PHYSICAL REVIEW E, VOLUME 64, 056301

Turbulence of polymer solutions

E. Balkovsky,[1] A. Fouxon,[2] and V. Lebedev[2,3]
[1]The James Franck Institute and the Department of Mathematics, University of Chicago, 5640 S. Ellis Ave., Chicago, Illinois 60637
[2]Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
[3]Landau Institute for Theoretical Physics, Kosygina 2, Moscow 117940, Russia

(Received 2 May 2001; published 12 October 2001)

We investigate high-Reynolds-number turbulence in dilute polymer solutions. We show the existence of a critical value of the Reynolds number, which separates two different regimes. In the first regime, below the transition, the influence of the polymer molecules on the flow is negligible, so they can be regarded as passively embedded in the flow. This case admits a detailed investigation of the statistics of the polymer elongations. The second state is realized when the Reynolds number is larger than the critical value. This regime is characterized by the strong back reaction of polymers on the flow. We establish some properties of the statistics of the stress and velocity in this regime and discuss its relation to the drag reduction phenomenon.

DOI: 10.1103/PhysRevE.64.056301 PACS number(s): 47.27.−i, 83.80.Rs, 61.25.Hq, 05.40.−a

I. INTRODUCTION

In this paper we present a theoretical investigation of turbulence in dilute polymer solutions. As opposed to Newtonian fluids, such solutions possess additional macroscopic degrees of freedom related to the elasticity of the polymer molecules. Relaxation times of elastic stresses can be comparable with time scales of the flow, which means that the relation between the stress and velocity gradient is nonlocal. Due to the nontrivial interaction of inertial and elastic degrees of freedom, the polymer solutions exhibit a variety of regimes. For example, a turbulentlike state has been recently observed at very low Reynolds numbers [1]. Here we will consider the more familiar situation of turbulence at high Reynolds numbers. Probably, the most striking effect of polymers on the high-Reynolds-number flows is the drag reduction phenomenon. The addition of long-chain polymers in concentrations as small as 10^{-5} by weight can induce a substantial reduction of the drag force needed to push a turbulent fluid through a pipe [2–4].

The reason why small amounts of polymer can significantly modify properties of the fluid is the flexibility of polymer molecules. At equilibrium a polymer molecule coils up into a spongy ball of a radius R_0. The value of R_0 depends on the number of monomers in the molecule, which is usually very large. For a dilute solution with the concentration, n, satisfying $nR_0^3 \ll 1$, the influence of equilibrium size molecules on the hydrodynamic properties of the fluid can be neglected. When placed in a nonhomogeneous flow, such a molecule is deformed into an elongated structure that can be characterized by its end-to-end distance R. If the number of monomers in a typical polymer molecule is large, the elongation R can be much larger than R_0. The influence of the molecules on the flow increases with their elongation and may become substantial when $R \gg R_0$.

The deformation of the molecule is determined by two processes, the stretching by the velocity gradients and the relaxation due to the elasticity of the molecule. To understand how a molecule resists deformation by the flow, let us consider its relaxation. Recent experiments with DNA molecules indicate that the relaxation is linear in a wide region of scales $R_0 \ll R \ll R_{\max}$, where $R_{\max}$ is the maximum extension [5]. In the case of polymer molecules, theoretical arguments and numerics presented in [6] support the linear relaxation. These results can be understood if we assume that at $R \gg R_0$, the role of excluded volume and hydrodynamic interactions between the monomers becomes negligible. Then the random walk arguments suggest that the entropy of polymer molecules is quadratic in the range $R_0 \ll R \ll R_{\max}$ implying linear relaxation. Whether the polymers are excited by the flow is determined by the softest relaxation mode that describes the dynamics of the elongation R. In the absence of stretching, the relaxation of R is described by $\dot{R} = -R/\tau$, where τ is the largest of the polymer relaxation times. The relaxation time is R independent at $R_0 \ll R \ll R_{\max}$. If the end-to-end distance R is of the order of the maximum extension, τ starts to depend on R and the dynamics of the molecule becomes nonlinear.

The behavior of the molecule in an inhomogeneous steady flow depends on the value of the Weissenberg number Wi defined as the product of the characteristic velocity gradient and τ. When a polymer molecule is placed in a flow, smooth at the scale R, the velocity difference between the end points is proportional to R multiplied by the characteristic value of velocity gradient. At $\mathrm{Wi} \ll 1$ relaxation is fast as compared to the stretching time and the polymer always relaxes to the equilibrium size, R_0. The behavior of the polymer at $\mathrm{Wi} \gtrsim 1$ depends on the geometry of the flow. For purely elongational flows the molecule gets aligned along the principal stretching direction. If the velocity gradient is larger than the inverse relaxation time, i.e., $\mathrm{Wi} \gtrsim 1$, the elastic response becomes too slow in comparison with the stretching and the molecule gets substantially elongated [7]. The sharp transition from the coiled state to the strongly extended state is called the coil-stretch transition. Rotation can suppress the transition and even damp it completely since the molecule does not always point in the stretching direction (see, e.g., [8]). For example, no coil-stretch transition occurs in the case of a shear flow, which is a combination of elongational and rotational flows.

In contrast to the steady flows, a polymer molecule moving in a smooth random flow alternately enters regions of high and low stretching. As the intensity of the flow increases, the effect of the stretching becomes more pronounced. One can generally assert the existence of the coil-stretch transition. This has been first demonstrated by Lumley [8], who considered the situation where the characteristic time of variations of the velocity gradient is much larger than the inverse of the characteristic value of the gradient. He showed that if the amplitude of velocity gradient fluctuations is large enough, the expectation value of R^2 grows with time, which signifies the coil-stretch transition. We have demonstrated in [9] that the coil-stretch transition occurs in any random flow and have established a general criterion for the transition. In particular, the transition occurs in the situation where the time of velocity gradient variation is of the order of the inverse of its characteristic value, which is likely to be the case for real flows. The coil-stretch transition in random flows is controlled by the parameter $\lambda_1\tau$, where λ_1 is the average logarithmic divergence rate of nearby Lagrangian trajectories, to be referred to as the principal Lyapunov exponent. It is positive for an incompressible flow [10,11]. The molecules are weakly stretched if $\lambda_1\tau<1$ and strongly stretched otherwise. Therefore, for random flows the parameter $\lambda_1\tau$ plays the role of the Weissenberg number.

To describe the behavior of a polymer molecule in turbulent flows, let us briefly review the basic properties of turbulence of incompressible Newtonian fluids. A high-Reynolds-number flow consists of chaotic motions from a wide interval of scales, $\eta\ll r\ll L$, where L is the scale at which the flow is excited and η is the viscous scale. The energy pumped at the scale L cascades down to the scale η, where it is dissipated. The size of polymer molecules is usually much smaller than the viscous scale. Viscosity makes the flow smooth at $r\ll\eta$, i.e., the velocity difference between two points is given by the velocity gradient multiplied by the distance. Then, the stretching of molecules is determined by the gradient of velocity, which should be considered random in a turbulent flow. The Lyapunov exponent can be estimated as the characteristic value of the velocity gradient, which is determined by the eddies at the viscous scale. As the Reynolds number increases, the velocity gradient increases, and so does $\lambda_1\tau$. At some value of the Reynolds number it reaches the value 1 and the coil-stretch transition occurs.

Several mechanisms can limit the stretching of polymers. The first one is the internal non-linearity of the elasticity of the polymer molecules. If this mechanism dominates, then above the transition the molecules are stretched up to the maximal elongation $R_{\max}$. An alternative explanation has been proposed by Tabor and de Gennes [12]. It is based on the fact that if the elongation of a polymer molecule is larger than the viscous length of turbulence, η, the elastic force always wins over the stretching. Estimates using the parameters of typical polymer solutions show that this situation is difficult to realize. Therefore we assume that the inequality $R\ll\eta$ is satisfied. It will also enable us to write local equations describing the dynamics of elastic stresses. Another mechanism is the back reaction of the polymers on the flow. It is caused by the collective contribution of coherently deformed polymer molecules into the stress tensor. This elastic part of the stress grows with the elongation of the molecules. When it becomes of the order of the viscous stresses existing in the flow, the polymers modify the flow around them and the stretching diminishes. As a result, a dynamic equilibrium is realized at a characteristic elongation, R_{back}. The total polymer stress is proportional to nR^2, so that R_{back} depends on the polymer concentration n. Therefore if the concentration is large enough, the value of R_{back} is much smaller than $R_{\max}$. We will consider the effect of the back reaction under the assumption $R_{\text{back}}\ll R_{\max}$. Probably, the condition $R_{\text{back}}\ll R_{\max}$ is necessary for the existence of a stationary state, because the polymer molecules stretched up to $R_{\max}$ are intensively destroyed by the flow.

Above the coil-stretch transition, the back reaction modifies the small-scale properties of turbulent flows, which leads to the emergence of a new scale, $r_*>\eta$. Large-scale eddies with the sizes $r>r_*$ do not excite elastic degrees of freedom, so that the usual inertial energy cascade is realized at these scales. At smaller scales inertial and elastic degrees of freedom exchange energy, which is dissipated mainly due to the polymer relaxation. The energy cascade terminates at r_*, so that r_* plays the role of a new dissipation scale.

The plan of the paper is as follows. In Sec. II we introduce a system of equations describing the coupled dynamics of inertial and elastic degrees of freedom. In Sec. III we study the situation when the back reaction of polymers is small and can be disregarded. We find the probability distribution function of the elastic stress tensor and examine its correlation functions. In Sec. IV we study the influence of the back reaction on the flow and establish some properties of the velocity and stress statistics. In the Conclusion we summarize our results and discuss their implications for the drag reduction phenomenon. In Appendix A we present a detailed derivation of the probability density function from Sec. III. Appendix B is devoted to a simple model illustrating some aspects of the interaction between the flow and polymers. Preliminary results of this work have been published in [9].

