. \tag{18}$$

To ensure that in the time-evolution equation of one-point functions only one-point functions appear, one must have

$$^i\!\mathcal{A}^{\,\alpha_i}_{\,\beta_1\cdots\beta_N}=\sum_j{}^i_j\!\mathcal{A}^{\,\alpha_i}_{\,\beta_j}. \tag{19}$$

Here too, the *gauge* transformation [Eq. (16)] does not change the right-hand side of Eq. (19), and hence the physics of the problem. It shoud be noted that Eq. (19) is also sufficient for n-point functions evolution equations to contain no more than n-point functions.

Some special cases

We now consider some special cases.

1. Single species case

In this case the matrices ${}^i_j\mathcal{A}$ are two by two. First consider the case $N=3$. The time-evolution equation for $\langle n_k\rangle$ will then be

$$\langle\dot{n}_i\rangle=-\alpha\langle n_i\rangle+\beta\langle n_{i+1}\rangle+\beta'\langle n_{i-1}\rangle+\gamma\langle n_{i+2}\rangle+\gamma'\langle n_{i-2}\rangle$$
$$+\delta. \tag{20}$$

δ can be eliminated using the redefinition

$$x_i:=\langle n_i\rangle-\frac{\delta}{\beta+\beta'+\gamma+\gamma'-\alpha}. \tag{21}$$

Then, introducing the generating function

$$G(z,t)=\sum_{-\infty}^{\infty}x_iz^i, \tag{22}$$

one arrives at

$$\dot{G}(z,t)=(-\alpha+\beta z^{-1}+\beta'z+\gamma z^{-2}+\gamma'z^2)G(z,t), \tag{23}$$

the solution to which is

$$G(z,t)=\exp[t(-\alpha+\beta z^{-1}+\beta'z+\gamma z^{-2}+\gamma'z^2)]G(z,0). \tag{24}$$

Using

$$\exp\left[\frac{u}{2}(z+z^{-1})\right]=\sum_{k=-\infty}^{\infty}\mathrm{I}_k(u)z^k, \tag{25}$$

where I_k is the modified Bessel function of order k, one arrives at

$$x_k(t)=e^{-\alpha t}\sum_{j,l=-\infty}^{\infty}\mathrm{I}_{k-j-2l}(2t\sqrt{\beta\beta'})\mathrm{I}_l(2t\sqrt{\gamma\gamma'})$$
$$\times(\beta'/\beta)^{(k-j-2l)/2}(\gamma'/\gamma)^{l/2}x_j(0). \tag{26}$$

A similar procedure can be done for more-than-three-neighboring-site interactions. The main difference will be the number of modified Bessel functions appearing in the expression.

To investigate the large-time behavior of the system, it is easier to use Eq. (24). Using this and Eq. (22), one has

$$x_k(t)=\oint\frac{\mathrm{d}z}{2\pi iz^{k+1}}e^{t\,f(z)}G(z,0), \tag{27}$$

where

$$f(z):=-\alpha+\beta z^{-1}+\beta'z+\gamma z^{-2}+\gamma'z^2, \tag{28}$$

and the integration contour is the unit circle. At large times, this integration can be done using the steepest descent method. The result would be

$$\delta x(t)\sim\frac{e^{t\,f(z_0)}}{t^{\Gamma}}. \tag{29}$$

Here δx is the deviation of x from its stationary value, z_0 is the point that maximizes f, and Γ may be 1/2 or 1/4: It is 1/2 if $f''(z_0)$ does not vanish, and 1/4 if it does. It is seen that if $f(z_0)$ is nonzero, the relaxation of the system toward its stationary state is exponential. If $f(z_0)=0$, this relaxation is a power law, generally like $t^{-1/2}$, but in exceptional cases it may be like $t^{-1/4}$. A similar argument for more-than-three-neighboring-site interactions shows that the large time behavior of the system may be exponential, or power law with powers $-1/2$, $-1/4$, $\ldots$, or $-1/(2s)$ for $(s+1)$-site interactions.

