

## Quantum effects in high-gain free-electron lasers

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A many-particle fully quantized theory for a free-electron laser which is valid in the high-gain regime is presented. We examine quantum corrections for the high-gain single-pass free-electron laser. It is shown that quantum effects become significant when the photon energy becomes comparable to the gain bandwidth. The initiation of the free-electron laser process from quantum fluctuations in the position and momentum of the electrons is considered, and the parameter regime for enhanced start-up is identified. Photon statistics of the free-electron laser radiation are discussed, and the photon number statistics for the self-amplified spontaneous emission are calculated.

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### I. INTRODUCTION

The free-electron laser (FEL) holds great potential as a source of intense coherent short-wavelength radiation. A single-pass high-gain FEL operating in the self-amplified spontaneous emission (SASE) mode [1] has received much attention recently as a candidate for the next generation light sources producing coherent x rays. Coherent x-rays have a wide range of applications such as x-ray spectroscopy, medical and biological imagery, holography, and analysis of ultrafast processes. Presently there are major proposals in the United States [2] and Europe [3] to construct a SASE FEL operating in the x-ray regime.

A conventional FEL amplifies coherent radiation by means of a relativistic electron beam passing through a periodic static magnetic field (magnetostatic undulator). The FEL process can be understood as the scattering of virtual undulator photons by the electron beam into photons of the radiation field, i.e., an exchange of photons between the undulator and the radiation, with the electrons providing the necessary momentum. This is a resonant process which emits radiation at the resonant wavelength

$$\lambda_r = \frac{\lambda_u}{2\gamma^2}(1 + K^2), \quad (1)$$

where  $\lambda_u$  is the undulator wavelength,  $\gamma$  is the electron beam Lorentz factor, and  $K$  is the undulator strength parameter (normalized vector potential of the undulator magnetic field). In the limit of an ultrarelativistic ( $\gamma \gg 1$ ) electron beam, the FEL process is analogous to Compton backscattering (with  $\lambda_u \approx \lambda_0/2$ , where  $\lambda_0$  is the incident photon wavelength).

As Eq. (1) indicates, production of short wavelength radiation requires either high-energy electron beams or short undulator wavelengths. In addition to x-ray production by conventional FEL's, there have been proposals and experimental work to generate x rays by stimulated Compton scat-

tering [4] of a comparatively low-energy electron beam, effectively replacing the conventional magnetostatic undulator with a counterpropagating high-intensity laser pulse (a short-wavelength electromagnetic undulator). Electrostatic undulators may also be considered for radiation generation, e.g., the interaction of a relativistic electron beam propagating through a plasma [5].

The first description of the gain mechanism of the FEL [6] relied on quantum recoil corrections to the frequencies of the emitted and absorbed photons, for which there is no classical analog. In the limit of small recoil, the main features of the FEL process are well described in terms of classical quantities (e.g., wave electric field amplitude), and, at present, the majority of calculations that deal with existing or proposed FEL devices use classical equations of motion. As experiments move toward the generation of shorter wavelength radiation, with shorter undulator wavelengths, corrections to the classical approximation for the FEL will become significant.

Previous quantum mechanical treatments [6–10] of the FEL have been successful in describing the weak-field noncollective regime. Madey [6] first described the small-signal FEL gain by calculating quantum mechanical transition rates using the Weizsäcker-Williams method. Bosco *et al.* [7] calculated relativistic electron wave functions using quantum electrodynamics in the weak-field regime. An extensive review of solving the single-electron Schrödinger equation through perturbation in the electron recoil was presented in the work by Dattoli and Renieri [10]. These results were derived assuming a small electron recoil due to emission and absorption of discrete photons, and focused on corrections in the small-signal noncollective regime of FEL operation.

In this paper we present a fully quantized (matter and radiation fields) many-particle theory of the FEL which is applicable in the high-gain collective regime. The paper is organized as follows. In Sec. II we present the Hamiltonian for the coupled electron-radiation field system. The details of the derivation of the Hamiltonian operator are presented in the Appendix. The theory is developed in a moving frame where the electron motion can be treated using nonrelativistic mechanics. In Sec. III we calculate the evolution of the expectation value of the photon number operator by solving

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the Heisenberg equations. In Sec. III B we calculate quantum corrections to the stimulated emission. Start-up of SASE is initiated by noise in the electron beam. Therefore, in Sec. III C, we consider quantum fluctuations in the position and momentum of the electrons as an effective source of noise for initiation of the FEL process. In Sec. IV the statistical properties of the FEL radiation are examined, and the photon number statistics for the SASE FEL radiation are calculated. A summary and some discussion of the results are offered in Sec. V.

## II. FEL HAMILTONIAN

The  $N_e$ -electron  $M$ -mode quantized Hamiltonian operator describing the FEL interaction in the frame moving at the mean velocity of the electron beam can be written as

$$H = \sum_{\lambda=1}^M \hbar \bar{\omega}_\lambda \left( a_\lambda^\dagger a_\lambda + \frac{1}{2} \right) + \sum_{j=1}^{N_e} \hbar \Omega \frac{\bar{p}_j^2}{2} + \sum_{\lambda=1}^M \hbar g_\lambda \left( a_\lambda^\dagger a_u \sum_{j=1}^{N_e} e^{-i\theta_{\lambda j}} + \text{H.c.} \right). \quad (2)$$

The parameter  $\Omega = \hbar \bar{k}_r^2 / m$  determines the electron recoil, and the parameter

$$g_\lambda = \frac{2\pi e^2}{mcV\sqrt{k_\lambda k_u}} \quad (3)$$

determines the strength of the coupling between the undulator and radiation field, where  $m = m_e(1 + K^2)^{1/2}$  is the renormalized electron mass,  $k_r = 2\pi/\lambda_r$ , and  $k_u = 2\pi/\lambda_u$ . The Hilbert space operators in Eq. (2) satisfy the commutation relations  $[\theta_{\lambda i}, \bar{p}_j] = i(\bar{k}_\lambda / \bar{k}_r) \delta_{ij}$  and  $[a_\lambda, a_\mu^\dagger] = \delta_{\lambda\mu}$ , where  $\theta_{\lambda j} = \bar{k}_\lambda z'_j$  is the FEL phase,  $\bar{p}_j = p'_j / (\hbar \bar{k}_r)$  is the normalized (to the recoil provided by a photon exchange between the undulator and the resonant radiation field) electron axial momentum, and  $a_\lambda$  ( $a_\lambda^\dagger$ ) are the photon annihilation (creation) operators of the radiation field. Here  $\bar{k}_\lambda = k'_\lambda + k'_u$  and  $\bar{\omega}_\lambda = \omega'_\lambda - \omega'_u$ . The primes indicate quantities in the electron beam rest frame.

The details of the derivation of the time-independent Hermitian Hamiltonian operator describing the FEL process [Eq. (2)] are presented in the Appendix. In particular we have assumed that the undulator field is much larger than the radiation field  $\langle a_u^\dagger a_u \rangle \gg \langle a_\lambda^\dagger a_\lambda \rangle$ , and that the number of undulator photons is very large  $\langle a_u^\dagger a_u \rangle \gg 1$ , such that we may treat the undulator field classically and replace the undulator creation and annihilation operators with  $c$  numbers. This is well-satisfied for undulators with  $K \sim 1$ . It should also be noted that Eq. (2) is a one-dimensional model. This one-dimensional model will be valid provided the electron beam phase space is smaller than the phase space occupied by the photon beam (i.e.,  $4\pi\epsilon_n < \lambda_r \gamma$ , where  $\epsilon_n$  is the normalized transverse emittance of the electron beam) and the Rayleigh range of the radiation is larger than the characteristic gain length of the radiation field, i.e., diffraction effects are small.