II. BASIC RELATIONS

Following Hinch [13] let us consider the dynamics of a polymer molecule in a smooth velocity field. The degree of freedom related to the elongation of the molecule is described by the vector $\boldsymbol{R}$, connecting the end points of the molecule. The equation describing the dynamics of $\boldsymbol{R}$ in the absence of a surrounding flow is

$$\partial_t R_i+\Gamma\frac{\partial E}{\partial R_i}=\zeta_i\,, \tag{2.1}$$

where E is the free energy of the molecule, ζ_i is the thermal noise, and Γ is the kinetic coefficient, which determines the relaxation of the molecule. The correlation function of ζ is

$$\langle\zeta_i(t)\zeta_j(t')\rangle=2k_BT\,\Gamma\,\delta_{ij}\delta(t-t'), \tag{2.2}$$

where T is temperature and k_B is the Boltzmann constant. If the size of molecules is much smaller than the viscous length, which we assume, the molecule moves in a spatially constant gradient flow. Its influence is described by the equation

$$\partial_t R_i - R_j \nabla_j v_i + \Gamma \frac{\partial E}{\partial R_i} = \zeta_i , \tag{2.3}$$

where the stretching term $-R_j \nabla_j v_i$ is added to Eq. (2.1) The velocity derivative must be evaluated at the position of the molecule. To avoid misunderstanding, note that we mean the flow that is ''external'' to the molecule, excluding the velocity induced by the relative motions of its chains.

The entropy of the molecule has a quadratic dependence on the elongation R in a wide interval [14]. It implies that the molecule can be treated in terms of elasticity theory with a Hook modulus K_0 so that $E = K_0 R^2/2$. This expression is correct provided $R \ll R_{\max}$, where $R_{\max}$ is the maximum elongation of the molecule. The equilibrium size of the molecule, R_0, can be estimated from the condition $E \sim k_B T$ as $\sqrt{k_B T/K_0}$. Substituting the energy into Eq. (2.3) we get

$$\partial_t R_i - R_j \nabla_j v_i + R_i/\tau = \zeta_i , \tag{2.4}$$

where $\tau = (\Gamma K_0)^{-1}$. We see that τ is the molecular relaxation time.

Generally, the kinetic coefficient Γ or the relaxation time τ is a function of R, which reflects the nonlinear character of the molecule relaxation [15] related to such effects as internal hydrodynamic interaction of chains in the polymer molecule. For example, the finitely extendible nonlinear elastic model [15] assumes $\tau \propto 1 - R^2/R_{\max}^2$. One expects that a dependence of Γ and τ on R can be disregarded at $R \ll R_{\max}$. Below we study this situation. Possible statistical consequences of the non-Hookean dependence of the free energy on R have been investigated in [16].

Equations (2.3) and (2.4) assume that R is the only mode related to the molecular deformation, which is an idealization. Actually, the molecule has many deformational degrees of freedom that have different relaxation times. They have been observed experimentally [5]. Nevertheless, in the turbulent flows, only the mode with the largest relaxation time can be strongly excited whereas other modes are at most weakly excited. Thus, Eq. (2.4) should be considered as the equation describing the principal mode.

A. Continuous media equations

To study the dynamics at scales much larger than the interpolymer distance, the polymer solution can be regarded as a continuous medium. The appropriate description is done in terms of macroscopic quantities, which are averages of microscopic variables over the volume. The polymer molecules are characterized by the average conformation tensor

$$A_{ij} = \langle R_i R_j \rangle . \tag{2.5}$$

The volume of averaging should contain a large number of polymer molecules and be smaller than the characteristic scales of the processes under consideration. The tensor A can also be interpreted as the average over the statistics of the thermal noise $\boldsymbol{\zeta}$. The equation for A_{ik} follows from Eqs. (2.2) and (2.4)

$$\partial_t A_{ij} + (\boldsymbol{v}\cdot\nabla) A_{ij} = A_{kj} \nabla_k v_i + A_{ik} \nabla_k v_j - \frac{2}{\tau}(A_{ij} - A_0 \delta_{ij}), \tag{2.6}$$

where $A_0 = k_B T/K_0$. Equation (2.6) is linear in A, which is correct provided $A \ll R_{\max}^2$.

Equation (2.6) should be supplemented by the equation for the fluid velocity. This equation is a consequence of the momentum conservation law. In order to derive it, one should take into account the contribution of the inner elastic forces of polymer molecules to the total stress tensor of the fluid. If Π_{ik} is the elastic stress tensor per unit mass, the polymer contribution is $\varrho \Pi_{ik}$, where ϱ is the mass density of the fluid. In the Hookean approximation

$$\Pi_{ik} = \frac{K_0 n}{\varrho} A_{ik} - \Pi_0 \delta_{ik} , \tag{2.7}$$

where $\Pi_0 = K_0 A_0 n/\varrho = (n/\varrho) k_B T$ originates from the thermal noise $\boldsymbol{\zeta}$ in Eq. (2.4). Here n is the concentration of the polymer molecules. If the flow is incompressible, i.e., ϱ is a constant and $\nabla\cdot\boldsymbol{v} = 0$, the momentum conservation law reads

$$\partial_t v_i + (\boldsymbol{v}\cdot\nabla) v_i + \varrho^{-1} \nabla_i P = \nu \nabla^2 v_i + \nabla_k \Pi_{ik} + f_i . \tag{2.8}$$

Here P is the pressure, ν is the kinematic viscosity of the solvent, and $\boldsymbol{f}$ is the external force (per unit mass) driving the flow. Equation (2.8) is a generalization of the Navier-Stokes equation to the case of viscoelastic fluids. To simplify the consideration we assume that $\boldsymbol{f}$ is homogeneously distributed over space. It is a common belief that this case does not differ qualitatively from that realized for real experimental setups, where pumping is usually related to the boundaries.

The applicability condition of Eq. (2.6) is $A \ll R_{\max}^2$. It can be rewritten in terms of elastic stress tensor as $\Pi \ll \Pi_{\max} \equiv K_0 n \varrho^{-1} R_{\max}^2$. We assume that this condition is satisfied for relevant fluctuations. The interaction between $\boldsymbol{v}$ and Π turns on if Π exceeds the viscous stress, $\nu \nabla_j v_i$. The latter can be estimated as $\nu\lambda_1$, where λ_1 is the average logarithmic divergence rate of nearby Lagrangian trajectories. Under the condition $\Pi_{\max} \ll \nu\lambda_1$ the polymer molecules exert no influence on the flow except for a small renormalization of the viscosity of the solution. Thus, the inequality $\Pi_{\max} \gg \nu\lambda_1$ is a necessary condition for the polymers to have nontrivial effects.

The free energy of the viscoelastic fluid is the sum of the kinetic and elastic contributions

$$\mathcal{F} = \int d\boldsymbol{r} \left\{ \frac{\varrho}{2} v^2 + \frac{n K_0}{2} [\operatorname{Tr} A - A_0 \ln(\det A/A_0)] \right\},$$

where the second term represents the entropy of the molecules. Then we find from Eqs. (2.6)–(2.8)

$$\frac{\partial\mathcal{F}}{\partial t}=\varrho\int d\boldsymbol{r}\,\boldsymbol{f}\cdot\boldsymbol{v}-\varrho\int d\boldsymbol{r}\left\{\nu(\nabla_i v_j)^2+\frac{1}{\tau}\mathrm{Tr}\,[\Pi(\Pi_0+\Pi)^{-1}\Pi]\right\}. \tag{2.9}$$

This equation provides a mathematical formulation of the energy balance: the force supplies the energy, which is then dissipated due to viscosity and relaxation of polymers. Note that the second integral in Eq. (2.9) has a definite sign, as it should be for a dissipative term. Relative contributions of the viscous and the elastic terms to the energy dissipation can be different.

If the forcing is statistically homogeneous, then a statistically homogeneous steady state is realized. It can be described in terms of correlation functions of $\boldsymbol{v}$ and A (or $\boldsymbol{v}$ and Π), which are averages over the statistics of the pumping force $\boldsymbol{f}$ or over space. In the steady state the average value of $\partial\mathcal{F}/\partial t$ is equal to zero. Therefore we get from Eq. (2.9)

$$\nu\langle(\nabla_i v_j)^2\rangle+\frac{1}{\tau}\langle\mathrm{Tr}\,[\Pi\,(\Pi+\Pi_0)^{-1}\Pi]\rangle=\epsilon, \tag{2.10}$$

where $\epsilon=\langle\boldsymbol{f}\cdot\boldsymbol{v}\rangle$ is the mean energy injection rate (per unit mass) by the external force.

Generally, the diffusion term $\kappa\nabla^2 A$ should also be added to the right-hand side of Eq. (2.6). The diffusion coefficient κ is small due to a large number of monomers. It is possible to show that the limit $\kappa\to 0$ is regular, so we can disregard the diffusion. The diffusion term can play a minor role for scales $r\lesssim\sqrt{\kappa/\lambda_1}$. If $\sqrt{\kappa/\lambda_1}$ is smaller than the intermolecular distance, then the diffusivity is irrelevant in the whole region of applicability of the macroscopic approach.

