Ising model on a small world network

Andrzej Pękalski
Institute of Theoretical Physics, University of Wrocław, plac Maxa Borna 9, 50-204 Wrocław, Poland
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A one-dimensional Ising model is studied, via Monte Carlo simulations, on a small world network, where each site has, apart from couplings to its two nearest neighbors, a certain probability to be linked to one of its farther neighbors. It is demonstrated that even a small fraction of such links enables the system to order at finite temperatures. The critical exponent β is smaller than the two-dimensional value, and seems to be independent of the concentration of the extra links. The dependence of the magnetization and the critical temperature on the concentration of the small world links is also presented.

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Static and dynamic properties of different models on small world networks have been extensively studied. Such networks interpolate between topologically ordered and random systems. In one extreme there is a one-dimensional (1D) chain where each lattice site is connected to its two nearest neighbors (NN). On the other end is a completely random system in which some NN bonds are exchanged for links to more distant neighbors. The model has been introduced by Watts and Strogatz [1]. It has been then applied to study disease transmission and probability of reaching epidemic behavior [2], percolation [3] or minimal path [4,5]. Some dynamic properties like diffusion [6] or relaxation [7] have been also investigated. An elegant mean-field-like solution of the small world network has been proposed in Ref. [8]. Watts and Strogatz [1] suggested that by changing a small amount of regular links to nearest neighbors into more distant ones the system undergoes a transition to a new, small world type of behavior. This has been challenged in Ref. [9], where the authors claim that the onset of the small world behavior is a crossover phenomenon, not a phase transition.

It is well known that the one-dimensional Ising model cannot exist in an ordered state at any finite temperature. A simple argument proving it (see, e.g., Ref. [10]) does not give a clear answer in the small world network, where the lattice sites (spins in the Ising model) are additionally linked by random couplings. Increasing the domain of overturned (say $S=-1$) spins until their number is equal that of $S=1$, may, or may not, cost energy, depending on the configuration of the long range links.

The problem of existence of a finite T_c in the Ising model on a small world network has been studied in Ref. [11]. The authors used analytic approach, but because of the mathematical problems they had to make some approximations. They concluded that for arbitrarily small but finite disorder and at sufficiently low temperatures, the system shows a mean-field-like low-temperature behavior. Analytic calculations have been augmented by numerical simulations on chains of up to 8000 spins. On that basis the authors claimed that any fraction of p results in ordering of the system. The early findings [12] seem, however, to suggest that adding finite long-range interactions to the 1D Ising system may not be sufficient to create a ferromagnetic state.

It seems, therefore, interesting to investigate the problem of existence of a ferromagnetic state in the Ising model on a small world network. In view of the complexity of the problem and following from it need for approximations in an analytic approach, we shall use Monte Carlo (MC) simulations performed on much longer chains. Although one cannot expect from MC simulations to prove or disprove the existence of finite T_c for, e.g., single additional link, such simulations give however valuable arguments.

Our model is constructed as follows. We consider a chain of L sites, each of them characterized by a two-valued function (spin) $\sigma_i=\pm 1$. We impose periodic boundary conditions ($\sigma_{i+L}\equiv\sigma_i$). Each spin interacts with a coupling, set as equal 1 for simplicity, with its two nearest neighbors. This corresponds to taking $k=1$ in the model of Watts and Strogatz [1]. In contrast to the Watts and Strogatz model, here the couplings between NN remain unchanged. We add new random links without modifying the old ones (see Fig. 1). In a sense this preserves the colloquial ''small world'' character, where we find unexpected mutual friends without loosing contact with the old ones. With a probability p a spin is linked additionally (with the same unit strength) to a randomly chosen spin, provided the two are not nearest neighbors. Each spin may have at most one such extra link. This type of construction is similar to the one used in Ref. [6]. The case $p=0$ corresponds to the 1D Ising model where every spin interacts only with its two nearest neighbors. For $p=1$ each spin has three couplings—two as before to NN and one to some more distant spin.

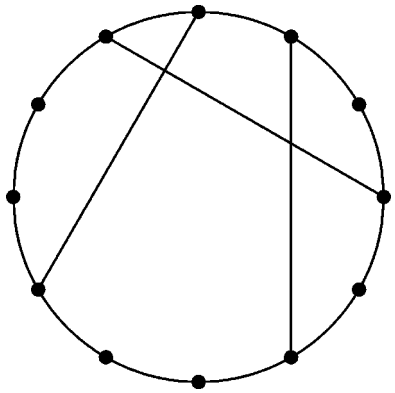

FIG. 1. Example of a small world network for $p=0.5$.