The total (electron and photon) momentum operator commutes with the Hamiltonian,

$$\left[ \left( \sum_{j=1}^{N_e} \bar{k}_r \bar{p}_j + \sum_{\lambda=1}^M \bar{k}_\lambda a_\lambda^\dagger a_\lambda \right), H \right] = 0, \quad (4)$$

and is a constant of motion. The emission of photons is balanced by the recoil of the electrons.

## III. HEISENBERG EQUATIONS

To study the FEL process we will use the Heisenberg picture and evolve the quantum mechanical operators. The time evolution of the operators is given by the Heisenberg equations

$$\frac{d\theta_{\lambda j}}{dt'} = \frac{1}{i\hbar} [\theta_{\lambda j}, H] = \frac{\bar{k}_\lambda}{\bar{k}_r} \Omega \bar{p}_j, \quad (5)$$

$$\frac{d\bar{p}_j}{dt'} = \frac{1}{i\hbar} [\bar{p}_j, H] = i \sum_{\lambda=1}^M \frac{\bar{k}_\lambda}{\bar{k}_r} g_\lambda (a_\lambda^\dagger a_u e^{-i\theta_{\lambda j}} - a_\lambda a_u^* e^{i\theta_{\lambda j}}), \quad (6)$$

$$\frac{da_\lambda}{dt'} = \frac{1}{i\hbar} [a_\lambda, H] = -i\bar{\omega}_\lambda a_\lambda - i g_\lambda a_u \sum_{j=1}^{N_e} e^{-i\theta_{\lambda j}}. \quad (7)$$

In analogy to classical FEL theory [1], it is convenient to define operators representing the observables of the collective motion of the electron beam. Consider a bunching operator

$$b_\lambda = \frac{1}{N_e} \sum_{j=1}^{N_e} e^{-i\theta_{\lambda j}} \quad (8)$$

and a collective momentum operator

$$\pi_\lambda = \frac{1}{N_e} \sum_{j=1}^{N_e} e^{-i\theta_{\lambda j}} \frac{p'_j}{\hbar \bar{k}_\lambda}, \quad (9)$$

which is the normalized axial momentum averaged over the FEL phase. Note that the collective operators satisfy the following commutation relations:  $[b_\lambda, b_\lambda^\dagger] = 0$ ,  $N_e [b_\lambda, \pi_\lambda^\dagger] = 1$ ,  $N_e [b_\lambda, \pi_\lambda] = N_e^{-1} \sum_{j=1}^{N_e} \exp(-2i\theta_{\lambda j})$ , and  $N_e [\pi_\lambda, \pi_\lambda^\dagger] = 1 + N_e^{-1} \sum_{j=1}^{N_e} 2p'_j / (\hbar \bar{k}_\lambda)$ . The Heisenberg equations for the time evolution of the collective operators are

$$\frac{da_\lambda}{d\tau} = i\delta_\lambda a_\lambda - i \left( \frac{N_e}{q_\lambda} \right)^{1/2} \frac{a_u}{[a_u]} b_\lambda, \quad (10)$$

$$\frac{db_\lambda}{d\tau} = i \frac{q_\lambda}{2} b_\lambda - i q_\lambda \pi_\lambda, \quad (11)$$

$$\begin{aligned} \frac{d\pi_\lambda}{d\tau} = & i\frac{q_\lambda}{2}\pi_\lambda - iq_\lambda \frac{1}{N_e} \sum_{j=1}^{N_e} e^{-i\theta_{\lambda j}} \left( \frac{p'_j}{\hbar \bar{k}_\lambda} \right)^2 \\ & - i \sum_{\nu=1}^M \left( \frac{1}{q_\nu N_e} \right)^{1/2} \left[ \frac{a_u^*}{|a_u|} a_\nu \frac{1}{N_e} \sum_{j=1}^{N_e} e^{-i(\theta_{\lambda j} - \theta_{\nu j})} \right. \\ & \left. - \frac{a_u}{|a_u|} a_\nu^\dagger \frac{1}{N_e} \sum_{j=1}^{N_e} e^{-i(\theta_{\lambda j} + \theta_{\nu j})} \right]. \end{aligned} \quad (12)$$

Here  $\tau = \rho \bar{k}_r c t' = 4\pi\rho N_u$  is the normalized number of undulator periods  $N_u$ ,  $\delta = (k_r - k)/(2k_r\rho)$  is the normalized frequency detuning, and

$$q_\lambda = \frac{\hbar \omega_\lambda}{\rho \gamma m_e c^2}, \quad (13)$$

where  $\rho$  is the dimensionless FEL parameter [1]:

$$\rho = \left( \frac{\pi e^2 N_e K^2 \gamma}{m_e V \gamma_\perp^4 \omega_r^2} \right)^{1/3}. \quad (14)$$

We shall refer to  $q_\lambda$ , defined in Eq. (13), as the quantum-recoil parameter. As we will show, the quantum-recoil parameter  $q_\lambda$  is a critical parameter for characterizing the quantum effects. For existing FEL devices, the quantum-recoil parameter is typically small:  $q_\lambda \ll 1$ . The parameter can be interpreted as the ratio of the axial displacement due to the emission or absorption of a discrete photon to the radiation wavelength. In classical FEL theory,  $q_r^{-1} = \rho \gamma m_e c^2 / (\hbar \omega_r)$  is approximately the number of resonant photons emitted per electron at saturation.

A multimode radiation field is explicitly considered in the derivation of the Heisenberg equations. The multimode Hamiltonian is necessary to study mode competition and the development of temporal coherence. We will now consider the simpler case of single-mode operation in the linear regime of amplification.

### A. Linear regime

The Heisenberg equations [Eqs. (10)–(12)] can be solved in the linear regime of amplification, which will be valid before the FEL process reaches saturation. For simplicity we will also consider an initially cold unbunched electron beam. Consider the linearized renormalized collective operators

$$B_\lambda = \sqrt{\frac{N_e}{q_\lambda}} \frac{a_u}{|a_u|} \frac{i}{N_e} \sum_{j=1}^{N_e} e^{-i\langle \theta_{\lambda j}(0) \rangle} (\langle \theta_{\lambda j}(0) \rangle - \theta_{\lambda j}), \quad (15)$$

$$P_\lambda = \sqrt{N_e q_\lambda} \frac{a_u}{|a_u|} \frac{1}{N_e} \sum_{j=1}^{N_e} e^{-i\langle \theta_{\lambda j}(0) \rangle} \frac{p'_j}{\hbar \bar{k}_\lambda}, \quad (16)$$

where we have assumed an initially cold,  $\langle p'_j(0) \rangle = 0$ , unbunched,  $N_e^{-1} \sum_{j=1}^{N_e} e^{-i\langle \theta_{\lambda j}(0) \rangle} = 0$ , electron beam. The commutation relations for the linearized collective operators are  $[B_\lambda, B_\lambda^\dagger] = 0$ ,  $[P_\lambda, P_\lambda^\dagger] = 0$ ,  $[B_\lambda, P_\lambda^\dagger] = 1$ , and  $[B_\lambda, P_\lambda]$