B. Lagrangian description

Equation (2.6) can formally be solved in the Lagrangian reference frame. Let us introduce $\tilde{A}(t,\boldsymbol{r})=A[t,\boldsymbol{x}(t,\boldsymbol{r})]$, where $\boldsymbol{x}(t,\boldsymbol{r})$ is the Lagrangian trajectory defined by the relations

$$\partial_t\boldsymbol{x}=\boldsymbol{v}(t,\boldsymbol{x}),\quad \boldsymbol{x}(t_0,\boldsymbol{r})=\boldsymbol{r}. \tag{2.11}$$

The condition $\boldsymbol{x}(t_0,\boldsymbol{r})=\boldsymbol{r}$ ensures that the fields A and $\tilde{A}$ coincide at $t=t_0$. The point $\boldsymbol{r}$ plays the role of a Lagrangian marker. The tensor $\tilde{A}(t,\boldsymbol{r})$ satisfies the matrix equation following from Eq. (2.6),

$$\partial_t\tilde{A}=\sigma\tilde{A}+\tilde{A}\sigma^T-\frac{2}{\tau}(\tilde{A}-A_0), \tag{2.12}$$

$$\sigma_{ij}(t,\boldsymbol{r})=\nabla_j v_i[t,\boldsymbol{x}(t,\boldsymbol{r})]. \tag{2.13}$$

Here σ is the tensor of the velocity derivatives in the Lagrangian frame and the superscript T denotes a transposed matrix. Due to causality, the value of the field $A(t,\boldsymbol{r})$ is determined by its dynamics at times $t'<t$. Therefore we will be interested in the backward in time Lagrangian evolution of A described by Eqs. (2.11) and (2.12).

A solution of Eq. (2.12) can be written in terms of the Lagrangian mapping matrix W defined by the relations

$$\partial_t W(t,t')=\sigma(t)W(t,t'),\quad W(t',t')=1. \tag{2.14}$$

The matrix W is defined for a given Lagrangian trajectory and therefore it depends on its marker $\boldsymbol{r}$. For brevity we omit the argument $\boldsymbol{r}$ in W. The matrix W describes the deformation of infinitesimal fluid volumes. For example, the separation $\delta\boldsymbol{x}$ between two close Lagrangian particles changes according to

$$\delta\boldsymbol{x}(t)=W(t,t')\;\delta\boldsymbol{x}(t'). \tag{2.15}$$

It follows from Eq. (2.15) that $W_{ij}(t,t_0,\boldsymbol{r})=\partial x_i(t,\boldsymbol{r})/\partial r_j$. The incompressibility condition $\nabla\cdot\boldsymbol{v}=0$ is formulated in terms of σ as $\mathrm{tr}\,\sigma=0$. A consequence of incompressibility is

$$\det W=1. \tag{2.16}$$

Using Eqs. (2.12) and (2.14) we obtain

$$\tilde{A}(t,\boldsymbol{r})=\frac{2A_0e^{-2t/\tau}}{\tau}\int_{-\infty}^{t}dt'\;W(t,t',\boldsymbol{r})W^T(t,t',\boldsymbol{r})e^{2t'/\tau}.$$

At $t=t_0$ this equation gives

$$A(t_0,\boldsymbol{r})=\frac{2A_0}{\tau}\int_0^{\infty}dt\;W(t_0,t_0-t,\boldsymbol{r})W^T(t_0,t_0-t,\boldsymbol{r})e^{-2t/\tau}. \tag{2.17}$$

It is easy to understand the meaning of Eq. (2.17). The polymers are advected along the Lagrangian trajectories being stretched by the velocity gradient and relaxing to their equilibrium shape due to elasticity. The value of the conformation tensor A is determined by the sum of the contributions of these processes at earlier times picked along the Lagrangian trajectory arriving at $\boldsymbol{r}$. The term WW^T describes the stretching and the exponential factor accounts for relaxation.

Expression (2.17) shows that when calculating correlation functions, the volume averages can be substituted by averages over the statistics of W.

III. PASSIVE ELASTIC TENSOR IN A RANDOM VELOCITY FIELD

In this section we consider the polymer molecules as passive objects, i.e., we assume that the inertial properties of the fluid are not perturbed by the polymer elasticity. In other words, we assume that the term $\nabla_k\Pi_{ik}$ in Eq. (2.8) can be disregarded so that the dynamics of $\boldsymbol{v}$ is independent of the polymer dynamics. Equation (2.6) determines the statistics of the conformation tensor A provided the statistical properties of $\boldsymbol{v}$ are known. We consider a high-Reynolds-number flow and assume that its statistics is stationary, spatially homogeneous, and isotropic.

One might think that in order to examine the correlation functions of A one needs to know the precise statistics of the velocity field described by the Navier-Stokes equation. However, we will show that the statistics of A is universal, i.e., it does not depend on the details of the velocity statistics. The crucial property underlying the universality is a finite Lagrangian correlation time of the velocity derivatives matrix σ [17].

Equation (2.6) shows that the Lagrangian dynamics of the polymer stress tensor is determined by velocity gradients, which are related to the viscous scale η of the turbulence. A typical value of the velocity gradient can be estimated as λ_1, where λ_1 is the logarithmic divergence rate of nearby Lagrangian trajectories. Then it follows from Eq. (2.8) that the feedback of the polymers on the flow can be neglected if $\Pi \ll \nu\lambda_1$. This is the applicability condition of the passive approach. As we show below, the condition $\Pi \ll \nu\lambda_1$ is satisfied for typical fluctuations if $\lambda_1\tau < 1$.

A formal solution of Eq. (2.6) is given by Eq. (2.17). The condition $\lambda_1\tau < 1$ means that the exponentially decaying factor $\exp(-2t/\tau)$ in Eq. (2.17) dominates over the product $WW^T \sim \exp(2\lambda_1 t)$. In this case, for a typical velocity fluctuation the integral over t converges at $t \sim \tau$ and therefore A fluctuates near A_0. In addition to the strong peak at $A \sim A_0$, the probability distribution function of A has a power tail. This has been demonstrated in [9] in terms of the molecule elongation $\boldsymbol{R}$. The fluctuations $A \gg A_0$ are formed if the product WW^T in Eq. (2.17) is anomalously large for a long time. Such events can be described in terms of a universal scheme (see Ref. [18]) shortly presented below.

A. Long-time Lagrangian statistics

Let us briefly review the long-time statistical properties of the Lagrangian mapping matrix W, determined by Eqs. (2.13) and (2.14). We consider $W(t_+, t_-)$ at $t_+ > t_-$ and assume that $t_+ - t_-$ is much larger than the Lagrangian correlation time τ_σ of the velocity derivatives matrix (2.13). If the velocity statistics is homogeneous in time, the probability distribution of $W(t_+, t_-)$ depends on the difference $t_+ - t_-$ only. Equation (2.14) implies that at $t_+ - t_- \gg \tau_\sigma$ the matrix W is a product of a large number of independent matrices. This is the main reason for the universality in the statistics of W.

It is convenient to decompose the matrix W as

$$W(t_+, t_-) = M\Lambda N, \tag{3.1}$$

where Λ is a diagonal matrix, and M and N are orthogonal matrices [19]. We denote the diagonal elements of Λ as e^{ρ_1}, e^{ρ_2}, and e^{ρ_3}, and assume that they are ordered: $\rho_1 > \rho_2 > \rho_3$. As a consequence of the constraint (2.16) we have $\rho_1 + \rho_2 + \rho_3 = 0$. Equation (2.14) can be rewritten in terms of ρ_i and the matrices M and N. The equations for ρ_i are

$$\frac{\partial \rho_i}{\partial t_+} = \tilde{\sigma}_{ii}, \tag{3.2}$$

where $\tilde{\sigma} = M^T \sigma M$ and no summation over the repeating index i is implied. The matrices M and N satisfy $\partial_t N = \Omega_1 N$ and $\partial_t M = M\Omega_2$, where

$$\Omega_{1ik} = \frac{\tilde{\sigma}_{ik} + \tilde{\sigma}_{ki}}{2\sinh(\rho_i - \rho_k)}, \qquad \Omega_{2ik} = \frac{\tilde{\sigma}_{ik} e^{2\rho_k} + \tilde{\sigma}_{ki} e^{2\rho_i}}{e^{2\rho_k} - e^{2\rho_i}},$$

for $i \neq k$ and $\Omega_{1ik} = \Omega_{2ik} = 0$ for $i = k$. It is possible to show that the eigenvalues of W repel each other, so that the inequalities $e^{\rho_1} \gg e^{\rho_2} \gg e^{\rho_3}$ are satisfied at $t_+ - t_- \gg \tau_\sigma$ [18]. Then the matrix Ω_1 tends to zero exponentially fast, i.e., N is determined by times of the order of τ_σ in the vicinity of t_-. The matrix Ω_2 becomes ρ independent at $t_+ - t_- \gg \tau_\sigma$ and the evolution of M is decoupled from that of ρ_i. Then the value of M is determined by the time of the order of τ_σ at $t \approx t_+$, i.e., at $t_+ - t_- \gg \tau_\sigma$ it becomes t_- independent.

The solution of Eq. (3.2) is

$$\rho_i = \int_{t_-}^{t_+} dt' \, \tilde{\sigma}_{ii}(t'), \tag{3.3}$$

where the right-hand side of Eq. (3.3) is an integral of a random process independent of ρ_i. Equation (3.3) shows that the variables ρ_i fluctuate around their average values $\lambda_i(t_+ - t_-)$. Here the constants λ_i are equal to $\langle \tilde{\sigma}_{ii} \rangle$. They are called the Lyapunov exponents of the flow. Generally, the spectrum of the Lyapunov exponents is nondegenerate: $\lambda_1 > \lambda_2 > \lambda_3$, which is a necessary condition for the formalism to be self-consistent. The incompressibility condition ensures the identity $\lambda_1 + \lambda_2 + \lambda_3 = 0$, which implies $\lambda_1 > 0$ and $\lambda_3 < 0$. Using the relation (2.15) one can show that λ_1 is indeed the average logarithmic divergence rate of two nearby Lagrangian trajectories,

$$\langle d \ln|\delta \mathbf{x}|/dt \rangle = \lambda_1 .$$

Similarly, $\lambda_1 + \lambda_2 = -\lambda_3$ is the average logarithmic rate of the area growth.