2. Generalizations of the Glauber model

Consider a two-state three-neighbor interaction of the form

$$^1A^{\alpha}_{\beta\gamma\lambda} = {}^3A^{\alpha}_{\beta\gamma\lambda} = 0. \tag{30}$$

This means that, similar to the Glauber model, any site interacts only with its neighboring sites. The interactions are

$$
\begin{aligned}
AAA &\rightarrow A\varnothing A, & \mu_1, \\
\varnothing\varnothing\varnothing &\rightarrow \varnothing A\varnothing, & \mu_2, \\
A\varnothing A &\rightarrow AAA, & \lambda_1, \\
\varnothing A\varnothing &\rightarrow \varnothing\varnothing\varnothing, & \lambda_2, \\
AA\varnothing &\rightarrow A\varnothing\varnothing, & \alpha_1, \\
\varnothing\varnothing A &\rightarrow \varnothing AA, & \alpha_2, \\
A\varnothing\varnothing &\rightarrow AA\varnothing, & \beta_1, \\
\varnothing AA &\rightarrow \varnothing\varnothing A, & \beta_2.
\end{aligned}
\tag{31}
$$

This is a generalization of the Glauber model. For the ordinary Glauber model,

$$\mu_1 = \mu_2 = 1 - \tanh\frac{J}{k_B T},$$

$$\lambda_1 = \lambda_2 = 1 + \tanh\frac{J}{k_B T}, \tag{32}$$

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1.$$

Criterion (19) for the closedness of the evolution equation for one-point functions results in the following relations between the rates in the generalized Glauber model:

$$\mu_i - \alpha_i = \beta_j - \lambda_j \quad \text{for any } i,j,$$

$$\alpha_1 + \beta_1 = \alpha_2 + \beta_2. \tag{33}$$

Thus there are four independent variables in terms of which the above eight parameters can be expressed. One can write the expressions as

$$\mu_1 = A - B - C - C',$$

$$\mu_2 = A + B,$$

$$\lambda_1 = A + B + C + C'$$

$$\lambda_2 = A - B, \tag{34}$$

$$\alpha_1 = A - B - C',$$

$$\alpha_2 = A + B + C,$$

$$\beta_1 = A + B + C',$$

$$\beta_2 = A - B - C.$$

Of course one of the parameters can be absorbed through a time rescaling. The resulting evolution equation for the average particle number is

$$\langle \dot{n}_i \rangle = A + B - 2A\langle n_i \rangle + C\langle n_{i+1} \rangle + C'\langle n_{i-1} \rangle, \tag{35}$$

which is easy to solve.

Note that these reaction rates are not necessirily consistent with detailed balance with respect to some local translationally invariant Hamiltonian, whereas the rates of the ordinary Glauber model do satisfy detailed balance. To see the reason for this, one can use a general nearest-neighbor-interaction translationally invariant Hamiltonian for a two-state system. This is basically an Ising Hamiltonian with an external magnetic field. From this, one can calculate the reaction rates consistent with detailed balance, and demand that they satisfy Eqs. (33), the criteria that the evolution equation for the one-point functions be closed. The result is that the external magnetic field should be zero. This means that the only two-state system, which is autonomous and whose rates satisfy detailed balance with respect to some local translationally invariant Hamiltonian, is the ordinary Glauber model.

This generalized Glauber model can further be generalized in two directions: when the number of the interacting sites is more than 3, and when the number of states of each site is more than 2. The first case means that the interaction is in a block of length N, resulting in the change of the state of a single specific site in that block. This rate of change depends on the states of this site and the states of the other $N-1$ sites. Let us label this specific site of the block by 0. (Usually the length of the block is considered to be an odd integer $(2k+1)$, and the evolving site is assumed to be the central one.) Denote the state of the site i by σ_i, where σ_i can take the values 1 (particle) or 0 (vacancy). Then the evolution equation for the average particle number is

$$
\langle \dot{n}_0 \rangle = -\left\langle \sum_{\vec{\sigma}} R(1,\vec{\sigma})\, n_0 \prod_{i\neq 0} [1 - n_i + \sigma_i(2n_i - 1)] \right\rangle
$$
$$
+ \left\langle \sum_{\vec{\sigma}} R(0,\vec{\sigma})(1-n_0) \prod_{i\neq 0} [1 - n_i + \sigma_i(2n_i - 1)] \right\rangle. \tag{36}
$$

Here the state of other interacting sites is denoted by $\vec{\sigma}$; $R(\sigma_0,\vec{\sigma})$ is the rate of change of the state of site 0, from σ_0 to $1-\sigma_0$, when the states of the other interacting sites is $\vec{\sigma}$.