$= (a_u^2/|a_u|^2) N_e^{-1} \sum_{j=1}^{N_e} \exp(-2i\langle \theta_{\lambda j}(0) \rangle) = 0$  for an initially unbunched electron beam. The linearized Heisenberg equations for a single mode (dropping the mode label  $\lambda$ ) take the forms

$$\frac{dB}{d\tau} = i\frac{q}{2}B - iP, \quad (17)$$

$$\frac{dP}{d\tau} = i\frac{q}{2}P - ia, \quad (18)$$

$$\frac{da}{d\tau} = i\delta a - iB. \quad (19)$$

Note that these equations may be derived from the linearized Hamiltonian operator

$$H_I = -\delta a^\dagger a + a^\dagger B + aB^\dagger - \frac{q}{2}(P^\dagger B + B^\dagger P) + P^\dagger P, \quad (20)$$

with momentum conservation expressed as

$$[B^\dagger P + P^\dagger B + a^\dagger a, H_I] = 0. \quad (21)$$

The Heisenberg equations for the linear collective operators can be combined to yield the following equation for the evolution of the annihilation operator:

$$\left[ \left( \frac{d}{d\tau} - i\frac{q}{2} \right)^2 \left( \frac{d}{d\tau} - i\delta \right) - i \right] a = 0, \quad (22)$$

which has the general solution

$$a(\tau) = g_1(\tau)a(0) + g_2(\tau)B(0) + g_3(\tau)P(0). \quad (23)$$

The coefficients in Eq. (23) are

$$\begin{aligned} g_1(\tau) = & \frac{(\mu_2 - \delta)(\mu_3 - \delta)e^{i\mu_1\tau}}{(\mu_1 - \mu_2)(\mu_1 - \mu_3)} + \frac{(\mu_1 - \delta)(\mu_3 - \delta)e^{i\mu_2\tau}}{(\mu_2 - \mu_3)(\mu_2 - \mu_1)} \\ & + \frac{(\mu_2 - \delta)(\mu_1 - \delta)e^{i\mu_3\tau}}{(\mu_3 - \mu_2)(\mu_3 - \mu_1)}, \end{aligned} \quad (24)$$

$$\begin{aligned} g_2(\tau) = & \frac{[(q/2) - \mu_1]e^{i\mu_1\tau}}{(\mu_1 - \mu_2)(\mu_1 - \mu_3)} + \frac{[(q/2) - \mu_2]e^{i\mu_2\tau}}{(\mu_2 - \mu_3)(\mu_2 - \mu_1)} \\ & + \frac{[(q/2) - \mu_3]e^{i\mu_3\tau}}{(\mu_3 - \mu_2)(\mu_3 - \mu_1)}, \end{aligned} \quad (25)$$

$$\begin{aligned} g_3(\tau) = & \frac{e^{i\mu_1\tau}}{(\mu_1 - \mu_2)(\mu_1 - \mu_3)} + \frac{e^{i\mu_2\tau}}{(\mu_2 - \mu_3)(\mu_2 - \mu_1)} \\ & + \frac{e^{i\mu_3\tau}}{(\mu_3 - \mu_2)(\mu_3 - \mu_1)}, \end{aligned} \quad (26)$$

where  $\mu_j$  (for  $j=1,2,3$ ) satisfies the dispersion relation

$$\mu^3 - (\delta + q)\mu^2 + (\delta q + q^2/4)\mu + 1 - \delta q^2/4 = 0. \quad (27)$$

In the classical limit (i.e.,  $\lim q \rightarrow 0$ ), Eq. (27) reduces to the characteristic cubic equation of classical FEL theory [1]. Note that this linear solution [Eqs. (23)–(27)] preserves the commutation relation for the annihilation and creation operators,  $[a, a^\dagger] = |g_1|^2 + g_2^* g_3 + g_3^* g_2 = 1$ , for all  $\tau$ .

The time evolution of the number of laser photons is given by the expectation value of the number operator  $a^\dagger a$ :

$$\begin{aligned} \langle a^\dagger a \rangle &= |g_1|^2 \langle a^\dagger(0) a(0) \rangle + |g_2|^2 \langle B^\dagger(0) B(0) \rangle \\ &\quad + |g_3|^2 \langle P^\dagger(0) P(0) \rangle + g_2^* g_3 \langle B^\dagger(0) P(0) \rangle \\ &\quad + g_3^* g_2 \langle P^\dagger(0) B(0) \rangle. \end{aligned} \quad (28)$$

Here we have assumed that initially there are no correlations between the electrons and photons and  $\langle B^\dagger(0) \rangle = \langle B(0) \rangle = \langle P^\dagger(0) \rangle = \langle P(0) \rangle = 0$ . Note that the first term on the right-hand side of Eq. (28) is the contribution due to stimulated emission, while the last four terms are due to spontaneous emission. A solution of this form [Eq. (28)] was originally studied by Bonifacio and Casagrande [11], although in their work the noncommutativity of the collective operators was neglected in the Heisenberg equations; therefore, they found only the classical FEL dynamics.

### B. Stimulated radiation

The solution to the Heisenberg equations in the regime of linear amplification has an unstable solution, leading to exponential growth in the mean number of photons. A stability analysis of the dispersion relation [Eq. (27)] indicates that instability is achieved for frequency detuning satisfying  $\delta < (3/2^{2/3})[1 + q/(54)^{1/3}]$ . The discrete electron recoil shifts the regime of stability.

In the high-gain regime ( $\tau > 1$ ), at resonance, the gain due to stimulated radiation can be expressed using Eqs. (24) and (27) as

$$\frac{\langle a^\dagger(\tau) a(\tau) \rangle_{st}}{\langle a^\dagger(0) a(0) \rangle} = |g_1|^2 \approx A(q) \exp[L/L_g(q)], \quad (29)$$

where  $L = N_u \lambda_u$  is the distance along the undulator in the laboratory frame,  $A$  is the gain coefficient, and  $L_g$  is the power gain length. The dependence of  $A$  and  $L_g$  on the resonant quantum-recoil parameter  $q$  is shown in Fig. 1. This figure shows the deviation from the classically predicted values as the quantum-recoil parameter  $q$  approaches unity and the FEL process moves from the classical to the quantum regime. In particular the figure shows the increase in the power gain length as  $q$  increases. This reduction in gain is due to the strong recoil of the electrons in the parameter regime where  $q \sim 1$ . The strong recoil moves the electrons off resonance after the emission of a photon, thereby decreasing the probability of emitting additional photons and reducing the gain. To lowest order in the quantum-recoil parameter  $q$ , the gain coefficient and power gain length satisfy  $A = (1/9)[1 - q/3 + \dots]$  and  $L_g = (2\sqrt{3}k_u \rho)^{-1}[1 + q^2/36 + \dots]$ . In the limit  $q \rightarrow 0$ , these quantities reduce to the classical one-dimensional results [1]:  $A_{class} = 1/9$  and  $L_{class} = (2\sqrt{3}k_u \rho)^{-1}$ . The exponential growth in Eq. (29) will

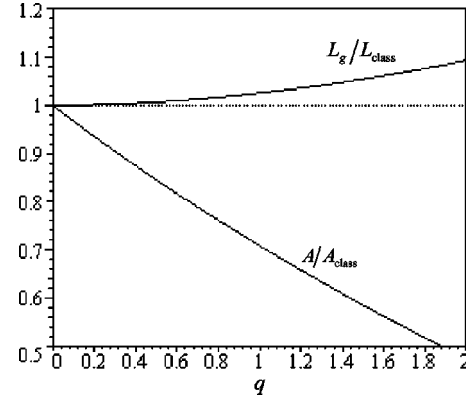


FIG. 1. The power gain length  $L_g$  and the gain coefficient  $A$  normalized to the classical values [ $L_{class} = (2\sqrt{3}k_u \rho)^{-1}$  and  $A_{class} = 1/9$ ] as functions of the quantum-recoil parameter for resonant radiation  $q$ .

end at saturation, which will occur when the beam becomes modulated at the resonant radiation wavelength  $\langle b_r^\dagger b_r \rangle \sim 1$ .