The MC simulation, based on the Metropolis algorithm, goes as follows. At the beginning, for a chosen value of p, the additional small world links are created between lattice sites. Then all the spins are given the same values -1 or $+1$. Next, a spin σ_i is randomly chosen and its energy is calculated as

$$\epsilon_i = -\sigma_i(\sigma_{i-1}+\sigma_{i+1}) - \sigma_i\sigma_j, \tag{1}$$

where the last term is present if there is a long-range link from the site i to the site j, and σ_j is the value of the spin at the site to which the site i is connected. If flipping the spin $\sigma_i \rightarrow -\sigma_i$ lowers the energy, the flip is accepted. Otherwise a random number r is generated and if it is smaller than

$$u = \exp[-(\epsilon_{ini}-\epsilon_{fin})/(k_B T)], \tag{2}$$

where the ϵ_{ini} and ϵ_{fin} are the energies before and after the flip and T is the dimensionless temperature reduced by the coupling constant, then the spin is also flipped. After choosing in such a way L spins, one MC step (MCS) has been accomplished. Time is measured in MCS. At a given time interval all spin values are added and the sum divided by L gives the net magnetization, $m(p,t)$, as a function of the concentration p of the extra links and the time t. To get the asymptotic ($t\rightarrow\infty$) value, a simple but efficient technique known in the high-temperature series expansion analysis has been used. $m(p,t)$ were plotted against $1/t$ and then linear fit gave the desired asymptotics. The runs were repeated for different configurations of links. The resulting asymptotic values of the magnetization have been then averaged. It turned out that the results, apart from the temperatures close to the critical one, do not depend on the initial distribution of links.

The simulations were made on a ring of typically $L=10^5$ sites and lasted till the system reached a stationary state where the magnetization fluctuated around the average value. Increasing the size to $L=10^6$ did not alter the results. At higher temperatures the stationary state was reached after several hundred MCS, while at lower temperatures several thousand were needed. In general, we run the simulations till $t=5\times10^3$ MCS, rejecting the first 2×10^3 MCS. For the Ising model without any additional links ($p=0$) we obtained for the magnetization values of the order $m\sim10^{-3}$ (for $T\leqslant0.3$) or smaller (for $T\geqslant0.4$). For $p>0$ we observe, however, a nonzero magnetization below a certain critical temperature (see Fig. 2).

The curves have a typical pattern and with increasing p the critical temperature moves to higher values. In numerical simulations it is of course very difficult, if possible at all, to determine T_c. Rather arbitrarily we have decided that $m\leqslant10^{-3}$ corresponds to a paramagnetic state, while $m\geqslant10^{-2}$ means a ferromagnetic state. Fortunately, as seen from Fig. 3, around T_c the magnetization grows rapidly and there is no doubt about the onset of the ordering.

The dependence of the critical temperature on the concentration of links p estimated in such a way, is shown in Fig. 4. It is clear that there is a crossover between two distinctive

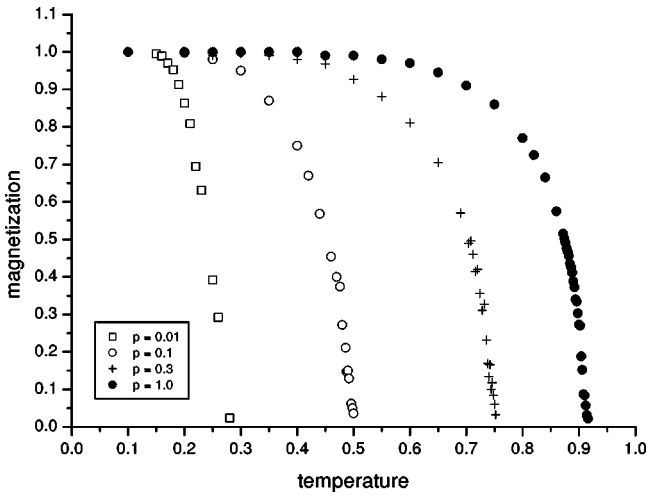

FIG. 2. Magnetization versus temperature for $p=0.01$(open squares), 0.1 (open circles), 0.3 (crosses), and 1 (full circles). Magnetization and temperature in this and the following figures are dimensionless.

regions—below and above $p=0.5$. In both the critical temperature as a function of the concentration p is given by a power law

$$t_c \sim p^{\alpha}, \tag{3}$$

but with different exponents. $\alpha\cong0.02$ for $p>0.5$ and $\alpha\cong0.36$ for $p>0.5$. When every second spin is linked to a distant neighbor, adding extra links has smaller effect on the shift of the critical temperature than when the number of the extra links is small.