Under the condition $t_+ - t_- \gg \tau_\sigma$ the quantity ρ_i can be considered as a sum of a large number of independent random variables. It is known from statistical mechanics (see, e.g., [20]) that the distribution of such quantities is given by the exponent of an extensive function. In our case the probability distribution function (PDF) of ρ_i is

$$\mathcal{P}(t, \rho_1, \rho_2, \rho_3) \propto \frac{1}{t} \exp\left[-tS\left(\frac{\rho_1 - \lambda_1 t}{t}, \frac{\rho_3 - \lambda_3 t}{t} \right) \right] \times \delta(\rho_1 + \rho_2 + \rho_3), \tag{3.4}$$

where $t = t_+ - t_-$ and $\rho_1 > \rho_2 > \rho_3$ is implied [18]. The main exponential factor of the PDF has a self-similar form described by the function S, which can be called the entropy function (see [18,21,22]). It is positive, convex and has a minimum at zero values of its arguments. The precise form of S is determined by details of the velocity statistics. The PDF has a sharp maximum at $\rho_i = \lambda_i t$. In its vicinity the function S has a quadratic expansion, i.e., the distribution of ρ is Gaussian. However, if one is interested in the expectation values of exponential functions of ρ_i, they are determined by the wings of the PDF where the Gaussian approximation is invalid. This entails the use of the whole entropy function.

To average the functions of ρ_1 only, one can introduce the reduced probability distribution function,

$$\mathcal{P}(t,\rho_1)\propto\frac{1}{\sqrt{t}}\exp\left[-tS_1\left(\frac{\rho_1-\lambda_1 t}{t}\right)\right], \qquad (3.5)$$

which is an integral of $\mathcal{P}(t,\rho_1,\rho_2,\rho_3)$ over ρ_2 and ρ_3. At small x the function $S_1(x)$ can be written as

$$S_1(x)\approx\frac{x^2}{2\Delta}. \qquad (3.6)$$

Here $\Delta=\int dt\,\langle\langle\tilde{\sigma}_{11}(t)\tilde{\sigma}_{11}(0)\rangle\rangle$ (where double brackets designate irreducible correlation function) determines the dispersion of ρ_1: $\langle(\rho_1-\lambda_1 t)^2\rangle\approx t\Delta$. Expansion (3.6) is sufficient to describe typical fluctuations of ρ_1, whereas the whole function S is needed to describe rare events.

In the passive regime the statistics of velocity gradients is determined by the fluctuations at the viscous scale η. The Lagrangian correlation time τ_σ is the turnover time at this scale. It can also be estimated as λ_1^{-1}. Using the expression $\epsilon=\nu\langle(\nabla_j v_i)^2\rangle$ for the energy dissipation rate one can write the estimates $\lambda_1\sim\sqrt{\epsilon/\nu}$ and $\Delta\sim\lambda_1$ for the Lyapunov exponent and the dispersion.

B. Single-point statistics

In this section we examine the single-point statistics of the conformation tensor A at $\lambda_1\tau<1$. As explained above, most of the time A fluctuates near A_0. We are interested in large values $A\gg A_0$ because it is only for large values of A that the polymers can possibly lead to noticeable effects. Large values are determined by the velocity fluctuations such that the product WW^T is anomalously large for a long time. To find the tail of the PDF of A let us substitute decomposition (3.1) into Eq. (2.17). We obtain

$$M^TAM=\frac{2A_0}{\tau}\int_0^\infty dt\,\Lambda^2(t)\exp\left[-\frac{2t}{\tau}\right],$$

where we have used the t independence of M at large t. Under the condition $e^{\rho_1}\gg e^{\rho_2}\gg e^{\rho_3}$ the tensor A is uniaxial,

$$A_{ij}\approx Tn_in_j. \qquad (3.7)$$

Here $\boldsymbol{n}$ is a unit vector, $n_i=M_{i1}$, uniformly distributed over a sphere and $T\equiv\mathrm{tr}\,A$,

$$T\approx\frac{2A_0}{\tau}\int_0^\infty dt\,\exp\left[2\rho_1(t)-\frac{2t}{\tau}\right]. \qquad (3.8)$$

Thus the matrix A is expressed via the scalar T, which is independent of ρ_2 and ρ_3. The statistics of T cannot be directly examined in terms of the single-time probability distribution function (3.5) because integral (3.8) involves different times. Nevertheless, it is possible to use PDF (3.5) to find the asymptotic behavior of the PDF of T at $T\gg A_0$. A rigorous derivation is presented in Appendix A (cf. [23,24]). Below we use a simple semiqualitative method.

For a typical fluctuation of the velocity, integral (3.8) converges at $t\sim\tau$, which gives $T\sim A_0$. To find the probability distribution for large deviations of T one should analyze rare events leading to a given value $T\gg A_0$, and find the event with the maximum probability. Let us establish the structure of such fluctuations. It is obvious that ρ_1 should initially grow faster than t/τ during some interval of time. To ensure nonzero value of the probability of such a configuration, ρ_1 should then return to its average value, $\lambda_1 t$. Since $\lambda_1<1/\tau$, the difference $\rho_1(t)-t/\tau$ has a maximum at some time $t=t_*$. At $T\gg A_0$ the maximum is sharp and integral (3.8) is determined by its vicinity with the logarithmic accuracy

$$\ln(T/A_0)\approx 2\rho_1(t_*)-2t_*/\tau. \qquad (3.9)$$

The probability of the event is also determined by a vicinity of $\rho_1(t_*)$ because it corresponds to the maximal deviation from the average value of ρ_1. In accordance with Eq. (3.5) it can be estimated as

$$\ln\mathcal{P}\approx -t_*S_1\left[\frac{\rho_1(t_*)}{t_*}-\lambda_1\right].$$

Substituting here $\rho_1(t_*)$ expressed via T from Eq. (3.9), and maximizing the result over t_* we get the condition

$$S_1\left(\beta+\frac{1}{\tau}-\lambda_1\right)-\beta S_1'\left(\beta+\frac{1}{\tau}-\lambda_1\right)=0, \qquad (3.10)$$

where $\beta=(2t_*)^{-1}\ln(T/A_0)$. Using the convexity of S_1, one can show that Eq. (3.10) together with the condition $\beta>0$ uniquely determines β. Then one finds

$$\ln\mathcal{P}\approx-\frac{\alpha}{2}\ln(T/A_0),\qquad \alpha=S_1'\left(\beta+\frac{1}{\tau}-\lambda_1\right). \qquad (3.11)$$

One can verify that the convexity of S_1 ensures the condition $\alpha>0$ if $\lambda_1<1/\tau$. Expression (3.11) determines the probability density function of $\ln(T/A_0)$. For the PDF of T we obtain

$$\mathcal{P}(T)\sim\frac{A_0^{\alpha/2}}{T^{1+\alpha/2}}. \qquad (3.12)$$

Since $A\propto R^2$, the power-law distribution of A can be used to obtain the power-law distribution of the molecular elongation R [9].

We see that the PDF is a power-law function with the exponent $1+\alpha/2$ that can be expressed via the entropy function S_1. Since the precise form of S_1 is generally unknown, it is impossible to find the precise dependence of α on the parameters of the flow. However, some general properties can still be inferred. As λ_1 increases, i.e., when the Reynolds number increases, α decreases and tends to zero when $\lambda_1\to 1/\tau$. One can easily establish the behavior of α in this region since then the approximation (3.6) is correct. Substituting Eq. (3.6) into Eqs. (3.10) and (3.11) we obtain

$$\alpha=\frac{2}{\Delta}\left[\frac{1}{\tau}-\lambda_1\right].$$

Note that the only characteristics of the flow entering this expression are the average value of ρ_1 and its dispersion.

Power tail (3.12) means a slow decay of the probability distribution of $T=\mathrm{tr}\,A$, which results in infinite values of its high moments. Namely, the moments $\langle T^n\rangle=\int dT\,\mathcal{P}(T)T^n$ diverge if $n>\alpha/2$. Moreover, the normalization integral $\int dT\,\mathcal{P}(T)$ converges at large T only if $\alpha>0$. Therefore an attempt to extend the passive consideration to $\lambda_1>1/\tau$ leads to the divergence of the normalization integral. It can be interpreted as the tendency of the polymer molecules to be stretched, i.e., the coil-stretch transition, and the breakdown of the passive approach.

As we have seen, the nth moment of the conformation tensor A formally diverges at $n\geqslant\alpha/2$. It signals the breakdown of the passive approach, i.e., the main contribution to the diverging moments comes from the configurations of the velocity such that the feedback of the polymers on the flow cannot be disregarded. As explained in the beginning of this section, the molecules can be considered as passive at $\Pi\ll\nu\lambda_1$. An account of the back reaction of the polymers on the flow leads to a much faster decrease of the PDF of Π at $\Pi\gtrsim\nu\lambda_1$, which ensures that the moments have finite values (in the framework of the simple model presented in Appendix B one can find the precise form of the PDF). Let us estimate the value of the diverging moments taking the feedback into account. It is more convenient to discuss the moments of Π, which are proportional to the moments of A [see Eq. (2.7)]. If $\alpha<2$ then Eq. (3.12) modified by the cutoff at $\Pi\sim\nu\lambda_1$ gives

$$\langle \mathrm{Tr}\,\Pi\rangle\sim\Pi_0^{\alpha/2}(\nu\lambda_1)^{1-\alpha/2}. \qquad (3.13)$$

Note that $\langle\mathrm{Tr}\,\Pi\rangle\gg\Pi_0$ because we assumed that $\Pi_0\ll\nu\lambda_1$. The equations (2.10) and (3.13) show that the elastic contribution to the energy dissipation, $\tau^{-1}\langle\mathrm{Tr}\,\Pi\rangle$, can be estimated as $\tau^{-1}\Pi_0^{\alpha/2}(\nu\lambda_1)^{1-\alpha/2}$. It becomes comparable to the viscous contribution, $\sim\nu\lambda_1^2$ just at the point of the coil-stretch transition, where $\alpha=0$.