### C. Spontaneous radiation

In the SASE mode of operation the spontaneous emission emitted in the undulator is coherently amplified by the FEL process. It was recognized by many authors [11,12] that the spontaneous FEL radiation can be affected by the quantum fluctuations of the electron beam. To evaluate the spontaneous emission it is necessary to know the initial expectation values of the collective operators in Eq. (28), which requires knowledge of the initial wave function of the electron beam. As an example, we may consider the case where the initial state of the electron beam is described by the product of  $N_e$  minimum-uncertainty wave packets [11], such that the conjugate operators for each electron  $p'_j$  and  $\theta_j$  satisfy the equality Heisenberg uncertainty relation  $\langle [\Delta \theta(0)]^2 \rangle \langle [\Delta p'(0)]^2 \rangle = (\hbar \bar{k}_\lambda)^2/4$ , where the variance operators for the position and momentum for each electron are  $\Delta \theta_{\lambda j} = \theta_{\lambda j} - \langle \theta_{\lambda j} \rangle$  and  $\Delta p'_j = p'_j - \langle p'_j \rangle$ . For this case the initial expectation values of the resonant linearized collective operators in Eq. (28), assuming an initially cold unbunched electron beam, are

$$\langle B^\dagger(0) B(0) \rangle = \frac{1}{q} [N_e |b_c(0)|^2 + \langle (\Delta \theta(0))^2 \rangle], \quad (30)$$

$$\langle B^\dagger(0) P(0) \rangle = \langle P^\dagger(0) B(0) \rangle = -\frac{1}{2}, \quad (31)$$

$$\langle P^\dagger(0) P(0) \rangle = q \langle (\Delta \bar{p}(0))^2 \rangle. \quad (32)$$

In Eq. (30), we have interpreted the centroid of the quantum wave packet as the classical position for each electron such that the classical bunching parameter is

$$b_c = \frac{1}{N_e} \sum_{j=1}^{N_e} \exp[-i \langle \theta_j \rangle]. \quad (33)$$

For an initially random longitudinal distribution of electron wave packet centroids in phase (i.e., classical shot noise),  $N_e|b_c(0)|^2=1$ .

With Eqs. (30)–(33), the expectation value of the photon number operator Eq. (28) for the spontaneous radiation can be expressed as

$$q\langle a^\dagger a \rangle_{\text{sp}} = N_e|b_c|^2|g_2|^2 + \langle (\Delta\theta)^2 \rangle |g_2|^2 - \frac{q}{2}(g_2^*g_3 + g_3^*g_2) + q^2\langle (\Delta\bar{p})^2 \rangle |g_3|^2. \quad (34)$$

The first term on the right hand side of Eq. (34) represents classical bunching, while the remaining terms represent the effective bunching due to quantum fluctuations in the position and momentum of the electrons. With no initial classical bunching  $|b_c(0)|^2=0$  (e.g., an ideal periodic or crystalline beam) and no initial input radiation  $\langle a^\dagger(0)a(0) \rangle=0$ , Eq. (34) gives the number of laser photons radiated starting from only quantum fluctuations in the electron beam. This is the minimum spontaneous radiation produced by any beam passing through the undulator.

In the classical limit [i.e., in the limit  $\langle (\Delta\theta)^2 \rangle=0$ ,  $\langle (\Delta\bar{p})^2 \rangle=0$ , and  $\lim q \rightarrow 0$ ], the spontaneous emission [Eq. (34)] reduces to

$$\hbar\omega\langle a^\dagger a \rangle = \frac{1}{9}e^{(2\sqrt{3}\rho k_u L)}\rho\gamma m_e c^2 N_e|b_c(0)|^2, \quad (35)$$

in the high-gain regime, which is the well-known result for a classical SASE FEL [1].

We can also consider the case where the initial minimum-uncertainty wave packet for each electron is localized such that the initial variance in position is  $\langle (\Delta\theta)^2(0) \rangle = 2\pi N_u \rho q$ . This initial state will minimize the expectation value of the variance in position over the length of the undulator of a free-space Gaussian wave packet, which evolves in time as  $\langle (\Delta\theta)^2(t') \rangle = \langle (\Delta\theta)^2(0) \rangle + (\hbar^2 \bar{k}_r^4 t'^2) / [4m^2 \langle (\Delta\theta)^2(0) \rangle]$ . With this near-classical initial condition, the expectation value for the number of photons [Eq. (34)] can be expressed as

$$q\langle a^\dagger a \rangle_{\text{sp}} \approx N_e|b_c|^2|g_2|^2 + \frac{q}{2}[(4\pi N_u \rho)|g_2|^2 - (g_2^*g_3 + g_3^*g_2) + (4\pi N_u \rho)^{-1}|g_3|^2]. \quad (36)$$

In the high-gain regime to first order in the quantum-recoil parameter,  $|g_2|^2 \approx (1+q/2)|g_1|^2$ ,  $|g_3|^2 \approx (1+q/3)|g_1|^2$ , and  $g_3^*g_2 + g_2^*g_3 \approx -|g_1|^2$ , where  $|g_1|^2$  is given by Eq. (29). For a FEL designed to reach saturation,  $N_u \rho \sim 1$ ; therefore we may expect an enhanced start-up from quantum fluctuations if  $q \sim N_e|b_c|^2$ . For the case of an electron beam initially seeded with classical shot noise ( $N_e|b_c|^2=1$ ), Eq. (36) indicates that the relative increase in start-up due to the effective bunching produced by the quantum fluctuations in the electron beam will be of the order of the quantum-recoil parameter.

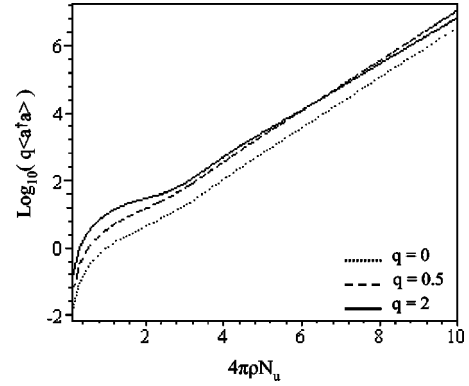


FIG. 2. The amplification of the normalized spontaneous radiation intensity  $q\langle a^\dagger a \rangle_{\text{sp}}$  vs the normalized undulator length  $4\pi\rho N_u$ , for quantum-recoil parameters:  $q=2$  (solid curve),  $q=0.5$  (dashed curve), and  $q=0$  (dotted curve).

The photon number expectation value [Eq. (36)] can be solved explicitly using Eqs. (24)–(27). The expectation value for the photon number operator, assuming an electron beam with initial bunching due to classical shot noise  $N_e|b_c(0)|^2=1$  and no initial radiation  $\langle a^\dagger(0)a(0) \rangle=0$ , is shown in Fig. 2. In Fig. 2 the dotted curve is the classical solution ( $q=0$ ), the dashed curve is the solution for a resonant quantum-recoil parameter of  $q=0.5$ , and the solid curve is the solution for a resonant quantum-recoil parameter of  $q=2$ . The figure shows the initial enhancement of the radiation due to the effective bunching from quantum fluctuations in the electron beam. The figure also shows the reduction in power gain length, compared to what is predicted by classical theory, owing to the strong electron recoil in the parameter regime  $q \sim 1$ .