Unfortunately we are unable to answer unequivocally the question whether there is a threshold value of p, below which the system will behave as 1D Ising model ($m=0$ for $T>0$) and above which it will have small world character ($m\neq0$ for $T>0$). At very low temperatures the convergence to a stationary state is extremely slow and the asymptotic magnetization depends on the topology of the

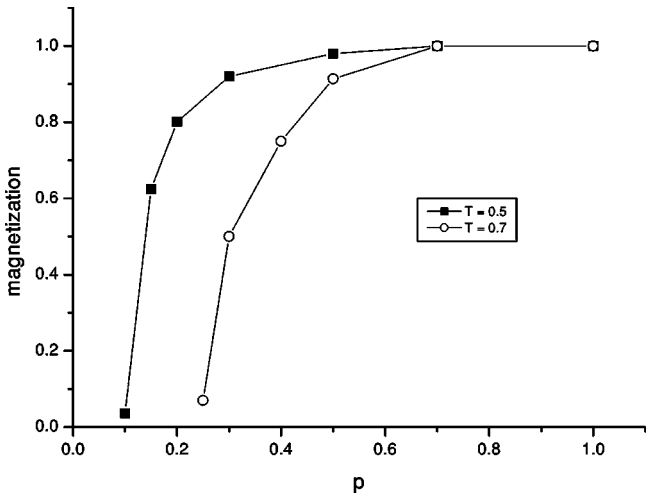

FIG. 3. Magnetization versus concentration p of extra links for $T=0.5$ and $T=0.7$.

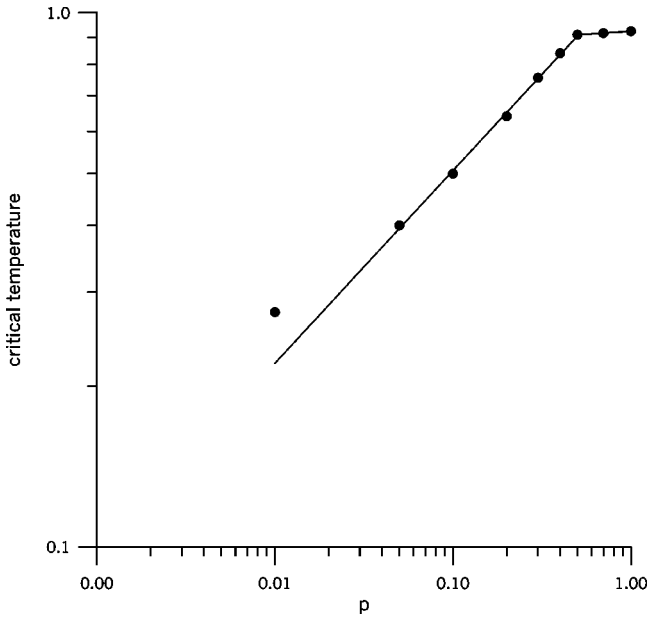

FIG. 4. Critical temperature versus p on the log-log plane. The lines are power fits.

additional links. The obtained data seem rather to suggest the existence of a threshold value. It is evident that the ordering in a small world Ising model exists and that it grows very rapidly with the amount of the extra links.

We have also estimated the β exponent, defining the critical behavior of the magnetization as a function of the temperature. Using the final size scaling technique [13] we obtained the results shown in Fig. 5. The slope seems to be the same for all values of the concentrations, p, of the additional links and is very close to zero ($\beta \sim 0.0001$ from the fit).