C. Correlation functions

Here we investigate simultaneous many-point correlation functions of A. Let us start with the two-point correlation function

$$G_2(\boldsymbol{r})=\langle T(t_0,\boldsymbol{r}_1+\boldsymbol{r})T(t_0,\boldsymbol{r}_1)\rangle, \qquad (3.14)$$

where $T=\mathrm{Tr}\,A$. The value of A at a given point is determined by the Lagrangian trajectory arriving at this point at $t=t_0$. Polymers separated by distances smaller than the viscous scale, η, are stretched coherently, whereas at larger separations the correlation is largely lost. Therefore η is the correlation length of A. For distances $r\gtrsim\eta$, the quantities T in Eq. (3.14) become weakly correlated and the correlation function tends to the product of averages $\langle T\rangle$. Nontrivial correlations occur at distances smaller than η. The correlation function is a monotonically decreasing function of the distance r.

Note that G_2 cannot be calculated in the framework of the passive approach if $\alpha<2$. Indeed, we can write $G_2\approx\langle T\rangle^2$ at $r\gg\eta$. Formal calculation of $\langle T\rangle$ using PDF (3.12) gives an infinite result. The same is true for $G(0)=\langle T^2\rangle$. Since $G_2(r)$ is a decreasing function of r, it follows that $G_2(r)$ is infinite in the framework of the passive approach. It means that the main contribution to the correlation function comes from configurations such that the back influence of the polymers on the flow is not small. On the other hand, if $\alpha>4$ the calculation of both $\langle T\rangle^2$ and $\langle T^2\rangle$ in the passive framework gives a value of the order of A_0^2. Hence $G_2(r)\sim A_0^2$ for all r, which means that the main contribution is made by the configurations where the polymers are close to the equilibrium state.

Let us consider the most interesting case $2<\alpha<4$. Then $\langle T\rangle\sim A_0$ is finite (and small), whereas $\langle T^2\rangle$ is infinite if calculated using the ''passive'' PDF (3.12). It means that at small distances the back reaction must be taken into account, whereas at larger distances the passive approach works well. If we calculate the correlation function in the framework of the passive approach, the result is valid for distances larger than some characteristic scale of the back reaction.

To calculate correlation function (3.14) one can substitute expression (2.17) for $A(\boldsymbol{r}_1+\boldsymbol{r})$ and $A(\boldsymbol{r}_1)$ and then average over $\boldsymbol{r}_1$, which can be considered as averaging over space. The distance between Lagrangian trajectories terminating at the points $\boldsymbol{r}_1+\boldsymbol{r}$ and $\boldsymbol{r}_1$ is an increasing function of t. It can be found from Eq. (2.15),

$$\delta\boldsymbol{x}(t_0-t)=W(t_0-t,t_0)\boldsymbol{r}=W^{-1}(t_0,t_0-t)\boldsymbol{r}. \qquad (3.15)$$

Expression (3.15) is correct provided $|\delta\boldsymbol{x}|<\eta$. Under the same condition the matrices W entering the expression (2.17) for $A(\boldsymbol{r}_1+\boldsymbol{r})$ and $A(\boldsymbol{r}_1)$ are identical, as follows from Eq. (2.14). Using the decomposition (3.1), rewritten as $W^{-1}(t_0,t_0-t)=N^T\Lambda^{-1}M^T$, and the inequalities $e^{\rho_1}\gg e^{\rho_2}\gg e^{\rho_3}$ we obtain $|\delta\boldsymbol{x}|\approx e^{-\rho_3}r$.

At $r\ll\eta$ we have $G_2\gg A_0^2$, i.e., the main contribution to G_2 is due to the rare events when the product WW^T is anomalously large during a long time. Then one can use the uniaxial approximation (3.7). The functions $\rho_1(t)$ in $A(\boldsymbol{r}_1+\boldsymbol{r})$ and $A(\boldsymbol{r}_1)$ are identical as long as $|\delta\boldsymbol{x}|<\eta$. When the separation $\delta\boldsymbol{x}$ becomes larger than η, the correlation between the Lagrangian trajectories becomes weak. The contribution of this stage to G_2 is given by the product of independent averages, $\langle T\rangle^2\sim A_0^2$, and can be neglected. We conclude that the main contribution to G comes from times when $|\delta\boldsymbol{x}|<\eta$. Using the expression $|\delta\boldsymbol{x}|\approx e^{-\rho_3}r$ we get

$$G\sim\left\langle\frac{A_0^2}{\tau^2}\left\{\int_0^s dt\,\exp(2\rho_1-2t/\tau)\right\}^2\right\rangle, \qquad (3.16)$$

where $\rho_3(s)=\ln(r/\eta)$. Since both ρ_1 and ρ_3 enter the integral, to evaluate G_2 one needs the joint PDF (3.4).

To ensure a large value of G_2, the function ρ_1 in Eq. (3.16) should first increase faster than t/τ and then return to its average value $\lambda_1 t$. Thus ρ_1-t/τ should have a maximum at $t_*<s$. A vicinity of $t=t_*$ makes the main contribution to G_2. In the absence of the constraint $t_*<s$ the value of G_2 grows exponentially as t_* increases, which corresponds to a formally infinite value of the second moment at $\alpha<4$. Therefore the optimum is achieved at $t_*\approx s$. It gives the estimate

$$G_2 \sim A_0^2 \int d\rho_1 \exp(4\rho_1 - 4s/\tau) \times \exp\left\{ -sS\left(\frac{\rho_1 - \lambda_1 s}{s}, \frac{\ln[r/\eta] - \lambda_3 s}{s} \right) \right\}, \quad (3.17)$$

where s is determined from the condition $\rho_3(s) = \ln(r/\eta)$. The integral can be calculated in the saddle-point approximation with the saddle point $\rho_1^* \propto s$. Next, one should optimize over s, which gives $s \propto \ln(r/\eta)$. The proportionality coefficients depend on the form of S. The substitution of the optimal values gives

$$G_2 \sim A_0^2 (\eta/r)^{\xi_2}. \quad (3.18)$$

The exponent ξ_2 in Eq. (3.18) can be found if the precise form of the entropy function S is known. We observe that $G_2 \sim A_0^2$ if $r \sim \eta$. It is natural since G_2 at $r \sim \eta$ can be estimated as $\langle T \rangle^2$. We also see that G_2 tends to infinity at $r \to 0$. This corresponds to a formally infinite value of the second moment.

All the conclusions concerning the pair correlation function of T are valid for correlation functions of separate components of A too, which follows from the single-axis substitution (3.7). Indeed, $\boldsymbol{n}$ is uniformly distributed over the unit sphere, which reduces correlation functions of A to correlation functions of its trace T.

Let us discuss the case $\alpha > 4$. Then the main contribution (3.14) to $G_2(r)$ at small r is equal to the second moment $\langle T^2 \rangle \sim A_0^2$. One can examine the r-dependent correction to the second moment $\langle T^2 \rangle - G_2(r)$. It can be done as above. The correction behaves as a positive power of r at $r \ll \eta$.

The proposed scheme can be generalized to higher-order correlation functions

$$G_n = \langle A(\boldsymbol{r}_1) \ldots A(\boldsymbol{r}_n) \rangle. \quad (3.19)$$

The behavior of G_n is similar to that of G_2. If the moment $\langle T^n \rangle$ calculated with the PDF (3.12) is infinite, the function G_n is a scaling function of the coordinates. The scaling exponent is negative, so the correlation function formally diverges at small distances. On the other hand, if the moment $\langle T^n \rangle$ is finite then the difference $G_n - \langle A^n \rangle$ scales with a positive exponent and is thus a small correction to $\langle A^n \rangle$.

Since the moments of Π are finite, we can assert that the growth of the correlation functions of Π observed at fusing points in Eq. (3.19) has to be saturated. For example, at $2 < \alpha < 4$ the pair correlation function (3.14) saturates at $G_2 \sim A_0^2 \Pi_0^{-2} \langle \Pi^2 \rangle \sim A_0^2 \Pi_0^{\alpha/2-2} (\nu\lambda_1)^{2-\alpha/2}$. One can say that the back reaction regularizes the correlation functions at small scales.

IV. STRONG BACK REACTION

Here we consider the dynamics of the polymer solutions above the coil-stretch transition, when the Reynolds number exceeds a critical value. Depending on the concentration of polymer molecules, two situations are possible. If the concentration is small, the elastic stresses are small in comparison with the viscous stresses. Then the polymers are stretched to their maximal elongation $R_{\max}$ and the properties of the fluid do not differ significantly from those of the pure solvent. Below we consider the more interesting case where the concentration of polymers is large enough, so that elastic stresses can be larger than the viscous stresses. Then the feedback due to the polymers substantially modifies the flow. The condition for the existence of the back reaction regime is $\Pi_{\max} \gg \nu/\tau$, where $\Pi_{\max}$ is the maximal value of the elastic stress tensor. It can be expressed in terms of microscopic parameters and the concentration of the polymers as $\Pi_{\max} = K_0 n \varrho^{-1} R_{\max}^2$ (see Sec. II A). Using estimates for the microscopic parameters proposed in Ref. [13] one can rewrite the condition as $n \gg (R_0 R_{\max}^2)^{-1}$.

Whereas in a pure solvent, typical gradients of the velocity grow unlimited as the Reynolds number increases, in polymer solutions the balance of inertial and elastic degrees of freedom fixes the characteristic value of the gradient at $1/\tau$. Indeed, if the instantaneous velocity gradient exceeds $1/\tau$, it extends the polymers, so that the elastic stress grows and damps the gradient. On the other hand, if the velocity gradient is much smaller than $1/\tau$, the molecules contract and do not produce any effect on the flow. Then the velocity gradients tend to grow to the value characteristic of the pure solvent, which is larger than $1/\tau$ above the transition. This explains the steady state realized above the transition. We now establish some general properties of this steady state.