#### IV. PHOTON STATISTICS

In this section we discuss the statistical properties of the FEL radiation. A description of the photon statistics of the radiation requires a fully quantized (matter and radiation fields) treatment of the FEL interaction. If the electron momentum operator is treated as a classical  $c$  number, the problem reduces to that of a classical current interacting with a quantized radiation field. It is well known that the photon field emitted by a classical current into a single mode is described by a coherent Glauber state [13], which obeys Poisson statistics. A comprehensive analysis of the statistical properties of the SASE FEL radiation based on classical radiation theory was presented in Ref. [14]. While photon statistics can be analyzed within the context of classical theory in terms of the statistical fluctuations in the coordinates and number of the radiating particles, the results are not generally consistent with the predictions of the fully quantized theory [15].

The departure from Poisson statistics of the stimulated emission can be calculated using Eq. (23):

$$[\langle (a^\dagger a)^2 \rangle_{\text{st}} - \langle a^\dagger a \rangle_{\text{st}}^2] - \langle a^\dagger a \rangle_{\text{st}} = |g_1|^4 (\langle a^{\dagger 2}(0)a^2(0) \rangle - \langle a^\dagger(0)a(0) \rangle^2). \quad (37)$$

Depending on the quantum state of the initial seeding radiation, the photon statistics of the stimulated radiation may be super- or sub-Poissonian (e.g., if the radiation is initially in a Fock state, then the stimulated FEL radiation will be sub-Poissonian). If the initial radiation seeding the FEL amplifier is in a coherent Glauber state of single mode,  $a(\tau)|\alpha\rangle = \alpha(\tau)|\alpha\rangle$ , then the field will remain in a coherent Glauber state, and the stimulated emission will have Poisson statistics; the probability of occupation of the photon number state  $|n\rangle$  is a Poisson distribution,

$$|\langle n|\alpha\rangle|^2 = \frac{\langle a^\dagger a \rangle_{\text{st}}^n}{n!} \exp[-\langle a^\dagger a \rangle_{\text{st}}], \quad (38)$$

with the number operator variance  $\langle (a^\dagger a)^2 \rangle_{\text{st}} - \langle a^\dagger a \rangle_{\text{st}}^2 = \langle a^\dagger a \rangle_{\text{st}} = |g_1|^2 \langle a^\dagger(0)a(0) \rangle$ .

The spontaneous radiation in the FEL process is due to a large number of independent sources (electrons). This has profound effects on the photon statistics. The Glauber quasiprobability function [13] (i.e., the diagonal coherent-state representation) for the spontaneous radiation  $\phi_{\text{sp}}(\alpha)$  is a convolution of the Glauber quasiprobability functions of the radiation emitted by each individual electron  $\phi_j(\alpha_j)$ . Provided there is a sufficiently large number of independent sources, the Glauber quasiprobability function for the spontaneous radiation will be a negative exponential distribution [13]

$$\begin{aligned} \phi_{\text{sp}}(\alpha) &= \int \delta^{(2)}\left(\alpha - \sum_{j=1}^{N_e} \alpha_j\right) \prod_{j=1}^{N_e} \phi_j(\alpha_j) d^2\alpha_j \\ &= \frac{1}{\pi \langle a^\dagger a \rangle_{\text{sp}}} \exp\left[-\frac{|\alpha(\tau)|^2}{\langle a^\dagger a \rangle_{\text{sp}}}\right]. \end{aligned} \quad (39)$$

This is a quantum optical analog to the central limit theorem. The probability of occupation of the photon number state  $|n\rangle$  is then a thermal (chaotic) distribution

$$|\langle n|0\rangle|^2 = \frac{1}{1 + \langle a^\dagger a \rangle_{\text{sp}}} \left(1 + \frac{1}{\langle a^\dagger a \rangle_{\text{sp}}}\right)^{-n}, \quad (40)$$

with number operator variance  $\langle (a^\dagger a)^2 \rangle_{\text{sp}} - \langle a^\dagger a \rangle_{\text{sp}}^2 = \langle a^\dagger a \rangle_{\text{sp}} (\langle a^\dagger a \rangle_{\text{sp}} + 1)$ . As a result, the spontaneous emission exhibits only first-order coherence. The thermal statistics of the spontaneous FEL radiation [Eq. (40)] was first shown by Becker and McIver [16]. The total quasiprobability distribution of the FEL radiation for a single mode is a convolution of the stimulated and spontaneous radiation quasiprobability distributions.

In general, the radiation produced by a SASE FEL will consist of many modes provided the FEL bandwidth is greater than the Fourier limited bandwidth  $\sigma_\omega > 1/\sigma_t$ , where  $\sigma_\omega$  is the FEL bandwidth and  $\sigma_t$  is the pulse duration. The number of modes present in the spectral distribution will be approximately  $M \sim \sigma_\omega \sigma_t$  [17]. The multimode spontaneous radiation Glauber quasiprobability function may be expressed as [13]

$$\phi(\{\alpha\}) = \prod_{\lambda=1}^M \frac{1}{\pi \langle a_\lambda^\dagger a_\lambda \rangle} \exp\left[-\frac{|\alpha_\lambda(\tau)|^2}{\langle a_\lambda^\dagger a_\lambda \rangle}\right], \quad (41)$$

where  $\{\alpha\} = \{\alpha_1, \alpha_2, \dots, \alpha_M\}$  is the set of annihilation operator eigenvalues for each mode. Note that the SASE FEL radiation with statistics defined by the positive-definite Glauber quasiprobability function [Eq. (41)] will exhibit only bunching of photoelectric detections. If the field is initially in a vacuum state, then the probability of occupation of a set of photon number states  $|\{n\}\rangle = \prod_\lambda |n_\lambda\rangle$  is

$$|\langle \{n\} | \{0\} \rangle|^2 = \prod_{\lambda=1}^M \frac{1}{1 + \langle a_\lambda^\dagger a_\lambda \rangle} \left[1 + \frac{1}{\langle a_\lambda^\dagger a_\lambda \rangle}\right]^{-n_\lambda}. \quad (42)$$

We define the total occupation number as the sum over all modes  $\langle n \rangle = \sum_{\lambda=1}^M \langle a_\lambda^\dagger a_\lambda \rangle$ . The probability of the total occupation number  $n$  is the sum over all possible sets of mode occupation numbers  $\{n_\lambda\}$  summing to  $n$ ,

$$\begin{aligned} |\langle n | \{0\} \rangle|^2 &= \sum_{\{n'\}} |\langle \{n'\} | \{0\} \rangle|^2 \delta_{nn'} \\ &= \int \prod_{\lambda=1}^M \left[ \frac{d^2\alpha_\lambda}{\pi \langle a_\lambda^\dagger a_\lambda \rangle} \exp\left(\frac{-|\alpha_\lambda|^2}{\langle a_\lambda^\dagger a_\lambda \rangle}\right) \right] \left( \frac{\beta^{2n}}{n!} e^{-\beta^2} \right), \end{aligned} \quad (43)$$

where  $\beta^2 = \sum_{\lambda=1}^M |\alpha_\lambda|^2$ . This is the general solution for the total photon number expectation [18].

