Perturbation theory for domain walls in the parametric Ginzburg-Landau equation

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We demonstrate that in the parametrically driven Ginzburg-Landau equation arbitrarily small nongradient corrections lead to qualitative differences in the dynamical properties of domain walls in the vicinity of the transition from rest to motion. These differences originate from singular rotation of the eigenvector governing the transition. We present analytical results on the stability of Ising walls, deriving explicit expressions for the critical eigenvalue responsible for the transition from rest to motion. We then develop a weakly nonlinear theory to characterize the singular character of the transition and analyze the dynamical effects of spatial inhomogeneities.

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I. INTRODUCTION

The analytical description of the dynamics of localized structures in nonlinear fields close to an instability threshold or under the action of perturbations is important in the understanding and application of such structures. Such a description is often obtained through the reduction of an infinitely dimensional partial differential equation to an ordinary differential equation for the order parameter. The latter is the amplitude of the critical, i.e., least damped, eigenvector in the spectrum of the localized structure.

One of the primary results of the present work is to demonstrate that a singular rotation of the critical eigenvector takes place when going from a gradient [1] to a nongradient system of equations by smooth variation of a control parameter. This singularity leads to a qualitative change in the dynamical equation for the order parameter and has drastic effects on the dynamics of the systems close to the instability threshold. The example considered is the parametrically driven Ginzburg-Landau equation (PGL), which is one of the prototype models used in nonlinear field theories. Our results demonstrate that approximate description of (even weakly) nongradient systems by a gradient model requires great care.

An energy functional of the form $E=\int_{-\infty}^{+\infty}dX[|\partial_X F|^2+V(F,F^*)]$ defines a gradient system whose evolution is described by the equation $\partial_\tau F=-\delta E/\delta F^*$. If the potential energy V is taken in the form $V_{gl}=-\gamma|F|^2+\frac{1}{2}|F|^4$, then the we obtain the Ginzburg-Landau equation, which describes the evolution of a system in the vicinity of a supercritical Hopf bifurcation at $\gamma=0$. Note that V_{gl} is invariant with respect to the gauge (phase rotation) symmetry $F\rightarrow Fe^{i\phi}$. The only *state* of the system invariant under such phase rotations is $F=0$, and it loses its stability for $\gamma>0$. This indicates that any asymptotic state for $\gamma>0$ has a broken gauge symmetry. If this system is now subjected to parametric forcing with frequency twice that of the Hopf oscillation, then V_{gl} is replaced by $V_{pgl}=V_{gl}-\mu(F^2+F^{*2})/2$. The gauge symmetry is now intrinsically broken, and replaced by the discrete symmetry $F\rightarrow -F$. Any nontrivial solution of this PGL necessarily has a counterpart flipped by a π phase shift. Perturbations breaking the latter symmetry can also be presented and induce interesting dynamics [2].

It is often important to include nongradient terms, in which case no energy functional E exists. The simplest correction of this type appears when the frequency of the parametric driving is detuned from the doubled Hopf frequency. This leads to the following *nongradient* PGL:

$$\partial_\tau F=\mu F^*+(\gamma+i\delta)F-|F|^2F+\partial_X^2 F, \tag{1}$$

which is the basic model we analyze. Here δ is (half) the frequency detuning. Note that Eq. (1) is not of gradient form if $\delta\neq 0$ (which makes physical sense only if μ is finite). Equation (1) has recently been derived for pulsed optical parametric oscillators with spectral filtering [3] and lasers with intracavity parametric amplification [4]. Its further generalizations, by inclusion, for example, of nonlinear frequency shift, dispersion, or second space dimension, are well known, not only in optics, but also in the contexts of surface water waves [5], ferromagnets [6,7], and liquid crystals [8,9].

Equation (1) is invariant under the joint transformation $F\rightarrow Fe^{-i\psi/2}$ and $\mu\rightarrow|\mu|$, where ψ is the phase of μ. Therefore, for concreteness and without restriction of generality, we assume below that μ is real and positive. It is known [6,5,7] that Eq. (1) has a family of domain walls of Ising type, connecting the two stable spatially homogeneous states with π phase difference. These stable states are given by $F=\sqrt{\kappa}e^{i\phi_I}$, where $\kappa=\gamma+\sqrt{\mu^2-\delta^2}$ and ϕ_I is found from $\sin 2\phi_I=\delta/\mu$. The existence of nontrivial homogeneous states requires $\kappa>0$ and therefore

$$|\delta|<\mu, \quad \gamma>-\sqrt{\mu^2-\delta^2}. \tag{2}$$

The Ising walls (or fronts) connecting these states are then given by

$$F_I=\sqrt{\kappa}g(X)e^{i\phi_I}, \quad g(X)=\tanh(\sqrt{\kappa/2}X). \tag{3}$$

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A characteristic feature of the Ising walls is that they are *symmetric* with respect to the change $(F,X)\rightarrow(-F,-X)$. As was shown in [6,7], Ising walls can become unstable and bifurcate to Bloch walls. This bifurcation is of pitchfork type, where the unstable Ising wall coexists with a pair of Bloch walls which are transformed into each other under the change $(F,X)\rightarrow(-F,-X)$. A formal difference between Ising and Bloch walls is that the real and imaginary parts of F pass through zero at the same point in space for Ising walls and at slightly different points for Bloch walls. The special case $\delta=0$, in which the PGL is a gradient system, is the only case in which analytic expressions for Bloch walls have so far been found (see, e.g., [6,7,3]). One important feature of the Bloch walls is that whenever $\delta\neq 0$ they exhibit spontaneous motion [7]. For δ close to zero their velocity has been calculated explicitly [10]. Walls that are symmetry partners under $(F,X)\rightarrow(-F,-X)$ move in opposite directions.

In Sec. II we demonstrate that, while the Ising-Bloch threshold of the PGL varies smoothly with δ, the mode responsible for the instability is drastically different in the gradient ($\delta=0$) and nongradient ($\delta\neq 0$) cases. The gradient limit is thus, in a dynamical sense, a singular one. We also find explicit expressions for the critical eigenvalue and eigenvector in the neighborhood of the transition. The eigenvalue is smooth in δ, while the eigenvector is singular.