Turbulence of Newtonian fluids can be characterized by two length scales: the integral scale L and the dissipation scale η. Energy pumped at the integral scale cascades without dissipation from larger to smaller eddies (coherent motions of the fluid) in the range $\eta < r < L$ called the inertial interval. Velocity difference between two points separated by the distance r from the inertial range diminishes slower than r, so that the characteristic value of the velocity gradient at the scale r grows downscales reaching a maximum at $r \sim \eta$ [25]. We assume $V\tau/L \ll 1$, where V is the velocity at the integral scale, then the gradient related to large eddies is smaller than τ^{-1}. Therefore, large eddies do not excite polymers, which means that the elastic stress tensor is not correlated at these scales. Since only coherent excitations of the elastic stress tensor can influence the velocity, we conclude that the elasticity is negligible for large eddies. The interaction of inertial and elastic degrees of freedom becomes essential at the scale r_*, determined from the condition $\boldsymbol{\nabla} v \sim 1/\tau$. The fluctuations of Π are correlated over the scale r_*. Because the value of the gradient cannot exceed $1/\tau$, the velocity difference scales linearly with r at $r \lesssim r_*$, i.e., the flow is smooth. Near the coil-stretch transition characteristic velocity gradient is determined by the viscous scale and is of the order of $1/\tau$, hence $r_* \sim \eta$. As the Reynolds number increases, velocity fluctuations increase, so that the scale r_* grows. Thus above the coil-stretch transition a new scale r_* separating the inertial and viscoelastic intervals arises. It is of the order of η near the transition and grows as the energy input increases.

Near the transition the viscous and elastic terms in Eq. (2.8) are of the same order, which gives $\Pi \sim \nu/\tau$. For dilute solutions ν/τ is much larger than Π_0, therefore all the terms in the energy balance equation (2.10) are of the order of ν/τ^2 near the transition. The energy pumping rate per unit mass, ϵ, can be estimated as V^3/L. Equating it to the dissipation rate, estimated as ν/τ^2, one finds the estimate $\mathrm{Re}_c = [L^2/(\nu\tau)]^{2/3}$ for the value of the Reynolds number at the transition. As the energy input increases the energy dissipation rate due to viscosity, $\nu(\nabla_j v_i)^2$, remains of the order of ν/τ^2. Therefore far above the transition, the principal part of the energy is dissipated by the polymer relaxation. Then the viscous term in Eq. (2.10) can be neglected and we obtain

$$\langle \mathrm{Tr}\,\Pi \rangle = \epsilon\tau. \tag{4.1}$$

We conclude that the energy is dissipated mainly by the elastic relaxation. Relation (4.1) means that the typical value of Π grows as the energy input increases, which can be interpreted as the increase in the effective (''elastic'') viscosity. It is defined as the proportionality coefficient between the polymer stress tensor Π and the strain tensor $\nabla_i v_k + \nabla_k v_i$, which remains of the order $1/\tau$. Using Eq. (4.1) one can estimate the ratio of the elastic term to the nonlinear inertial term in Eq. (2.8) as $V\tau/L \ll 1$, which shows that the elasticity is indeed negligible at large scales.

The strong interaction between the elastic and inertial degrees of freedom imposes a restriction on the Lagrangian statistics of velocity. To demonstrate it, observe that Eq. (2.6) gets simplified under the condition $\Pi \gg \Pi_0$ satisfied in the strong back reaction regime. Neglecting the terms proportional to A_0 and Π_0 in Eqs. (2.6) and (2.7) we obtain

$$\partial_t \Pi_{ij} + (\boldsymbol{v}\cdot\nabla)\Pi_{ij} = \Pi_{kj}\nabla_k v_i + \Pi_{ik}\nabla_k v_j - \frac{2}{\tau}\Pi_{ij}. \tag{4.2}$$

Expressing the solution of Eq. (4.2) in terms of the Lagrangian quantities $\boldsymbol{x}$ and W introduced by Eqs. (2.11) and (2.14) we obtain

$$\Pi(t,\boldsymbol{r}) = W(t,0,\boldsymbol{r})\Pi[0,\boldsymbol{x}(0,\boldsymbol{r})]W^T(t,0,\boldsymbol{r})e^{-2t/\tau}. \tag{4.3}$$

The Lagrangian correlation time at the scale r_* is τ. Therefore at $t \gg \tau$ the eigenvalues of W are strongly separated so that Π is uniaxial,

$$\Pi_{ij} = n_i n_j \,\mathrm{tr}\,\Pi, \tag{4.4}$$

where $\boldsymbol{n}$ is a unit vector. Then Eq. (4.3) gives

$$2\rho_1 = \frac{2t}{\tau} + \ln[\mathrm{Tr}\,\Pi(t,\boldsymbol{r})] - \ln[\mathrm{Tr}\,\Pi(0,\boldsymbol{x}(0,\boldsymbol{r}))]. \tag{4.5}$$

The stationarity of Π implies that $\rho_1 - t/\tau$ has a stationary distribution. In particular, we conclude that the principal Lyapunov exponent λ_1 of the flow is equal to $1/\tau$ exactly. The stationarity of $\rho_1 - t/\tau$ is very different from the situation for the Newtonian fluids, described by Eq. (3.5). The reason is the anticorrelations in the temporal dynamics of σ due to its interaction with Π, which were qualitatively described in the beginning of the section. They lead to vanishing dispersion Δ of $\rho_1 - t/\tau$, $\Delta = \int dt \langle\langle \tilde{\sigma}_{11}(t)\tilde{\sigma}_{11}(0)\rangle\rangle$, which is nonzero for Newtonian turbulence.

Averaging Eq. (4.2) one can obtain the exact relation

$$\langle \Pi_{ik}(\boldsymbol{r})\nabla_k v_i(\boldsymbol{r})\rangle = \frac{\langle \mathrm{Tr}\Pi\rangle}{\tau}. \tag{4.6}$$

Consider now $\langle \Pi_{ik}(\boldsymbol{r})\nabla_k v_i(\boldsymbol{r}')\rangle$ as a function of the separation $l = |\boldsymbol{r}-\boldsymbol{r}'|$. Its value at $l=0$ is given by Eq. (4.6) and can be shown to be much larger than the value at the pumping scale, $l \sim L$. Indeed, consider the correlation function averaged over a ball of size L centered at $\boldsymbol{r}$, i.e., $\langle \Pi_{ik}(\boldsymbol{r})\int_V d\boldsymbol{r}'\nabla_k v_i(\boldsymbol{r}')\rangle/V$. The velocity gradient averaged over the scale L is determined by the external forces. Using Eq. (4.1) one can estimate the value of the averaged correlation function as $\epsilon V\tau/L \ll \epsilon$. It follows that $\langle \Pi_{ik}(\boldsymbol{r})\nabla_k v_i(\boldsymbol{r}')\rangle$ decays at scales larger than r_*. Below r_* the fluctuations of Π and ∇v are strongly correlated. The decay of the correlation function at $r_* < r < L$ can be used to derive Kolmogorov's four-fifths law [25] at these scales. The latter states that the third order longitudinal structure function is equal to $-4\epsilon l/5$ in the inertial interval. All the above conclusions are in agreement with the general picture presented in the beginning of the section.

Expression (4.1) gives the typical value of the stress tensor. As we argued above, the fluctuations with $\Pi \gg \epsilon\tau$ relax rapidly due to the back reaction, which leads to a fast decrease of the PDF of Π at $\Pi \gg \epsilon\tau$. On the other hand, the probability to have $\Pi \ll \epsilon\tau$ is also small. The rough details of the behavior of the PDF can be understood on the basis of a simple model presented in Appendix B. The solution of the model shows that the PDF of Π has an exponential tail at large values of Π and power-law behavior at small values of Π. We believe that a similar qualitative behavior is realized for the stress described by Eqs. (2.8) and (4.2). The model also explicitly demonstrates the finite value of the Lagrangian correlation time of Π and ∇v. This property holds despite a strong modification of the Lagrangian dynamics due to the back reaction.

Note, that the concentration of the polymer molecules, n, does not enter the system of equations (2.8) and (4.2). Therefore the dynamics of polymer solutions with different values of n will be identical in the strong back reaction regime. Moreover, using the equation $\partial_t n + (\boldsymbol{v}\cdot\nabla)n = 0$ for the concentration, it is possible to show that Eq. (4.2) is also valid for n being inhomogeneous in space. Thus the hydrodynamic properties of spatially inhomogeneous solutions do not differ from the homogeneous ones. This assertion holds if local n is large enough for $\Pi_{\max}(n)$ to be larger than the local value of Π prescribed by the dynamics.

The uniaxial form (4.4) of the tensor Π allows one to rewrite Eqs. (2.8) and (4.2) in the form similar to the equations of the magnetic hydrodynamics. The field $\hat{\boldsymbol{n}}\sqrt{\mathrm{Tr}\,\Pi}$ satisfies the induction equation with linear damping. In addition one can show that the field is solenoidal. This analogy helps understand the dynamics of fluctuations at the scales $r \ll r_*$, which occur on the background of the relatively slow stresses excited at $r \sim r_*$. These small-scale fluctuations are elastic waves similar to the Alfven waves propagating in the presence of a large-scale magnetic field in magnetic hydrodynamics [26,27]. The dispersion relation of the waves is $\omega = (\boldsymbol{k}\cdot\hat{\boldsymbol{n}})\sqrt{\mathrm{Tr}\,\Pi}$. Thus the velocity of these waves is $\sqrt{\mathrm{Tr}\,\Pi}$, which can be estimated as $\sqrt{\epsilon\tau}$. There exist two mechanisms of the elastic waves attenuation: polymer relaxation and viscous dissipation. The first mechanism leads to the scale-independent attenuation τ^{-1}, which is smaller than the frequency, at $kr_* \gg 1$. The second mechanism produces the attenuation $\sim \nu k^2$, which is much smaller than the frequency for $k\eta_* \ll 1$ where $\eta_* = \nu(\epsilon\tau)^{-1/2}$. Thus the elastic waves are well defined in the interval $r_*^{-1} \lesssim k \lesssim \eta_*^{-1}$. This interval can be called the elastic dissipation range.