We are looking for those rates $R(\sigma_0,\vec{\sigma})$ that make the right-hand side of this evolution equation a linear combination of $\langle n_i \rangle$'s. The claim is that the general form of these rates is

$$R(\sigma_0,\vec{\sigma}) = A + (-1)^{\sigma_0}\left(B + \sum_{i\neq 0} C_i\,\sigma_i\right). \tag{37}$$

Inserting this ansatz into the evolution equation (36), we obtain

$$\langle \dot{n}_0 \rangle = B + A \langle 1 - 2n_0 \rangle + \sum_{i \neq 0} C_i \langle n_i \rangle. \qquad (38)$$

Thus it is clear that ansatz (37) leads into a closed set of evolution equations for the average particle number. It remains to prove that this ansatz is the most general one satisfying this property. To see this, one considers Eq. (36). On the right-hand side of this equation there are 2^N terms (the expectation of monomials in terms of n_i's). Of these, we desire that the coefficients of all be zero, except for the coefficients of the constant term and linear terms. Thus there are $2^N - (N+1)$ equations to be satisfied for the rates (consisting themselves of 2^N unknowns). This shows that the rates satisfying the desired property contain $N+1$ independent variables, and it is clear that ansatz (37) contains $N+1$ independent variables. Thus it is the most general solution.

Now consider the second generalization, when the number of possible states at each site is more than 2, say q, and each block consists of N sites. The state of the site i is denoted by σ_i, which can take q values. That site, the evolution of its state being considered, is denoted by 0. The rate of change of the state of the site 0 from β to α is denoted by $R^\alpha_{\beta,\vec{\sigma}}$. The evolution equation for the average numbers is then

$$\langle \dot{n}_0^\alpha \rangle = \left\langle \sum_{\vec{\sigma}} \sum_{\beta \neq \alpha} R^\alpha_{\beta,\vec{\sigma}} n_0^\beta \prod_{i \neq 0} n_i^{\sigma_i} \right\rangle$$

$$- \left\langle \sum_{\vec{\sigma}} \sum_{\beta \neq \alpha} R^\beta_{\alpha,\vec{\sigma}} n_0^\alpha \prod_{i \neq 0} n_i^{\sigma_i} \right\rangle. \qquad (39)$$

Defining

$$R^\alpha_{\alpha,\vec{\sigma}} := - \sum_{\beta \neq \alpha} R^\beta_{\alpha,\vec{\sigma}}, \qquad (40)$$

the evolution equation reads

$$\langle \dot{n}_0^\alpha \rangle = \sum_{\vec{\sigma}} R^\alpha_{\beta,\vec{\sigma}} \left\langle n_0^\beta \prod_{i \neq 0} n_i^{\sigma_i} \right\rangle. \qquad (41)$$

It is easy to see that for the right-hand side of this equation be expressible in terms of one-point functions, one should have

$$R^\alpha_{\beta,\vec{\sigma}}, = {}_0 R^\alpha_\beta + \sum_{i \neq 0} {}_i R^\alpha_{\sigma_i}. \qquad (42)$$

Note that this is nothing but

$${}^0 \mathcal{A}^\alpha_{\beta,\vec{\sigma}} = \sum_i {}^0_i \mathcal{A}^\alpha_{\sigma_i}, \qquad (43)$$

obtained in Eq. (19). In this case, other ${}^i \mathcal{A}$'s vanish. The evolution equation of the one-point functions is then

$$\langle \dot{n}_0^\alpha \rangle = \sum_i {}_i R^\alpha_\beta \langle n_i^\beta \rangle. \qquad (44)$$

Previous discussion showed that this is the most general form of the rates, for which the evolution equations of the one-point functions contain only one-point functions. Here too, there is a *gauge* freedom in choosing ${}_i R$'s, namely,

$${}_i R^\alpha_\beta \rightarrow {}_i R^\alpha_\beta + {}_i B^\alpha s_\beta, \quad \text{with} \quad \sum_i {}_i B^\alpha = 0. \qquad (45)$$

This is the same *gauge* freedom encountered with earlier [Eq. (16)].

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[1] F.C. Alcaraz, M. Droz, M. Henkel, and V. Rittenberg, Ann. Phys. (N.Y.) **230**, 250 (1994).

[2] K. Krebs, M.P. Pfannmuller, B. Wehefritz, and H. Hinrichsen, J. Stat. Phys. **78**, 1429 (1995).