In Sec. III we demonstrate some qualitative differences between the dynamics of gradient and nongradient systems. We derive approximate dynamical equations for the order parameter close to the Ising-Bloch transition of the PGL. We show that the transition is a pitchfork bifurcation with the front velocity as the order parameter, and find an analytic expression for the velocity of stable Bloch fronts as a function of a bifurcation parameter. By considering slow spatial variations of a parameter we find dynamical equations governing front dynamics, and demonstrate that gradient and nongradient PGL systems respond to perturbations in qualitatively different ways. For example, we predict oscillations of Bloch walls in a weak parabolic potential for $\delta\neq 0$, whereas oscillatory dynamics is forbidden in the gradient case. We demonstrate very satisfactory agreement between our approximate analytic and numerical results in this and in all other cases analyzed.

In Sec. IV we conclude with a summary of our results and a brief discussion of their experimental implications.

II. LINEAR STABILITY ANALYSIS OF THE ISING WALLS

The singularity of the gradient limit of the PGL is associated with the eigenvector responsible for the transition from Ising to Bloch walls. Therefore a linear stability analysis of the Ising walls is a natural starting point. In the process we will show that the stability threshold and critical eigenvalue for the Ising walls can be found in a closed analytical form for any value of δ. These were previously known only for $\delta=0$ [7,10],

We look for solutions of Eq. (1) in the form of a perturbed Ising wall,

$$F(x,t)=\sqrt{\kappa}[\tanh(x)+\tilde{u}(x,t)+i\tilde{w}(x,t)]e^{i\phi_I}, \qquad (4)$$

where we have introduced the convenient space-time scalings $t=\kappa\tau/2$, $x=\sqrt{\kappa/2}X$, and where $\tilde{u}$ and $\tilde{w}$ are real. The resulting equations for $\tilde{u}$ and $\tilde{w}$ are

$$\partial_t\vec{U}=\hat{\mathcal{L}}\vec{U}+\vec{N}, \qquad (5)$$

where $\vec{U}=(\tilde{u},\tilde{w})^T$,

$$\hat{\mathcal{L}}=\begin{bmatrix}\hat{D}_1 & -4\delta/\kappa\\ 0 & \hat{D}_2-3+4\gamma/\kappa\end{bmatrix},$$

$$\hat{D}_1=\partial_x^2+2-6\tanh^2(x),$$

$$\hat{D}_2=\partial_x^2+1-2\tanh^2(x),$$

$$\vec{N}=-2\tanh(x)\begin{bmatrix}3\tilde{u}^2+\tilde{w}^2\\ 2\tilde{u}\tilde{w}\end{bmatrix}-2\begin{bmatrix}\tilde{u}^3+\tilde{u}\tilde{w}^2\\ \tilde{w}^3+\tilde{w}\tilde{u}^2\end{bmatrix}.$$

The following linearly independent solutions of $\hat{D}_1 z_1=0$ and $\hat{D}_2 z_2=0$ are important in our analysis:

$$z_{11}=\text{sech}^2(x),$$
$$z_{12}=\sinh(x)\cosh(x)+\frac{3}{2}\,\text{sech}(x)\{\sinh(x)+x\,\text{sech}(x)\}, \qquad (6)$$

and

$$z_{21}=\text{sech}(x), \quad z_{22}=\sinh(x)+x\,\text{sech}(x). \qquad (7)$$

Assuming that $\tilde{u}=u(x)e^{\lambda t}+u^*(x)e^{\lambda^* t}$, $\tilde{w}=w(x)e^{\lambda t}+w^*(x)e^{\lambda^* t}$, where u and w are small, we linearize and find that the stability of the Ising walls is determined by the eigenvalue problem $\hat{\mathcal{L}}\vec{\xi}=\lambda\vec{\xi}$, where $\vec{\xi}=(u,w)^T$. The Ising wall is stable provided that $\hat{\mathcal{L}}$ has no eigenvalue with positive real part.

It follows from the translational symmetry of $\hat{\mathcal{L}}$ that it possesses a Goldstone mode everywhere in the region of existence of the Ising walls. This mode is given by

$$\vec{\xi}_x=(\text{sech}^2(x),0)^T, \qquad (8)$$

which obeys the neutral mode equation $\hat{\mathcal{L}}\vec{\xi}_x=\vec{0}$. It is clear that $\vec{\xi}_x$ is just the gradient of the Ising wall, and so excitation of this neutral mode results in motion of the wall. The Ising-Bloch transition is characterized by an eigenvalue λ_{ib} that changes sign. Thus exactly at threshold we might expect there to be a second null eigenvector $\vec{\xi}_{ib}$ of $\hat{\mathcal{L}}$. As we will see, the actual behavior is generally more complex and interesting.

A. Gradient case: $\delta=0$

For $\delta=0$ our system is gradient and therefore $\hat{\mathcal{L}}$ is self-adjoint. The vector eigenvalue problem $\hat{\mathcal{L}}\vec{\xi}=\lambda\vec{\xi}$ splits now into two scalar eigenvalue problems for the Schrödinger operators with $\tanh^2$ potentials. Both discrete and continuum parts of the spectra for such problems can be found analytically [11]. The eigenvector and eigenvalue governing the Ising-Bloch transition are

$$\vec{\xi}_{ib}^{(g)}=(0,\mathrm{sech}(x))^T, \quad \lambda_{ib}^{(g)}=\frac{\gamma-3\mu}{\gamma+\mu}, \tag{9}$$

respectively. (The superscript stands for *gradient*.) The Ising-Bloch transition takes place at $\mu=\mu_{ib}^{(g)}=\gamma/3$ [6,7]. Note that the inner product of the translational and Ising-Bloch modes is zero, i.e., $\langle\vec{\xi}_{ib}^{(g)}|\vec{\xi}_x\rangle=0$, where $\langle\cdots|\cdots\rangle$ denotes the scalar product in L_2. This is not surprising, since the eigenfunctions of a self-adjoint operator form an orthogonal basis.

B. Nongradient case: $\delta\neq 0$

1. Instability threshold and singular rotation of the critical eigenvector

For $\delta\neq 0$ the eigenvalue problem is more complicated. We first consider the system exactly at the bifurcation threshold with $\lambda_{ib}=0$. Knowing that for $\delta\neq 0$ Bloch walls move [7], we assume that the Ising-Bloch transition for $\delta\neq 0$ is caused by an eigenvector *parallel* to the translational mode, i.e., we will make the assumption $\vec{\xi}_{ib}=\vec{\xi}_x$ and verify its correctness *a posteriori*. This assumption seems rather paradoxical, given that in the gradient limit the Ising-Bloch transition is caused by an eigenvector *orthogonal* to the translational mode. One might expect small nongradient effects to cause only a small rotation of $\vec{\xi}_{ib}$, which would thus acquire only a small projection along $\vec{\xi}_x$. Instead we are proposing that, as soon as δ becomes nonzero, there is a discontinuous rotation of $\vec{\xi}_{ib}$ in function space, from orthogonal to parallel to $\vec{\xi}_x$.