Our equations are valid as long as $\Pi \ll \Pi_{\max}$. The relation (4.1) allows us to reformulate this condition as $\epsilon \ll \Pi_{\max}/\tau$. Another limitation of our scheme is related to the inequality $R \ll r_*$, under which the flow is smooth at the scale R. Using Eqs. (2.5), (2.7), and (4.1) one can write the estimate $R^2 \sim \varrho\epsilon\tau(K_0 n)^{-1}$ for the typical size of a polymer molecule, R.

Let us estimate the parameters introduced above within the framework of Kolmogorov's theory (K41) [28]. Though the theory is, rigorously, incorrect [25], it is satisfactory for rough estimates. The characteristic velocity difference $\delta_r v$ between two points separated by the distance r from the inertial interval is given in K41 by $(\epsilon r)^{1/3}$, where ϵ is the energy input. Writing $|\nabla v| \sim \delta_r v/r \sim \epsilon^{1/3} r^{-2/3}$, one finds $r_* \sim \sqrt{\epsilon\tau^3}$. In the K41 theory the condition $R \ll r_*$ can be rewritten as $\varrho(K_0 n\tau^2)^{-1} \ll 1$. Note that in the framework of K41 theory the ratio R/r_* is independent of the Reynolds number.

Our analysis assumes that the characteristic size of the molecules, R, is much smaller than their maximal size $R_{\max}$. As Re increases, the typical elongations eventually become of the order of $R_{\max}$, and further elongation becomes impossible. In this case the molecules behave as hard rods, modifying the effective viscosity of the fluid [29]. Therefore at large enough Re we return to the case of Newtonian fluid. However, this regime is expected to be unstable because polymer molecules are intensively destroyed by strong flows.

We have shown that in the steady state the velocity gradients in the bulk do not exceed τ^{-1}. Consider now the situation where the boundary forces tend to produce gradients larger than $1/\tau$ at $r \sim L$. Then the elastic reaction should lead to formation of a boundary layer where the value of velocity gradient diminishes from the value imposed by the forcing to the value τ^{-1} in the bulk. Then $r_* \sim L$, i.e., the inertial range and energy cascade are absent. This situation is similar to the elastic turbulence regime [1].

Finally, let us consider the role of other modes of the polymer molecules relaxation. They are characterized by the relaxation times $\tau_i < \tau$. We have shown that the interaction of the fluid with the principal relaxation mode fixes the value of the principal Lyapunov exponent at $\lambda_1 = \tau^{-1}$. The inequality $\lambda_1\tau_i < 1$ then implies that other modes are always only weakly excited by the flow, so the interaction is fully determined by the softest relaxation mode. We conclude that Eqs. (2.8) and (4.2), based on the single relaxation mode approximation correctly describe the solution hydrodynamics above the coil-stretch transition.

V. CONCLUSION

We have examined properties of turbulence in dilute polymer solutions. Our results support the theory of Lumley [8] who argued the existence of the coil-stretch transition in turbulent flows, which occurs at a critical Reynolds number, $\mathrm{Re_c}$. The polymer molecules are typically weakly stretched, so that their elasticity only weakly influences the flow in the regime realized below the transition, at $\mathrm{Re} < \mathrm{Re_c}$. At $\mathrm{Re} > \mathrm{Re_c}$ the polymer molecules are substantially stretched and strongly modify the small-scale flow.

At $\mathrm{Re} < \mathrm{Re_c}$ the polymer molecules are passively advected and stretched by the flow. This regime occurs under the condition $\lambda_1\tau < 1$ where τ is the polymer relaxation time and λ_1 is the principal Lyapunov exponent. The Lyapunov exponent is defined as the logarithmic rate of the divergence of nearby Lagrangian trajectories and can be estimated as the inverse turnover time at the viscous scale of turbulence. The majority of the molecules in this regime fluctuates near the equilibrium. There also exists a small number of strongly elongated molecules, which appear due to rare large fluctuations in the rate of strain. Even though the number of substantially elongated molecules is small, they may be relevant in some situations due to the relatively slow power-law decrease of the probability density function of elongations of molecules Eq. (3.12).

In the second regime, at $\mathrm{Re} > \mathrm{Re_c}$, most of the molecules are substantially elongated. It leads to a strong interaction between the elasticity and the flow, which modifies the flow below the scale r_*. At $r \gtrsim r_*$ the properties of turbulence are the same as in Newtonian fluids. The energy cascades downscales from the pumping scale and dissipates due to polymer relaxation at $r \sim r_*$. The scale can be considered as a new dissipation scale. The flow is smooth at $r \lesssim r_*$ with the Lyapunov exponent λ_1 fixed at the value $1/\tau$ by the interaction.

The smoothness of the flow at $r \lesssim r_*$ leads to the conclusion that the velocity spectrum $E(k)$ decreases faster than k^{-3} at $kr_* \gtrsim 1$. The precise form of $E(k)$ in this interval is related to the elastic waves propagating at these scales. As both spectral transfer time and the decay time are scale independent one can expect a power-law spectrum.

The properties of the polymer statistics near $\mathrm{Re_c}$ were examined numerically by Kronjäger and Eckhardt [30] in the framework of Eqs. (2.6) and (2.8). The results indicate the power PDF tail for the polymer elongations at $\mathrm{Re} < \mathrm{Re_c}$ and a substantial modification of the PDF at $\mathrm{Re} > \mathrm{Re_c}$, in agreement with our results.

Let us discuss implications of our results for the drag reduction. A description of the experimental situation can be found in the works [2–4]. It has been observed that the onset of the drag reduction at increasing Re depends on the concentration of the polymer molecules, whereas asymptotically the friction force falls on a curve, which is independent of the concentration. This curve is usually referred to as the MDR (maximum drag reduction) asymptote. A discussion of the MDR can be found in the recent work [31]. A natural explanation of the n independence of the MDR asymptote can be formulated in the framework of Eqs. (2.8) and (4.2) describing the strong back reaction regime. Indeed, the system contains no n-dependent parameters. The n-dependence of the onset can also be explained in our scheme. The drag force is formed in the boundary layer, which has a complicated structure [32]. Whereas gradients of the average velocity grow toward a wall, the amplitude of the velocity fluctuations decreases. Therefore one can expect that the polymer molecules are strongly extended in bulk and weakly extended near the walls. Then the structure of the boundary layer will be sensitive to the polymer concentration n. The situation corresponds to the transient regime (which is sensitive to the polymer concentration) from the Newtonian behavior to the MDR asymptote. The asymptote itself corresponds to the case when the polymer molecules are strongly extended everywhere.

A striking property of polymer solutions is the so-called elastic turbulence regime discovered by Groisman and Steinberg [1]. It is a chaotic state that is realized at small Reynolds numbers Re. Its existence is made possible by the large value of Weissenberg number $\mathrm{Wi}=\tau V/L$, which implies a strong nonlinearity of the system. This state can also be investigated in the framework of our scheme. The results will be published elsewhere.

Let us give numerical values of parameters appearing in our theory for a typical experimental arrangement. For the number of monomers 10^6-10^7 one has $R_0\sim 10^{-5}$ cm, $R_{\mathrm{max}}\sim 10^{-2}$ cm, and $\tau\sim 10^{-2}-10^{-1}$ s. Then using $n_{\mathrm{c}}\sim(R_{\mathrm{max}}^2 R_0)^{-1}$ one can obtain 0.1 ppm for the concentration n_{c} below which polymers have no effect on the flow. Let us assume that the polymer concentration is 10 ppm, the integral length is $L\sim 10$ cm, and take the water viscosity, $\nu\sim 10^{-2}$ cm^2/s. Then the critical Reynolds number $\mathrm{Re}_{\mathrm{c}}\sim[L^2/(\nu\tau)]^{2/3}$ is of the order of 10^4. Above the coil-stretch transition the characteristic size of polymers is given by $R\sim R_0\sqrt{\mathrm{Re}^3\varrho\tau\nu^3/(k_{\mathrm{B}}TnL^4)}\sim 10^{-5}R_0\mathrm{Re}^{3/2}$. We obtain that $R\sim 10^2 R_0$ in the vicinity of the transition, which is in agreement with the assumption $R_0\ll R\ll R_{\mathrm{max}}$. Using Kolmogorov's estimate $\eta\sim L\mathrm{Re}^{-3/4}$ we find that at the transition $\eta\sim 10^{-2}$ cm, which is of the order of R_{max}. Thus the assumption $R\ll\eta$ is satisfied. These estimates seem to fit the existing experimental data.

ACKNOWLEDGMENTS

We thank M. Chertkov, B. Eckhardt, G. Falkovich, A. Groisman, I. Kolokolov, Y. Rabin, V. Steinberg, and M. Stepanov for valuable discussions. V.L. acknowledges the grants of the Israel Science Foundation and the Minerva Foundation. A.F. acknowledges the support of Arc-en-Ciel. E.B. acknowledges the support by the National Science Foundation under Grants Nos. 9971332, 9808595, and 0094569.