If we neglect the spectral distribution of the SASE radiation and assume the occupation number expectation values of all  $M$  modes are equal, i.e.,  $\langle n \rangle \approx M \langle a_\lambda^\dagger a_\lambda \rangle$  for any  $\lambda$ , then Eq. (43) can be evaluated analytically. With this equal occupation number assumption, the probability of occupation of the total photon number state  $|n\rangle$ , given an initial vacuum state, is a negative binomial distribution

$$|\langle n | \{0\} \rangle|^2 = \frac{\Gamma(n+M)}{\Gamma(M)\Gamma(n-1)} \frac{[1+M/\langle n \rangle]^{-n}}{[1+\langle n \rangle/M]^M}, \quad (44)$$

with joint photon number state occupation variance  $\langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle^2 / M + \langle n \rangle$ . For a typical SASE FEL,  $\langle n \rangle = \sum_{\lambda=1}^M \langle a_\lambda^\dagger a_\lambda \rangle \gg M > 1$ . The negative binomial distribution [Eq. (44)] was suggested previously by several authors [19] as an adequate description of the SASE FEL radiation. We expect this approximation will be valid in the asymptotic limit  $\sigma_t \rightarrow \infty$ .

The equal occupation approximation used to derive Eq. (44) does not take into consideration the spectral distribution of the FEL radiation. The spectral distribution of the FEL radiation, determined by the dispersion relation [Eq. (27)], is approximately Gaussian

$$a_\lambda^\dagger a_\lambda = a_r^\dagger a_r \exp[-(\omega_\lambda - \omega_r)^2 / (2\sigma_\omega^2)]. \quad (45)$$

Although the general probability distribution [Eq. (43)] is difficult to evaluate analytically for arbitrary spectral distribution, we may use Eq. (43) to calculate the normalized

moments of  $n$ . The factorial moments of the photon number statistics  $\langle n^{(m)} \rangle = \langle n(n-1) \cdots (n-m+1) \rangle$  can be calculated using a simple relation [20] between the photon counting correlations and counting moments  $\langle \exp(n\xi) \rangle = \langle \exp[(e^\xi - 1) \sum_{\lambda=1}^M |\alpha_\lambda|^2] \rangle$ . With this relation, and assuming a Gaussian spectral distribution [Eq. (45)], the expectation values of the first few factorial moments of the total photon occupation number  $n$  for the SASE radiation are

$$\frac{\langle n^{(2)} \rangle}{\langle n \rangle^2} = 1 + \frac{1}{M}, \quad (46)$$

$$\frac{\langle n^{(3)} \rangle}{\langle n \rangle^3} = 1 + \frac{3}{M} + \frac{4\sqrt{3}}{3M^2}, \quad (47)$$

$$\frac{\langle n^{(4)} \rangle}{\langle n \rangle^4} = 1 + \frac{6}{M} + \frac{(9+16\sqrt{3})}{3M^2} + \frac{6\sqrt{2}}{M^3} \quad (48)$$

for  $M \gg 1$ . The higher-order photon number counting moments, e.g., Eqs. (47) and (48), deviate from those generated by the negative binomial probability distribution [Eq. (44)], which predicts

$$\frac{\langle n^{(m)} \rangle}{\langle n \rangle^m} = \frac{\Gamma(m+M)}{M^m \Gamma(M)}. \quad (49)$$

As expected, in the limit  $\sigma_\omega \sigma_t \rightarrow \infty$  the expectation values of the photon number counting moments predicted by the negative binomial distribution are a good approximation to the expectation values of the exact photon number counting moments for the Gaussian spectral distribution of the SASE FEL.

## V. DISCUSSION AND SUMMARY

In this paper we have shown that quantum effects manifest themselves in a FEL when the quantum-recoil parameter, defined in Eq. (13), approaches unity, i.e., when the photon energy is comparable to the gain bandwidth  $\hbar\omega \sim \rho \gamma m_e c^2$ . In this parameter regime the axial displacement due to the emission or absorption of a single photon is comparable to the radiation wavelength.

Proposed x-ray SASE FEL's based on conventional magnetostatic undulators, e.g., Refs. [2,3], will typically have undulators with  $\lambda_u \sim 1$  cm and  $K \sim 1$ . Production of 1-Å radiation will then require an electron beam energy of  $\gamma \sim 10^4$ . These conventional x-ray sources will have efficiencies of the order of  $\rho \sim 10^{-4}$ , where the FEL parameter (device efficiency) is given by Eq. (14). For these parameters,  $q \sim 10^{-3}$ , and we expect the conventional FEL x-ray sources, to a good approximation, to be in the classical regime.

X-ray production using a high-power optical laser pulse such as an electromagnetic undulator, with, for example, a laser wavelength of  $\lambda_0 = 1$  μm, and a normalized vector potential of  $K \sim 1$  (i.e., a peak laser intensity of  $I \sim 10^{18}$  W cm<sup>-2</sup>), would require an electron beam energy of  $\gamma \sim 10^2$  to produce 1-Å radiation. If we consider an electro-

magnetic undulator device with comparable efficiency  $\rho \sim 10^{-4}$ , then the quantum-recoil parameter is  $q \sim 1$ . In this regime we can expect deviations from the predictions of classical theory owing to the discrete electron recoil and the quantum fluctuations of the electron beam, as discussed in Secs. III B and III C. Specifically, the theory presented in this paper predicts a reduction in the gain, given in Eq. (29), and enhanced start-up due to the effective bunching produced by quantum fluctuations, given by Eq. (36).

In this paper we have presented a many-particle quantum theory of the FEL. The Heisenberg equations were solved for a single-mode radiation field before saturation. The stimulated amplification of the radiation was computed in the high-gain collective regime. For FEL parameters satisfying  $q \sim 1$ , the gain was shown to decrease compared to the classically predicted value, owing to the strong electron recoil. The initiation of spontaneous radiation due to quantum fluctuations in the position and momentum of the electron beam was examined. The minimum spontaneous radiation emitted by the beam passing through the undulator was calculated. For an initial electron beam wave function in the near-classical regime, the effective bunching of the beam due to initial quantum fluctuations was shown to increase by a factor of  $\sim q$  from the classical shot noise value. The photon statistics of the FEL radiation was also examined, and the photon counting statistics of the SASE radiation was calculated including the effects of the FEL spectral distribution.

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## APPENDIX

In this appendix we derive the Hermitian Hamiltonian operator for the FEL [Eq. (2)]. The classical Hamiltonian describing the matter-radiation interaction is

$$H = \sum_{j=1}^{N_e} \sqrt{m_e^2 c^4 + (c\vec{P}_j + e\vec{A})^2}, \quad (A1)$$

where  $\vec{P}_j$  is the canonical momentum,  $\vec{A}$  is the field vector potential,  $N_e$  is the number of electrons in the beam,  $m_e$  is the mass of the electron,  $-e$  is the charge of the electron, and  $c$  is the speed of light in vacuum. We will assume that the electron beam is sufficiently dilute such that space charge effects are negligible.