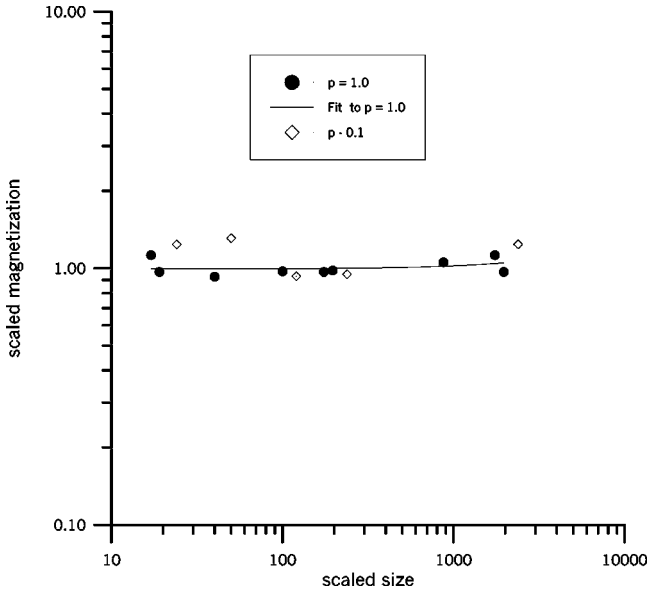

FIG. 5. Finite size scaling estimate of the β exponent as the slope of the data. The magnetization (vertical axis) is scaled by $L^{\beta/\nu}$ and the size of the system (horizontal axis) is scaled by $\epsilon=(T_c-T)/T_c$. Two values of $p, p=1$ (full circles) and $p=0.1$ (open diamonds). The line is the fit to the $p=1$ data.

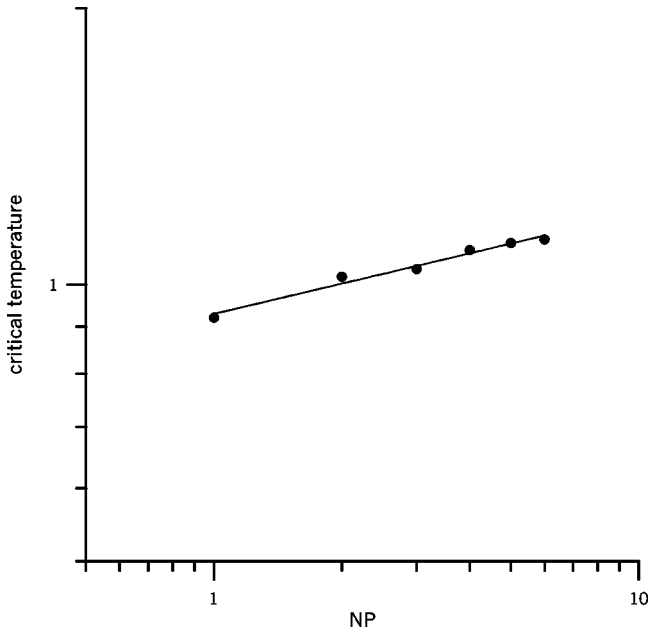

FIG. 6. Critical temperature versus number of extra links per site.

Since the mean standard deviation in determining the magnetization ranged from $\Delta m=0.001$ to $\Delta m=0.07$, close to T_c, the precise determination of β is a very delicate problem. It is, however, clear that the obtained value is different from the 2D (square lattice) value $\beta=1/8$ [14]. Before claiming that the small world Ising model belongs to a different universality class, other critical exponents should be also investigated, using also different than Monte Carlo simulation methods.

Suppose now that a site may have not just one but a certain number (NP) links to farther than NN sites. The dependence of the critical temperature on the number of links each site may have, is shown in Fig. 6. For $NP=0$ we have of course $T_c=0$. The case $p=1$ of the previous model corresponds to $NP=1$ here. The model is not completely random, since each site has the same number of extra links but to randomly chosen neighbors. Again, like in the previous model, we have found the power law dependence of the reduced critical temperature versus the number of the small world links [Eq. (3)]. Here, however, the exponent $\alpha \cong 0.109$, a value between the two found previously. Adding the first extra links has more effect on the behavior of the system. Subsequent increasing the number of connections modifies only slightly the pattern.

Our investigations of the 1D Ising model on a small world network show that adding more distant than NN interactions changes entirely the behavior of the system. The net magnetic moment is different from zero if the temperature is low enough and (probably) if the number of extra links is above a certain threshold. The dependence of the magnetization on the temperature follows a typical 2D pattern.

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