One alternative scenario is that for $\delta\neq 0$ a secondary bifurcation takes place in the vicinity of $\mu=\mu_{ib}^{(g)}$, very close to the gradient limit. In other words, perhaps the transitions from Ising to Bloch walls and from rest to motion are separate bifurcations. However, this turns out not to be the case.

The condition $\vec{\xi}_{ib}=\vec{\xi}_x$ implies that the critical Ising-Bloch eigenmode $\vec{\xi}_{ib}$ of $\hat{\mathcal{L}}$ exactly coincides with the translational mode at threshold, as a double zero eigenvalue with just a single eigenvector. This happens if and only if [11] the operator $\hat{\mathcal{L}}$ possesses a *root vector* or *generalized eigenvector* $\vec{\xi}_r$. This vector, which plays an important role in the subsequent derivations, is a solution of the equation

$$\hat{\mathcal{L}}\vec{\xi}_r+\vec{\xi}_x=\vec{0}. \tag{10}$$

The solvability condition of Eq. (10) requires $\vec{\xi}_x$ to be orthogonal to the corresponding eigenvector $\vec{a}_x$ of the adjoint operator $\hat{\mathcal{L}}^\dagger$, defined by $\hat{\mathcal{L}}^\dagger\vec{a}_x=0$:

$$\langle\vec{\xi}_x|\vec{a}_x\rangle=0. \tag{11}$$

Note that this condition can never be fulfilled for a self-adjoint operator, and in particular cannot hold in the gradient limit of the PGL.

We assert that the condition (11) can be considered as a criterion for the transition from an Ising wall at rest to a Bloch wall in motion. Indeed, it has previously been found to identify similar bifurcations of localized structures in lasers [12] and polarization fronts in intracavity second harmonic generation [13]. The crucial point here is that in the parametric Ginzburg-Landau model the Ising-Bloch threshold can be continued to the gradient limit, and the interplay between gradient and nongradient effects can be analyzed. This was not possible in the models of Refs. [13,12]. The gradient limit is the exceptional case of a transition from an Ising wall at rest to a Bloch wall that is also at rest, and the criterion (11) does not apply.

The components of the vector $\vec{a}_x=(a_{x1},a_{x2})^T$ obey the system of equations

$$\hat{D}_1 a_{x1}=0, \quad (\hat{D}_2-3+4\gamma/\kappa)a_{x2}=4\delta a_{x1}/\kappa. \tag{12}$$

As follows from Eqs. (6), the only nontrivial spatially localized solution of $\hat{D}_1 a_{x1}=0$ is

$$a_{x1}=Q\,\mathrm{sech}^2(x), \tag{13}$$

where Q is a constant. Therefore a_{1x} coincides, up to a constant, with the first component of the translational mode $\vec{\xi}_x$. Thus the only possibility for $\vec{a}_x$ to be both different from zero and orthogonal to $\vec{\xi}_x$ is that $Q=0$ and the equation $(\hat{D}_2-3+4\gamma/\kappa)a_{x2}=0$ has a nontrivial spatially localized solution. It is clear from Eqs. (7) that such a solution exists if $\kappa=\kappa_{ib}=4\gamma/3$, or equivalently $\mu=\mu_{ib}$, where

$$3\mu_{ib}=\sqrt{9\delta^2+\gamma^2}. \tag{14}$$

If this condition holds, we can set $\vec{a}_x=(0,\mathrm{sech}(x))^T$, and can proceed to solve Eq. (10), which takes the form

$$\hat{D}_2\xi_{r2}=0, \quad \hat{D}_1\xi_{r1}=g_1, \tag{15}$$

where $g_1(x)=3\delta\xi_{r2}/\gamma-\mathrm{sech}^2(x)$. From Eqs. (7) it is clear that $\xi_{r2}=R\,\mathrm{sech}(x)$, where R is a constant to be determined. The solvability condition of the second equation of system (15) requires orthogonality of g_1 to z_{11}, yielding $R=8\gamma/(9\pi\delta)$ and

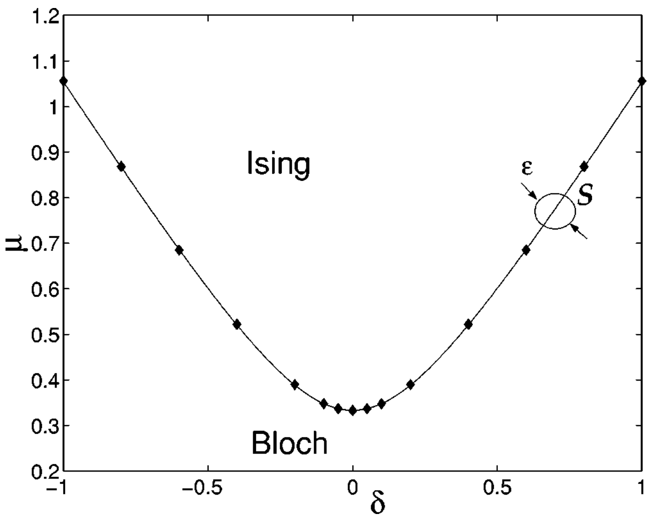

FIG. 1. Threshold of the Ising-Bloch transition in the plane (δ,μ) for $\gamma=1$. Full line corresponds to Eq. (14). Diamonds correspond to the numerically found locations of the double degeneracy of the translational mode $\vec{\xi}_x$, $\hat{\mathcal{L}}\vec{\xi}_x=\vec{0}$. The circle $\mathcal{S}$ schematically indicates the region of validity of the asymptotic theory developed in Sec. II B.

$$\vec{\xi}_r=\begin{bmatrix} \dfrac{8}{3\pi}I_{11}(x)-I_{12}(x) \\ \\ \dfrac{8\gamma}{9\pi\delta}\operatorname{sech}(x)\end{bmatrix}, \tag{16}$$

where

$$I_{mn}(x)=\frac{1}{W_m}\left[z_{m2}(x)\int_{x_0}^{x}\operatorname{sech}^n(x')z_{m1}(x')dx' - z_{m1}(x)\int_{x_0}^{x}\operatorname{sech}^n(x')z_{m2}(x')dx'\right],$$

$m=1,2$, $n=1,2$, and $W_1=4$, $W_2=2$. (The general definition of I_{mn} will be used at several places in the later development.)