The Coulomb gauge  $\nabla \cdot \vec{A} = 0$  is chosen, and we assume the radiation fields propagate along the  $\hat{z}$  direction. The vector potential may be decomposed such that  $\vec{A} = \vec{A}_L + \vec{A}_u$ , where  $\vec{A}_u$  is the vector potential describing the undulator and  $\vec{A}_L$  is the vector potential describing the laser field which may consist of  $M$  modes. The vector potentials of the laser and undulator may be written as

$$\vec{A}_L = \sum_{\lambda=1}^M \sqrt{\frac{2\pi c \hbar}{k_\lambda V}} [a_\lambda \hat{e} e^{i(k_\lambda z - \omega_\lambda t)} + \text{c.c.}], \quad (\text{A2})$$

$$\vec{A}_u = \sqrt{\frac{2\pi c \hbar}{k_u V}} [a_u \hat{e} e^{-i(k_u z + \omega_u t)} + \text{c.c.}], \quad (\text{A3})$$

where  $\omega_\lambda = k_\lambda c$  is the frequency of the radiation mode with index  $\lambda$ . If the undulator is magnetostatic then  $k_u = 2\pi/\lambda_u$  and  $\omega_u = 0$ , where  $\lambda_u$  is the undulator wavelength. If the undulator is electromagnetic (e.g., optical laser pulse), then  $\omega_u = k_u c = 2\pi c/\lambda_u$ . Here  $a_\lambda$  and  $a_u$  are the complex amplitudes of the fields, which are contained in the volume  $V$ . For definiteness, both the laser and undulator fields are assumed to be circularly polarized (i.e., helical undulator) such that  $\hat{e} = (\hat{x} + i\hat{y})/\sqrt{2}$ . We will assume that diffraction effects are small, and therefore, the fields can be described by plane wave solutions.

We consider an electron beam that is initially injected into the undulator in the axial  $\hat{z}$  direction. The transverse canonical momentum  $\vec{P}_\perp$  is a constant of motion; therefore  $\vec{P}_j \cdot \vec{A} = 0$ , and the Hamiltonian [Eq. (A1)] reduces to

$$H = \sum_{j=1}^{N_e} \sqrt{m_e^2 c^4 + p_j^2 c^2 + e^2 (\vec{A}_u + \vec{A}_L)^2}, \quad (\text{A4})$$

where  $p_j$  is the axial electron momentum.

### 1. Electron beam rest frame

In the laboratory frame, the relativistic electron beam is moving along the direction of propagation of the radiation field, with axial velocity  $\beta_{\parallel} c$ . The equivalence between magnetostatic undulators and electromagnetic undulators can easily be seen in the frame moving with the electron beam. For an ultrarelativistic electron beam ( $\beta_{\parallel} \approx 1$ ), both the magnetostatic and electromagnetic undulators appear as counter-propagating electromagnetic waves in the electron beam rest frame.

We consider a Lorentz transformation to a frame moving with velocity

$$\beta_f = \left(1 - \frac{\omega_u}{\omega_r}\right) \left(1 + \frac{k_u}{k_r}\right)^{-1} \quad (\text{A5})$$

in the  $\hat{z}$  direction with respect to the laboratory frame. We will assume  $k_r \gg k_u$  in the laboratory frame. The resonant frequency  $\omega_r = k_r c$  is defined with respect to the energy of the electron beam such that

$$\omega_r = 2(c k_u + \omega_u) \gamma_{\parallel}^2, \quad (\text{A6})$$

where  $\gamma_{\parallel} = (1 - \beta_{\parallel}^2)^{-1/2}$  is the Lorentz factor owing to the axial velocity of the electron beam. With the definition of the resonant frequency [Eq. (A6)], the frame moving at  $\beta_f$  is the electron beam rest frame.

In this frame, the FEL phase becomes

$$(k_u + k_\lambda) z_j - (\omega_\lambda - \omega_u) t = \bar{k}_\lambda z'_j - \bar{\omega}_\lambda t', \quad (\text{A7})$$

where the primes indicate coordinates in the frame moving at  $\beta_f$ , and

$$\begin{aligned} \bar{k}_\lambda &= k'_\lambda + k'_u = \gamma_f [k_u + k_\lambda - \beta_f (k_\lambda - \omega_u/c)] \\ &= \bar{k}_r + k'_r \left( \frac{k_\lambda - k_r}{k_r} \right), \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} \bar{\omega}_\lambda &= \omega'_\lambda - \omega'_u = \gamma_f [c k_\lambda (1 - \beta_f) - \omega_u - c k_u \beta_f] \\ &= \omega'_r \left( \frac{k_\lambda - k_r}{k_r} \right). \end{aligned} \quad (\text{A9})$$

Here  $\bar{k}_r = (k_r + k_u)/\gamma_{\parallel} \approx \sqrt{2k_r(k_u + \omega_u/c)}$ . Note that, for a conventional magnetostatic undulator,  $\omega_u = 0$ , and for an electromagnetic undulator  $\omega_u = k_u c$ .

### 2. Strong undulator regime

We will consider the case where the number of undulator photons is much greater than the number of laser photons. In particular we assume that

$$\frac{e^2 |A_L|^2}{m_e^2 c^4} \ll \frac{e^2}{m_e^2 c^4} |A_L| |A_u| \ll \frac{e^2 |A_u|^2}{m_e^2 c^4} = K^2 \sim 1, \quad (\text{A10})$$

where  $K$  is the undulator strength parameter. Operational FEL's based on conventional magnetostatic undulators will typically have undulator strength parameters of order  $K \sim 1$ . Present high-intensity laser pulses have the capability to produce even larger normalized vector potentials,  $K \geq 1$ . Therefore, Eq. (A10) will be satisfied for most FEL devices being considered.

The Hamiltonian [Eq. (A4)] may be expressed as

$$H = \sum_{j=1}^{N_e} [m^2 c^4 + p_j^2 c^2 + e^2 (\vec{A}_L \cdot \vec{A}_u + \vec{A}_u \cdot \vec{A}_L) + e^2 \vec{A}_L^2]^{1/2}, \quad (\text{A11})$$

where  $m$  is the renormalized mass:

$$m = m_e \sqrt{1 + K^2} = m_e \gamma_{\perp}. \quad (\text{A12})$$

Here  $\gamma_{\perp} = \sqrt{1 + K^2}$  is the Lorentz factor associated with the quiver motion of the electrons due to the transverse undulator field, and the total beam energy is  $m_e c^2 \gamma = m_e c^2 \gamma_{\parallel} \gamma_{\perp} = m c^2 \gamma_{\parallel}$ . With the assumption of Eq. (A10), we may neglect the term quadratic in the laser vector potential. Substituting the vector potentials [Eqs. (A2) and (A3)] into the Hamiltonian yields

$$\begin{aligned} H &= m c^2 \sum_{j=1}^{N_e} \left[ 1 + \frac{p_j^2}{m^2 c^2} \right. \\ &\quad \left. + \sum_{\lambda=1}^M \frac{2\hbar g_\lambda}{m c^2} (a_\lambda a_u^* e^{i[(k_\lambda + k_u)z_j - (\omega_\lambda - \omega_u)t]} + \text{c.c.}) \right]^{1/2}, \end{aligned} \quad (\text{A13})$$



where the strength of the coupling between the undulator and radiation field is determined by the parameter

$$g_\lambda = \frac{2\pi e^2}{mcV\sqrt{k_\lambda k_u}}. \quad (\text{A14})$$

In the electron beam rest frame and assuming Eq. (A10), the interaction Hamiltonian becomes

$$H = \sum_{j=1}^{N_e} \frac{p_j'^2}{2m} + \sum_{\lambda=1}^M \hbar g_\lambda \left( a_\lambda a_u^* \sum_{j=1}^{N_e} e^{i(\bar{k}_\lambda z_j' - \bar{\omega}_\lambda t')} + \text{c.c.} \right). \quad (\text{A15})$$

Note that the condition  $[e^2|A_L||A_u|/(m_e^2c^4)] \ll 1$  may be expressed as  $|a_\lambda|^2/N_e \ll 1/(\rho^4 q_\lambda)$ , where  $q_\lambda$  and  $\rho$  are given by Eqs. (13) and (14), respectively. This condition will always be well satisfied provided  $\rho \ll 1$ , since the FEL process saturates at  $|a_\lambda|^2 \sim N_e q_\lambda^{-1}$ .