[3] H. Simon, J. Phys. A **28**, 6585 (1995).

[4] V. Privman, A.M.R. Cadilhe, and M.L. Glasser, J. Stat. Phys. **81**, 881 (1995).

[5] M. Henkel, E. Orlandini, and G.M. Schütz, J. Phys. A **28**, 6335 (1995).

[6] M. Henkel, E. Orlandini, and J. Santos, Ann. Phys. (N.Y.) **259**, 163 (1997).

[7] A.A. Lusknikov, Sov. Phys. JETP **64**, 811 (1986).

[8] M. Alimohammadi, V. Karimipour, and M. Khorrami, Phys. Rev. E **57**, 179 (1998).

[9] F. Roshani and M. Khorrami, Phys. Rev. E **60**, 3393 (1999).

[10] F. Roshani and M. Khorrami, preprint, cond-mat/0007352.

[11] M. Alimohammadi, V. Karimipour, and M. Khorrami, J. Stat. Phys. **97**, 373 (1999).

[12] A. Aghamohammadi and M. Khorrami, J. Phys. A **33**, 7843 (2000).

[13] R. Schoonover, D. Ben-Avraham, S. Havlin, R. Kopelman, and G. Weiss, Physica A **171**, 232 (1991).

[14] D. Toussaint and F. Wilczek, J. Chem. Phys. **78**, 2642 (1983).

[15] K. Kong and S. Render, Phys. Rev. Lett. **52**, 955 (1984).

[16] D. Ben-Avraham, V. Privman, and D. Zhong, Phys. Rev. E **52**, 6889 (1995).

[17] I. Ispolatov, P.L. Krapivsky, and S. Redner, Phys. Rev. E **52**, 2540 (1995).

[18] J.W. Lee and V. Privman, J. Phys. A **30**, L317 (1997).

[19] Z. Koza, preprint, cond-mat/9601072.

[20] V. Karimipour, Phys. Rev. E **59**, 205 (1999).

[21] K. Kim and K.H. Chang, J. Phys. Soc. Jpn. **68**, 1450 (1999).

[22] A. Aghamohammadi, A.H. Fatollahi, M. Khorrami, and A. Shariati, Phys. Rev. E **62**, 4642 (2000).

[23] M. Khorrami and V. Karimipour, J. Stat. Phys. **100**, 999 (2000).

[24] G. Odor and N. Menyhard, Phys. Rev. E **61**, 6404 (2000).

[25] M. Alimohammadi and N. Ahmadi, Phys. Rev. E **62**, 1674 (2000).

[26] F.H. Jafarpour, preprint, cond-mat/0004357.

[27] F. Roshani and M. Khorrami, Phys. Rev. E **64**, 011101 (2001).

[28] M. Henkel and H. Hinrichsen, J. Phys. A **34**, 1561 (2001).

[29] G.M. Schütz, J. Stat. Phys. **79**, 243 (1995).

[30] A. Aghamohammadi and M. Khorrami, J. Phys. A **34**, 7431 (2001).

[31] R.J. Glauber, J. Math. Phys. **4**, 294 (1963).

[32] K. Kawasaki, Phys. Rev. **145**, 224 (1966).

[33] J.G. Amar and F. Family, Phys. Rev. A **41**, 3258 (1990).

[34] T. Vojta, Phys. Rev. E **55**, 5157 (1997).

[35] R.B. Stinchcombe, J.E. Santos, and M.D. Grynberg, J. Phys. A **31**, 541 (1998).

[36] C. Godrèche and J.M. Luck, J. Phys. A **33**, 1151 (2000).

[37] M. Khorrami and A. Aghamohammadi, Phys. Rev. E **63**, 042102 (2001).

[38] L.L. Gonçalves, M. Lopez de Haro, and J. Taguena-Martinez, Phys. Rev. E **63**, 026114 (2001).

[39] M. Droz, Z. Rácz, and J. Schmidt, Phys. Rev. A **39**, 2141 (1989).

[40] B.C.S. Grandi and W. Figueiredo, Phys. Rev. E **53**, 5484 (1996).

[41] S. Artz and S. Trimper, Int. J. Mod. Phys. B **12**, 2385 (1998).

[42] M. Khorrami and A. Aghamohammadi, e-print, cond-mat/0103090.