There is a corresponding degeneracy for $\hat{\mathcal{L}}^{\dagger}$ with a single corresponding eigenvector

$$\vec{a}_x=\vec{a}_{ib}=(a_{ib1},a_{ib2})^T=(0,\operatorname{sech}(x))^T. \tag{17}$$

We have now shown that Eq. (14) implies a bifurcation, but not that it is an Ising-Bloch transition. We will do so below, when we derive the explicit form of $\vec{\xi}_{ib}$ in the neighborhood of the transition, but note that this bifurcation threshold coincides precisely with the Ising-Bloch transition in the gradient limit $\delta=0$.

Comparing Eq. (14) and the existence conditions (2), one can conclude that an Ising-Bloch transition is possible only for $\gamma>0$, i.e., when there is linear gain in the system. In the three-parameter space (μ,γ,δ) the critical surface defined by Eq. (14) is a half cone separating regions of stability of Bloch and Ising walls. Figure 1 shows a cross section of this cone for fixed δ.

The singularity of the gradient limit of the PGL with regard to the Ising-Bloch transition is now clear. Everywhere on the critical surface the critical Ising-Bloch mode is orthogonal to its adjoint mode, except on the intersection of the cone with the plane $\delta=0$, where $\hat{\mathcal{L}}$ becomes self-adjoint, and thus the critical mode is necessarily parallel to its adjoint. Specifically, in Fig. 1 $\langle\vec{\xi}_{ib}|\vec{a}_{ib}\rangle$ is zero everywhere on the critical parabola, except precisely at its lowest point (where it is equal to 2).

We found excellent agreement between Eq. (14) and the numerical solutions of the eigenvalue problem $\hat{\mathcal{L}}\vec{\xi}=\lambda\vec{\xi}$; see Fig. 1. Detailed numerical studies of the eigenvalue problem $\hat{\mathcal{L}}\vec{\xi}=\lambda\vec{\xi}$ have confirmed that the condition $\mu=\mu_{ib}$ gives the only instability threshold for the Ising walls. These facts exclude the existence of secondary bifurcations in the neighborhood of the gradient limit. We can conclude that arbitrarily small deviations from the variational limit result in the rotation of the critical eigenvector governing the Ising-Bloch transition from being perpendicular to being parallel to the translational mode. This rotation is not just a mathematical curiosity but can have drastic effects on the dynamics described by Eq. (1) as proved below.

2. Critical eigenvalue and eigenvector

At any point close to the Ising-Bloch threshold the critical eigenvalue λ_{ib} is small (of order ϵ). It is readily verified that to lowest order in ϵ, i.e., close enough to the surface on which Eq. (14) is satisfied,

$$\vec{\xi}_{ib}=\vec{\xi}_x-\lambda\vec{\xi}_r+O(\epsilon^2) \tag{18}$$

is an eigenvector of $\hat{\mathcal{L}}$ with eigenvalue λ_{ib}. On the critical surface $(\epsilon=0)$ this eigenvalue goes to zero, and the eigenvector coincides with $\vec{\xi}_x$, exactly as described above. Using our explicit expression for the second component of $\vec{\xi}_r$, we can now see directly from Eq. (18) that we are indeed dealing with an Ising-Bloch transition, with $w(x)$ finite in the perturbation that distorts the Ising wall. Exactly as for the gradient case, the effect of finite $w(x)$ is to split the points at which the real and imaginary parts of F pass through zero, turning the Ising wall into a Bloch wall.

Expression (18) is valid within a sphere $\mathcal{S}$ of radius of order $|\epsilon|$, in the three-dimensional parameter space (μ,γ,δ) (see Fig. 1). $\mathcal{S}$ is centered on a chosen bifurcation point on the critical surface, i.e., for parameter values specified by the condition Eq. (14), and $\vec{\xi}_r$ is to be evaluated at that point, or at any point on the critical surface lying within $\mathcal{S}$.

Taking the scalar product of both sides of $\hat{\mathcal{L}}\vec{\xi}=\lambda\vec{\xi}$ with $\vec{a}_x$ it follows that

$$\langle\lambda\vec{\xi}|\vec{a}_x\rangle=0 \tag{19}$$

for any eigenvector $\vec{\xi}$ within the entire range of existence of the domain walls. Using Eq. (18) we find that up to the second order of perturbation, i.e., within $\mathcal{S}$, Eq. (19) implies, to lowest order,

$$\lambda\langle\vec{\xi}_x|\vec{a}_x\rangle=\lambda^2\langle\vec{\xi}_r|\vec{a}_{ib}\rangle, \tag{20}$$

where the right side has to be evaluated on the critical surface. The trivial root of this quadratic equation, $\lambda=\lambda_x=0$, corresponds to the translational mode, while the other determines the critical Ising-Bloch eigenvalue

$$\lambda=\lambda_{ib}=\frac{\langle\vec{\xi}_x|\vec{a}_x\rangle}{\langle\vec{\xi}_r|\vec{a}_{ib}\rangle} \tag{21}$$

corresponding to the Ising-Bloch mode. The numerator of this expression can be evaluated anywhere within $\mathcal{S}$, and the denominator anywhere on the part of the threshold surface inside $\mathcal{S}$.

Using Eqs. (16) and (17) we can calculate $\langle\vec{\xi}_r|\vec{a}_{ib}\rangle=\int_{-\infty}^{+\infty}dx\,\xi_{r2}a_{ib2}=16\gamma/(9\pi\delta)$. Thus to calculate an explicit expression for λ_{ib} we need to find a first order approximation for $\vec{a}_x$ within $\mathcal{S}$. We consider μ as the bifurcation parameter, and set $\kappa(\mu)=\kappa_{ib}+\epsilon\mu_1\partial_\mu\kappa+O(\epsilon^2)$, where $\epsilon\mu_1=\mu-\mu_{ib}$. We now write

$$\vec{a}_x=\vec{a}_{ib}+\epsilon\vec{b}+O(\epsilon^2). \tag{22}$$

Here $\vec{b}=(b_1,b_2)^T$ and $b_{1,2}$ obey

$$\hat{D}_1b_1=0,\quad \hat{D}_2b_2=g_2, \tag{23}$$

where $g_2=3\delta b_1/\gamma+27\mu_{ib}\mu_1/(4\gamma^2)\mathrm{sech}(x)$, the second term arising from the κ dependence of $\hat{\mathcal{L}}$. It follows from Eqs. (6) that $b_1=Q\,\mathrm{sech}^2(x)$, consistent with Eq. (13). However, the amplitude Q has now to be determined from the solvability condition of the second equation of (23). The latter requires orthogonality of g_2 to z_{21} and provides $Q=-9\mu_{ib}\mu_1/(\pi\gamma\delta)$. Completing the solution, we obtain

$$\vec{a}_x=\begin{bmatrix}-9\mu_1\mu_{ib}/(\pi\gamma\delta)\mathrm{sech}^2(x)\\ \mathrm{sech}(x)+27\mu_1\mu_{ib}/(4\gamma^2)\left\{I_{21}(x)-\frac{4}{\pi}I_{22}(x)\right\}\end{bmatrix}+O(\epsilon^2). \tag{24}$$