In addition, we can consider a canonical transformation to remove the time dependency in the Hamiltonian. Using an action-angle generating function the Hamiltonian becomes

$$H = \sum_{\lambda=1}^M \hbar \bar{\omega}_\lambda \frac{1}{2} (a_\lambda^* a_\lambda + a_\lambda a_\lambda^*) + \sum_{j=1}^{N_e} \left[ \frac{p_j'^2}{2m} + \sum_{\lambda=1}^M \hbar g_\lambda (a_\lambda a_u^* e^{i\theta_{\lambda j}} + \text{c.c.}) \right], \quad (\text{A16})$$

where the phase is  $\theta_{\lambda j} = \bar{k}_\lambda z_j'$ .

We will construct a quantum Hamiltonian operator from the classical Hamiltonian through the Dirac prescription for quantization: the quantum Hamiltonian is assumed to have the form of the classical Hamiltonian according to the correspondence principle, and the canonical dynamical variables are associated with Hilbert space operators. The Hermitian Hamiltonian describing the multimode FEL process is

$$H = \sum_{\lambda=1}^M \hbar \bar{\omega}_\lambda \left( a_\lambda^\dagger a_\lambda + \frac{1}{2} \right) + \sum_{j=1}^{N_e} \hbar \Omega \frac{\bar{p}_j^2}{2} + \sum_{\lambda=1}^M \hbar g_\lambda \left( a_\lambda^\dagger a_u \sum_{j=1}^{N_e} e^{-i\theta_{\lambda j}} + \text{H.c.} \right). \quad (\text{A17})$$

Here  $\Omega = \hbar \bar{k}_r^2/m$  determines the strength of the electron recoil, and  $\bar{p}_j = p_j'/(\hbar \bar{k}_r)$  is the electron axial momentum normalized to the recoil provided by a photon exchange between the undulator and the resonant laser. The operators satisfy the commutation relations

$$[\theta_{\lambda i}, \bar{p}_j] = i \frac{\bar{k}_\lambda}{k_r} \delta_{ij}, \quad (\text{A18})$$

$$[a_\lambda, a_\mu^\dagger] = \delta_{\lambda\mu} \quad (\text{A19})$$

for the phase and momentum operators and the non-Hermitian photon annihilation  $a_\lambda$  and creation  $a_\lambda^\dagger$  operators. Note that these commutation relations are satisfied for all

time since the Hamiltonian is Hermitian. In Eq. (A17) we assume that the number of undulator photons is large  $\langle a_u^\dagger a_u \rangle \gg 1$ , and we treat the undulator field classically (i.e., there exists an infinite reservoir of undulator photons to scatter into laser photons). This approximation is justified provided the number of undulator photons is nearly unaffected by the interaction (photon exchange), which is satisfied for  $\langle a_u^\dagger a_u \rangle \gg \langle a_\lambda^\dagger a_\lambda \rangle$ . Equation (A17) is the Hamiltonian operator used in the main body of this work.

In this work it is also assumed that the wave functions of the electrons do not overlap and they may be treated as distinguishable particles, i.e., the number of available states is much larger than the number of electrons, and therefore there will be no degeneracy in the electron beam wave function. This will be valid provided the phase space volume of the electrons is sufficiently dilute, i.e.,  $\varepsilon_\parallel \varepsilon_\perp^2 > N_e \lambda_c^3$ , where  $\varepsilon_\parallel$  and  $\varepsilon_\perp$  are the normalized longitudinal and transverse emittance of the electron beam, respectively, and  $\lambda_c$  is the Compton wavelength. The effects of electron beam wave function degeneracy on the FEL process were considered by Kim [21].

### 3. Weak undulator regime

For the case of a weak undulator field, the laser and undulator fields satisfy  $e^2|A_L|^2/m_e^2c^4 \ll 1$  and  $e^2|A_u|^2/m_e^2c^4 = K^2 \ll 1$ . In this regime we cannot approximate the number of undulator photons as constant. In the electron beam rest frame, the Hermitian Hamiltonian operator describing the multimode FEL process in the weak undulator regime is

$$H = \sum_{j=1}^{N_e} \frac{p_j'^2}{2m_e} + \hbar g_{uu}^{(w)} \left( a_u^\dagger a_u + \frac{1}{2} \right) + \sum_{\lambda=1}^M \hbar (\bar{\omega}_\lambda + g_{\lambda\lambda}^{(w)}) \times \left( a_\lambda^\dagger a_\lambda + \frac{1}{2} \right) + \sum_{\lambda=1}^M \hbar g_{\lambda u}^{(w)} \left( a_\lambda^\dagger a_u \frac{1}{N_e} \sum_{j=1}^{N_e} e^{-i\theta_{\lambda j}} + \text{H.c.} \right) + \sum_{\lambda=1}^M \sum_{\nu > \lambda} \hbar g_{\lambda\nu}^{(w)} \left( a_\lambda a_\nu^\dagger \frac{1}{N_e} \sum_{j=1}^{N_e} e^{i(\theta_{\lambda j} - \theta_{\nu j})} + \text{H.c.} \right). \quad (\text{A20})$$

The coupling between the fields is determined by the parameter

$$g_{\lambda\nu}^{(w)} = \frac{\omega_p^2}{2c\sqrt{k_\lambda k_\nu}}, \quad (\text{A21})$$

where  $\omega_p^2 = 4\pi(N_e/V)e^2/m_e$  is the plasma frequency of the electron beam. Here we have quantized the undulator field, and the undulator creation  $a_u^\dagger$  and annihilation  $a_u$  operators obey the usual commutation relation  $[a_u, a_u^\dagger] = 1$ . In addition to the laser-undulator interaction, the Hamiltonian operator also describes the interaction between laser modes through the last term on the right-hand side of Eq. (A20).

The Hamiltonian contains the following conservation laws: the total (undulator and laser) photon number operator

$$\left[ a_u^\dagger a_u + \sum_{\lambda=1}^M a_\lambda^\dagger a_\lambda, H \right] = 0, \quad (\text{A22})$$

and the total linear (photon and electron) momentum operator

$$\left[ \sum_{j=1}^{N_e} p_j' + \sum_{\lambda=1}^M \hbar \bar{k}_\lambda a_\lambda^\dagger a_\lambda, H \right] = 0. \quad (\text{A23})$$

Equations (A22) and (A23) can be combined to yield

$$\left[ \sum_{j=1}^{N_e} p_j' - \hbar k_u' a_u^\dagger a_u + \sum_{\lambda=1}^M \hbar k_\lambda' a_\lambda^\dagger a_\lambda, H \right] = 0. \quad (\text{A24})$$

The physical interpretation of Eqs. (A22)–(A24) is clear; the FEL interaction consists of the annihilation (creation) of an undulator photon and the creation (annihilation) of a laser photon, with the necessary momentum provided by the recoil of the electrons.

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