Thus $\langle\vec{\xi}_x|\vec{a}_x\rangle=\int_{-\infty}^{+\infty}dx\,\xi_{x1}b_1+O(\epsilon^2)=4Q/3+O(\epsilon^2)$ and finally

$$\lambda_{ib}=\frac{27(\mu_{ib}-\mu)\mu_{ib}}{4\gamma^2}+O(\epsilon^2). \tag{25}$$

Equation (25) explicitly shows that Ising walls are stable for $\mu>\mu_{ib}$. Figure 2 shows excellent agreement between numerical and analytical results for λ_{ib}. Note that Fig. 2 shows no evidence of singular behavior as $\delta\to 0$. There is no singularity: in the gradient limit $\mu_{ib}=\gamma/3$ the general expression (25) reduces to precisely that was found for $\lambda_{ib}^{(g)}$ [see Eq. (9) and [6,7]].

As already remarked, the singular behavior in δ is associated with the critical eigenvector $\vec{\xi}_{ib}$. The extreme sensitivity of the eigenmodes to δ is illustrated in Fig. 3, which shows the dependence of the normalized scalar product $|\langle\vec{\xi}_x|\vec{a}_x\rangle|/(\langle\vec{\xi}_x|\vec{\xi}_x\rangle\langle\vec{a}_x|\vec{a}_x\rangle)^{1/2}$ on μ for different values of δ. The magnitude of the scalar product is close to unity, i.e., the translational mode and its adjoint are nearly parallel, everywhere except within a valley close to $\mu=\mu_{ib}$ (at which the scalar product goes through zero). The decreasing width of the valley for $\delta\to 0$ indicates that the scalar product changes very rapidly for small $|\delta|$. In the limit $\delta\to 0$ the change becomes critical and it takes place suddenly at a single point, i.e., it has zero measure. The singularity also shows itself in $\vec{\xi}_r$, as a consequence of which the first order term in the expansion for $\vec{\xi}_{ib}$ has coefficient $\sim\lambda_{ib}/\delta$. This implies that

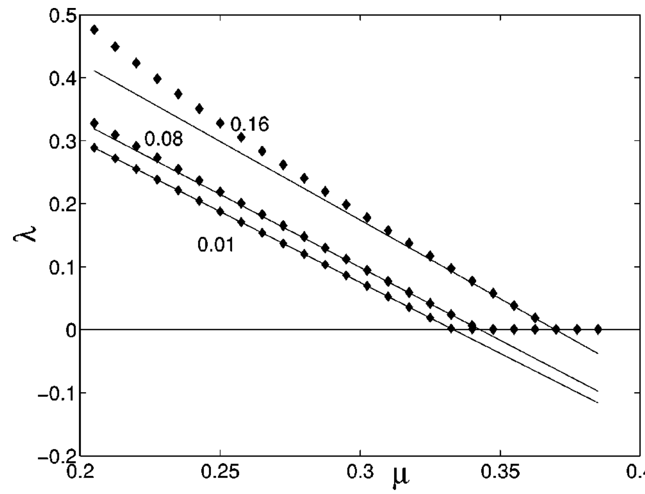

FIG. 2. Eigenvalue λ_{ib} governing the Ising-Bloch transition vs the parametric pump μ for $\gamma=1$. Numerically found eigenvalues of $\hat{\mathcal{L}}$ with maximal real part are shown by diamonds, while the solid lines are calculated from the analytic formula presented in Eq. (25). The numbers near the curves indicate the corresponding values of δ.

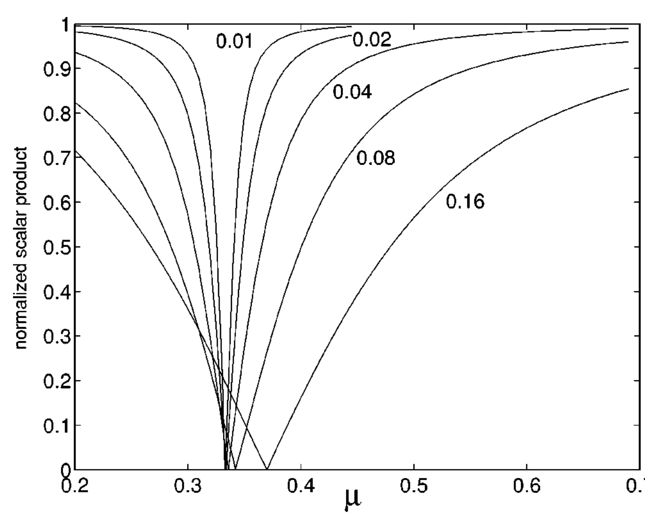

FIG. 3. Normalized scalar product $|\langle\vec{\xi}_x|\vec{a}_x\rangle|/(\langle\vec{\xi}_x|\vec{\xi}_x\rangle\langle\vec{a}_x|\vec{a}_x\rangle)^{1/2}$ vs μ: $\gamma=1$. Numbers near the curves indicate corresponding values of δ. Variations of the scalar product become faster when the linear detuning δ approaches zero.

the region of validity of the asymptotic expansion (18) for $\vec{\xi}_{ib}$ shrinks in the gradient limit and disappears for $\delta=0$, where one needs to use Eq. (9) for the critical eigenvector.

III. WEAKLY NONLINEAR THEORY AND ROLE OF SPATIAL INHOMOGENEITIES

In real systems the translational invariance of Eq. (1) is often, if not invariably, broken by inhomogeneities of the medium, pump, and boundary conditions and by defects. In such situations one expects that domain walls should drift toward the nearest minimum of an effective potential created by the inhomogeneities, leading to a pinning effect. If we are far from the Ising-Bloch threshold and inhomogeneities are weak, they act directly on the velocity of the wall, and the effective gradient (Aristotelian) ''force'' acting on the domain wall is easily calculated by projecting the inhomogeneities onto the translational mode of $\hat{\mathcal{L}}$ by taking its scalar product with $\vec{a}_x$.

If, however, we are close to an Ising-Bloch transition, then the dynamics become less trivial because the translational degree of freedom is now coupled to the dynamics of the amplitude of the critical mode. In particular, we will show below that the singular rotation of the critical mode described in the previous section has a profound influence on the wall dynamics. To this end, we derive the dynamical equations for the order parameters of domain walls close to the Ising-Bloch transition for both the gradient and nongradient cases and demonstrate their qualitative difference.

The presence of inhomogeneities leads to a drift of the domain wall. Therefore we assume that its position x_0 is an adiabatic function of time, $x_0=x_0(\epsilon^3 t)$. For convenience we switch to the frame of coordinates moving with the wall center, $x\to x+x_0$ [previously we implicitly assumed $x_0=0$, e.g., in Eq. (3)]. We assume that the parametric pump in Eq. (1) is a function of X and make the following substitution:

$$\mu\to\mu+\frac{\epsilon^3\kappa_{ib}}{2}\mu_3(\sqrt{\kappa_{ib}/2}X)e^{2i\phi_I}. \tag{26}$$

Below we restrict ourselves to the following form of $\mu_3(x)$:

$$\mu_3=(a_r+ia_i)x+(b_r+ib_i)x^2. \tag{27}$$

The term linear in x describes any inhomogeneities of gradient type while the quadratic term approximates, e.g., a smooth Gaussian pump of an optical resonator. We allow for complex perturbations to μ, because the global symmetry that allowed us to assume μ real is (weakly) violated by these weak inhomogeneities.

We can rewrite Eq. (1) in the form

$$\partial_t\vec{U}=\hat{\mathcal{L}}\vec{U}+\partial_t x_0(\vec{\xi}_x+\partial_x\vec{U})+\vec{N}+\epsilon^3\hat{B}(\vec{G}+\vec{U}), \tag{28}$$

where $\vec{G}=(\tanh(x),0)^T$, and

$$\hat{B}=\begin{bmatrix}\mathrm{Re}\,\mu_3(x+x_0) & \mathrm{Im}\,\mu_3(x+x_0)\\ \mathrm{Im}\,\mu_3(x+x_0) & -\mathrm{Re}\,\mu_3(x+x_0)\end{bmatrix}. \tag{29}$$

We analyze this equation by making suitable expansions of its terms, first for the gradient and then for the nongradient case.

A. Gradient case: $\delta=0$

Assuming that the deviation of κ from κ_{ib} is second order in the perturbation we can set $\hat{\mathcal{L}}=\hat{\mathcal{L}}_0+\hat{\mathcal{L}}_2$, $\hat{\mathcal{L}}_0=\hat{\mathcal{L}}(\kappa=\kappa_{ib}^{(g)})$, where $\hat{\mathcal{L}}_2\equiv(\hat{\mathcal{L}}-\hat{\mathcal{L}}_0)\sim\epsilon^2$. We now introduce the slowly varying amplitude of the critical mode, i.e., the order parameter, $A=A(\epsilon^2 t)\sim\epsilon$ and search for solutions of Eq. (28) of the form

$$\vec{U}=A\vec{\xi}_{ib}^{(g)}+A^2\vec{\xi}_2+A^3\vec{\xi}_3+\cdots. \tag{30}$$

At first order we have $\hat{\mathcal{L}}_0\vec{\xi}_{ib}^{(g)}=0$. The equation for the second order is $\hat{\mathcal{L}}_0\vec{\xi}_2=-\vec{N}_2$, where $\vec{N}_2=-2\tanh(x)(\mathrm{sech}^2(x),0)^T$ and thus $\vec{\xi}_2=(\xi_{21},\xi_{22})^T=(-\frac{1}{2}x\,\mathrm{sech}^2(x),0)^T$.

The main information, as usual, is obtained from the third order equation

$$-\epsilon^3\hat{\mathcal{L}}_0\vec{\xi}_3=A^3\vec{N}_3+\partial_t x_0\vec{\xi}_x+\epsilon^3\hat{B}\vec{G}-(\partial_t A-A\hat{\mathcal{L}}_2)\vec{\xi}_{ib}^{(g)}, \tag{31}$$

where $\vec{N}_3=2\,\mathrm{sech}^3(x)(0,x\tanh(x)-1)^T$. The solvability conditions of Eq. (31) give a system of equations for A and x_0:

$$2\partial_t A=\frac{9(\mu_{ib}^{(g)}-\mu)}{2\gamma}A-2A^3+\pi a_i+2\pi b_i x_0, \tag{32}$$

$$\frac{4}{3}\partial_t x_0+a_r+2b_r x_0=0. \tag{33}$$

These are the dynamical equations we seek. In this gradient case there are separate equations governing the Ising-Bloch transition and the wall motion. As expected, the location of the wall is unaffected by the amplitude A of the critical mode. Any real positive gradient of the pump, i.e., $a_r>0$, $b_r=0$, results in motion of the $\tanh(x)$-like domain wall with negative velocity $-3a_r/4$ and of the $-\tanh(x)$-like domain wall with positive velocity $3a_r/4$. If $b_r\neq 0$ and $a_r b_r<0$ then the wall will be pinned at the point $x_0=-a_r/(2b_r)$. The pinning position is stable if $b_r>0$ and unstable otherwise.

In the absence of perturbations the equation for A describes a classic pitchfork bifurcation. A couples only to the imaginary part of the spatially dependent perturbation which, if present, makes the Ising-Bloch transition an imperfect pitchfork bifurcation. A spatially uniform perturbation has previously been shown to render the pitchfork imperfect, leading to interesting interaction dynamics of front pairs [2].

B. Nongradient case: $\delta\neq 0$

For arbitrary small deviations from the gradient limit the critical mode collapses onto the translational one and therefore the order parameter evolution is expected to be described by a single equation for the position of the wall. In this case it is more convenient to proceed along the lines of the method applied in Sec. II B, which does not require the asymptotic expansion of the operator $\hat{\mathcal{L}}$. We simply assume that the system parameters are sufficiently close to the Ising-Bloch transition. The distance from the transition boundary can be conveniently measured by the value of $\langle \vec{\xi}_x | \vec{a}_x \rangle$, which should be taken to be of order ϵ^2. Thus the weakly nonlinear theory developed below has narrower region of validity than the linear theory of Sec. II B. In order to guarantee consistency with the rest of the expansion, we introduce additional assumptions about the time derivatives of the position, taking $\partial_t x_0 \sim \epsilon$ and $\partial_t^2 x_0 \sim \epsilon^3$. To solve Eq. (28) we insert the asymptotic expansion

$$\vec{U} = \partial_t x_0 \vec{\xi}_r + (\partial_t x_0)^2 \vec{\xi}_2 + (\partial_t x_0)^3 \vec{\xi}_3 + \cdots \tag{34}$$

and find

$$\begin{aligned} -\hat{\mathcal{L}}[\partial_t x_0 \vec{\xi}_r + (\partial_t x_0)^2 \vec{\xi}_2 + (\partial_t x_0)^3 \vec{\xi}_3] &= -\partial_t^2 x_0 \vec{\xi}_r + \partial_t x_0 [\vec{\xi}_x + \partial_t x_0 \partial_x \vec{\xi}_r + (\partial_t x_0)^2 \partial_x \vec{\xi}_2] \\ &\quad + (\partial_t x_0)^2 \vec{N}_2 + (\partial_t x_0)^3 \vec{N}_3 + \epsilon^3 \hat{B} \vec{G} + O(\epsilon^4), \end{aligned} \tag{35}$$

where $\vec{N}_2$ and $\vec{N}_3$ are second and third order nonlinear terms.

The qualitative difference in the leading order terms in Eqs. (30) and (34) indicates that no smooth transition between weakly nonlinear dynamical theories for gradient and nongradient cases can be expected. This is again because in the nongradient case there is in fact only one, though degenerate, critical mode, while in the gradient case there are two different and mutually orthogonal modes.

Since $\hat{\mathcal{L}}$ is a singular operator, we must impose a solvability condition, by requiring the right hand side of Eq. (35) to be orthogonal to $\vec{a}_x$. Using symmetry arguments it is possible to demonstrate that all terms of order ϵ^2 are equal to zero so that we are left with a single differential equation for the wall position:

$$\begin{aligned} \langle \vec{\xi}_r | \vec{a}_x \rangle \partial_t^2 x_0 &= \langle \vec{\xi}_x | \vec{a}_x \rangle \partial_t x_0 + \langle \partial_x \vec{\xi}_2 + \vec{N}_3 | \vec{a}_x \rangle (\partial_t x_0)^3 \\ &\quad + \langle \hat{B} \vec{G} | \vec{a}_x \rangle, \end{aligned} \tag{36}$$

where $\vec{N}_3 = (n_{31}, n_{32})^T$, $\vec{\xi}_2 = (\xi_{21}, \xi_{22})^T$, $n_{32} = -4 \tanh(x)(\xi_{r1}\xi_{22} + \xi_{21}\xi_{r2}) - 2\xi_{r2}^3 - 2\xi_{r2}\xi_{r1}^2$, $\xi_{22} = (8\gamma/9\pi\delta) f_1(x)$, $\xi_{21} = f_2(x) - (x/2)(8\gamma/9\pi\delta)^2 \operatorname{sech}^2(x)$, and $f_{1,2}$ solve the parameter independent equations $\hat{D}_2 f_1 = \tanh(x)\operatorname{sech}(x)(1 + 4\xi_{r1})$ and $\hat{D}_1 f_2 = (8/3\pi) f_1 - \partial_x \xi_{r1} + 6 \tanh(x) \xi_{r1}^2$. After some algebra one can find that Eq. (36) transforms into the desired equation governing the front dynamics in the vicinity of a nongradient Ising-Bloch transition:

$$\begin{aligned} 2\partial_t^2 x_0 &= \frac{27(\mu_{ib} - \mu)\mu_{ib}}{2\gamma^2} \partial_t x_0 - \left(2\left[\frac{8\gamma}{9\pi\delta}\right]^2 + C \right) (\partial_t x_0)^3 \\ &\quad + \left[\frac{9\pi\delta}{8\gamma}\right] \pi(a_i + 2x_0 b_i), \end{aligned} \tag{37}$$

where $C \simeq 0.72$ is a constant that can be evaluated numerically.

This is a fully explicit amplitude equation governing the dynamics of the domain walls in the vicinity of the Ising-Bloch transition and in the presence of spatial inhomogeneities. The complex calculations leading to the manifestly negative coefficient of $(\partial_t x_0)^3$ are amply repaid, because we obtain the important result that the Ising-Bloch transition in the parametric Ginzburg-Landau equation is always supercritical.

The next important consequence of Eq. (37) results if we ignore spatial inhomogeneities and look for a stationary solution, i.e., constant front velocity. We find that the equilibrium velocity v of the Bloch walls in the vicinity of the Ising-Bloch transition obeys

$$v^2 = \frac{27(\mu_{ib} - \mu)\mu_{ib}}{2\gamma^2(2[8\gamma/9\pi\delta]^2 + C)}. \tag{38}$$

This expression is characteristic of a pitchfork bifurcation, with v as the order parameter. Expressions for the velocity of the Bloch walls obtained previously [7,10] are valid only for small δ and far from the Ising-Bloch transition, since they were derived under the assumption of small deviations from the gradient limit. Our expression Eq. (38) is instead valid only in the vicinity of the transition, but arbitrarily far from the gradient limit. It thus represents a substantial extension of previous results. Note that in the original coordinates X and τ Eq. (38) should be replaced by $\kappa v^2/2$. Taking this into account, we can compare the two expressions for the wall velocity, our Eq. (38) and Eq. (22) from Ref. [10], in the

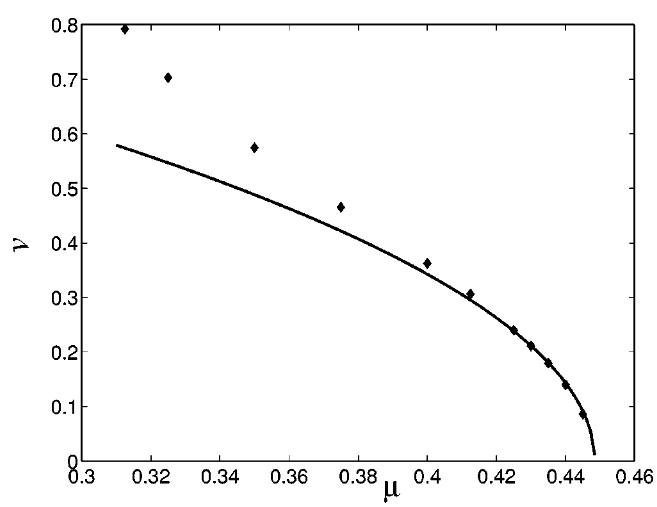

FIG. 4. Comparison between numerical (diamonds) and analytical results (full line) given by Eq. (38) for velocity of the Bloch wall. $\gamma = 1$, $\delta = 0.3$.

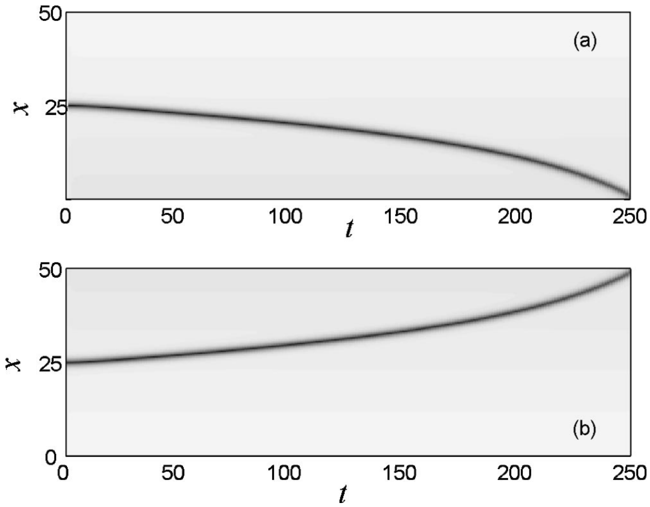

FIG. 5. Spatiotemporal evolution of the Ising wall in the presence of a positive linear gradient for opposite signs of the detuning δ: (a) $\delta=-0.4$, (b) $\delta=0.4$. Other parameters are $\gamma=1$, $\mu=0.5$, $a_r=b_r=b_i=0$, $a_i=0.0055$. The darkest areas correspond to the smallest values of $|F|$.

limit $|\delta|\ll 1$, $|\mu_{ib}-\mu|\ll 1$ where both formulas are valid. Making the appropriate expansions and substitutions we find precise correspondence between the two expressions [14]. Comparison of Eq. (38) with values of the velocity computed by the direct modelling of Eq. (1) is shown in Fig. 4.

Considering now the response to spatial inhomogeneities, we note that in the nongradient case the wall motion is driven by phase perturbations to μ, again in complete contrast to the gradient limit in which it is the real part of μ that induces motion—see Eq. (33). It follows from Eq. (37) that where the Ising wall is stable a phase gradient perturbation causes it to drift in the direction determined by the sign of the product $a_i\delta$. Figure 5 shows examples from numerical simulations of Eqs. (1) and (27) that confirm and illustrate this prediction. If $b_i\neq 0$ then Eq. (37) has a time independent solution $x_0=-a_i/(2b_i)$, i.e., a pinning point. Stability of the pinning point is determined by the roots of the characteristic equation $\lambda^2=\lambda_{ib}\lambda+b_i\pi[9\pi\delta/8\gamma]$. If $b_i\delta<0$ and $\lambda_{ib}<0$, then the domain wall relaxes to its pinning position either monotonically or, for $\lambda_{ib}^2<4b_i\pi[9\pi\delta/8\gamma]$, with oscillations. Thus, sufficiently close to the Ising-Bloch transition, the Ising walls exhibit damped oscillations, a dynamics that is impossible in the gradient limit of the PGL. When crossing the transition point at which $\lambda_{ib}=0$, the wall becomes Hopf unstable, leading to undamped oscillations of its position near the pinning point (see Fig. 6). If $b_i\delta>0$ then the domain walls cannot be pinned by the inhomogeneity. Note that oscillating fronts in the vicinity of the transition from rest to motion have also been previously reported in the special case of a reaction diffusion system [15].

By comparing the results of Secs. III A and III B we can conclude that arbitrarily small nongradient effects lead to the appearance of time periodic dynamics of a pinned domain wall, which is instead prohibited in the gradient limit. The origin of this effect is the singular rotation of the critical eigenvector, which leads to a qualitatively different asymptotic expansion to be used in the two cases.

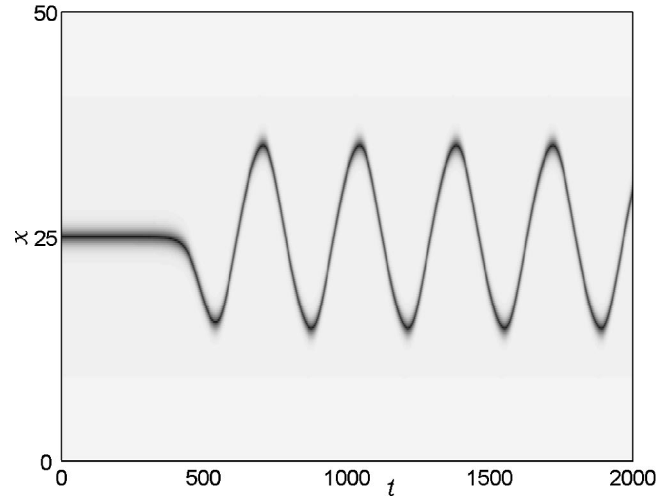

FIG. 6. Oscillations of the Bloch wall in the parabolic potential: $\gamma=1$, $\mu=0.4$, $\delta=0.25$, $a_r=b_r=a_i=0$, $b_i=0.000\,15$. The darkest areas correspond to the smallest values of $|F|$.

IV. SUMMARY

We have considered dynamical and stability properties of domain walls in the vicinity of the Ising-Bloch transition and have demonstrated that the critical eigenvector responsible for the transition undergoes a singular rotation when arbitrarily small nongradient effects are included. We showed that the Ising-Bloch threshold and critical eigenvalue vary smoothly between the gradient and nongradient cases, and we derived explicit analytic expressions for the threshold and eigenvalue. Weakly nonlinear asymptotic theories have been developed in the vicinity of the transition and in the presence of spatial inhomogeneities. An explicit expression for the velocity of Bloch walls, valid even far from the gradient limit, has been derived. It was shown that an Ising wall trapped by a pinning potential may exhibit damped oscillations on approach to the pinning point, while a Bloch wall may oscillate spontaneously around a pinning point. Such oscillatory dynamics is perhaps the clearest practical signature of the singular Ising-Bloch dynamics identified in this work. It should be practically accessible in experiments similar to that described in [8,9] where spontaneous oscillation of Bloch walls in a layer of liquid crystal were observed in the presence of inhomogeneous magnetic fields. Suppression of transverse instabilities [8,9] can be achieved in quasi-one-dimensional geometries.

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