APPENDIX A

Here we consider the statistical properties of the integral

$$I=\int_0^\infty dt\,\exp\left[\int_0^t dt'\ \xi(t')\right], \tag{A1}$$

where $\xi(t)$ is a random process with a finite correlation time τ_ξ and a negative average $\xi_0<0$. For this purpose it is convenient to introduce the auxiliary object

$$I(t)=\int_t^\infty dt'\exp\left[\int_t^{t'} dt''\xi(t'')\right].$$

Due to the stationarity of ξ, the statistics of $I(t)$, it is independent of t. Separating the integration interval one finds the relation

$$I(t-\delta t)=I(t)\exp\left[\int_{t-\delta t}^{t}\xi(t')dt'\right]+\int_{t-\delta t}^{t}dt'\exp\left[\int_{t-\delta t}^{t'}\xi(t'')dt''\right].$$

It follows that

$$\ln I(t-\delta t)=\ln I(t)+\int_{t-\delta t}^{t}\xi(t')dt'+\ln\left\{1+\frac{\int_{t-\delta t}^{t}dt'\exp\left[\int_{t-\delta t}^{t'}\xi(t'')dt''\right]}{I(t)\exp\left[\int_{t-\delta t}^{t}\xi(t')dt'\right]}\right\}. \tag{A2}$$

If $I(t)$ is sufficiently large, it is possible to neglect the last term on the left-hand side. The exact condition is formulated below. Observe that $\ln I(t)$ depends on the values of the noise at times larger than t so that the second term is independent of the first provided $\delta t\gg\tau_\xi$. Therefore the probability distribution function $P(z)$ of $z(t-\delta t)\equiv\ln I(t-\delta t)$ is given by the convolution of the distributions of $z(t)$ [which is also equal to $P(z)$] and $\int_{t-\delta t}^{t}\xi(t')dt'$. The latter has a probability function similar to Eq. (3.5). We thus obtain the integral equation

$$P(z)=\int\frac{dz'}{\sqrt{2\pi t\Delta}}P(z')\exp\left[-\delta t S_\xi\left(\frac{z-z'-\xi_0\delta t}{\delta t}\right)\right],$$

where $\xi_0=\langle\xi\rangle$ and S_ξ is the entropy function characterizing $\xi(t)$. Since the kernel of the integral operator depends on the difference $z-z'$ only, the solution of this equation is $P(z)\propto\exp[-\alpha z]$. We obtain the following expression for the tail of the PDF of $I\equiv e^z$

$$P(I)\sim I^{-\alpha-1}. \tag{A3}$$

Here α is determined from the condition

$$\int\frac{dx}{\sqrt{2\pi t\Delta}}\exp\left[\alpha x-\delta t S_\xi\left(\frac{x-\xi_0\delta t}{\delta t}\right)\right]=1. \tag{A4}$$

The solution $\alpha=0$ should be rejected. This integral can be evaluated by the saddle-point method so that its value is determined by the maximum of the exponent. Taking its value at $x=\xi_0\delta t$ we conclude that in order to satisfy condition (A4), α and ξ_0 must have different signs. Thus if $\xi_0<0$, then $\alpha>0$ and the normalization integral for PDF (A3) converges at $I\to\infty$. On the other hand, if $\xi_0>0$, there exists no well-defined distribution of I.

The two equations that implicitly define α are given by the saddle-point condition

$$\alpha=S'_\xi(\beta-\xi_0),$$

where β is the saddle-point value of $(z-z')/\delta t$, and the condition

$$S_\xi(\beta-\xi_0)-\beta S'_\xi(\beta-\xi_0)=0,$$

which follows from the condition that the integral (A4) is equal to 1. One should reject the formal solution $\beta=\xi_0$ of these equations corresponding to $\alpha=0$. It is easy to see that β is positive together with α. Now we may formulate the condition for the applicability of the power tail. It is valid provided the third term in Eq. (A2) is indeed much smaller than the second for those $I(t)$ that determine the PDF of $I(t-\delta t)$. From $z-z'=\beta\delta t$ it follows that $I(t)=I(t-\delta t)\exp[-\beta\delta t]$, so that we arrive at the condition $I\gg\exp(\beta\delta t)/(\beta^2\delta t)$ (ξ is estimated as β). The increment δt is constrained by the condition $\delta t\gg\tau_\xi$. There are two cases to be considered. If $\beta^{-1}\gg\tau_\xi$ one can use the choice minimizing the above ratio $\delta t\sim\beta^{-1}$, so that Eq. (A3) is valid for $I\gg\beta^{-1}$. In the opposite case $\beta^{-1}\ll\tau_\xi$ the power tail is valid for $I\gg\exp[\beta\tau_\xi]/(\beta^2\tau_\xi)$.

At small ξ_0 one can use quadratic expansion for $S_\xi(x)\approx x^2/(2\Delta)$, which gives

$$\beta=-\xi_0,\quad \alpha=-\frac{2}{\Delta}\xi_0. \tag{A5}$$

The entropy function becomes quadratic in the limit $\tau_\xi\to0$. Thus the expression (A5) is valid for any ξ_0 in the case of a short-correlated process ξ (cf. [23]).

APPENDIX B: MODEL OF THE BACK REACTION

Let us introduce a simple model that captures the most robust features of the interaction between elastic and inertial degrees of freedom. The model is formulated in terms of the system of equations for two variables σ and x. The equations are

$$\frac{dx}{dt}=\sigma x+x_0,\quad \sigma=-x+\xi, \tag{B1}$$

$$\langle\xi\rangle=a,\quad \langle\langle\xi(t_1)\xi(t_2)\rangle\rangle=2\delta(t_1-t_2), \tag{B2}$$

where double brackets denote the irreducible part of the correlation function. The variable σ models the rate of strain subtracted by $1/\tau$ and x models the elastic stress. The time derivative in Eq. (B1) represents the full derivative $\partial_t+\boldsymbol{v}\cdot\boldsymbol{\nabla}$, i.e., we consider the Lagrangian dynamics. The product σx represents the combination on the right-hand side of Eq. (4.2) and x_0 stands for A_0 in Eq. (2.6). The second equation in system (B1) represents the Navier-Stokes equation (2.8). Since we consider dynamics at the scale r_*, all the spatial derivatives can be estimated as $1/r_*$. The term $-x$ describes the back reaction and ξ models the influence of larger scales, exciting the motion at $r\sim r_*$. The average $a\equiv\langle\xi\rangle$ is negative below the coil-stretch transition and positive above. If $a>0$ the term x_0 on the right-hand side of the first equation in Eq. (B1) can be disregarded.

Starting from the system of equations (B1) and (B2) one can derive the Fokker-Planck equation for the PDF of x

$$\partial_t\mathcal{P}=\partial_x[x\partial_x(x\mathcal{P})]-\partial_x[\{x_0+(a-x)x\}\mathcal{P}]. \tag{B3}$$

The normalized stationary solution of Eq. (B3) is

$$\mathcal{P}_0(x)=\frac{1}{Z}x^{a-1}\exp\left(-x-\frac{x_0}{x}\right), \tag{B4}$$

where $Z=2x_0^{a/2}K_a(2\sqrt{x_0})$ is the normalization factor. Here K_a is the MacDonald function.

At $a<0$, which corresponds to systems (2.6)–(2.8) below the transition, the properties of σ are only insignificantly modified by the interaction with the variable x. For example, in the limit $x_0\ll1$ one finds that $\langle\sigma\rangle=a$, which is the same value as without the back reaction. However, the back reaction is important for rare events when ξ is large. The interaction leads to the exponentially decaying tail, which makes all the moments of x finite. This corresponds to the picture presented in the main body of the text. Note that the power tail is universal, i.e., force-independent, whereas the exponential tail is an artifact of a zero correlation time of ξ [18].

In the case $a>0$, i.e., above the transition, the limit $x_0\to0$ is regular. One obtains

$$\mathcal{P}_0(x)=\frac{x^{a-1}\exp(-x)}{\Gamma(a)}, \tag{B5}$$

where $\Gamma(x)$ is the Euler Γ function. We observe that all the positive moments of x exist because the back reaction stops the growth of x. The average value of x is given by $\langle x\rangle=a$ so that $\langle\sigma\rangle=0$. These facts correspond to the statements $\langle\Pi\rangle=\epsilon\tau$ and $\lambda_1=\tau^{-1}$ from Sec. IV.

Let us now investigate nonsimultaneous correlation functions of x above the transition, i.e., when $a>0$. Then we can assume $x_0=0$. We need the Green function $\mathcal{G}(t,x,y)$ of Eq. (B3), which satisfies

$$\partial_t\mathcal{G}-\partial_x[x\partial_x(x\mathcal{G})]+\partial_x[(a-x)x\mathcal{G}]=\delta(t)\delta(x-y), \tag{B6}$$

with the condition $\mathcal{G}(t<0)=0$. The Green function should be regular at $x=0$ and decrease faster than any power of x at $x\to\infty$. Using $\mathcal{G}$ one can find nonsimultaneous correlation functions of x in the steady state,

$$\langle f_1[x(t)]f_2[x(0)]\rangle=\int dx\,dy\,\mathcal{P}_0(y)\mathcal{G}(t,x,y)f_1(x)f_2(y),$$

where f_1 and f_2 are arbitrary functions and $\mathcal{P}_0(x)$ is defined by Eq. (B5).

The Laplace transform of $\mathcal{G}(t,x,y)$ is

$$G(\lambda,x,y)\equiv\int_0^\infty dt\,\exp(-\lambda t)\mathcal{G}(t,x,y). \tag{B7}$$

It satisfies the equation

$$\lambda G=\partial_x[x\partial_x(xG)]-\partial_x[(a-x)xG]+\delta(x-y), \tag{B8}$$

following from Eqs. (B6) and (B7). The solution of Eq. (B8) can be expressed in terms of two independent solutions of the homogeneous equation

$$\lambda G_{1,2}=\partial_x[x\partial_x(xG_{1,2})]-\partial_x[(a-x)xG_{1,2}]. \tag{B9}$$

Two independent solutions of Eq. (B9) are

$$G_1=x^{k_1}e^{-x}F(-k_2-1,k_1-k_2+1,x), \tag{B10}$$

$$G_2=x^{k_2}e^{-x}\Psi(-k_1-1,k_2-k_1+1,x), \tag{B11}$$

where

$$\Psi(\alpha,\beta,x)=\frac{\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)}F(\alpha,\beta,x)+\frac{\Gamma(\beta-1)}{\Gamma(\alpha)}x^{1-\beta}\times F(\alpha-\beta+1,2-\beta,x). \tag{B12}$$

and $F(\alpha,\beta,x)$ is the confluent hypergeometric function [33]. The functions G_1 and G_2 satisfy the boundary conditions at $x=0$ and $x\to\infty$ correspondingly. The parameters $k_{1,2}$ are

$$k_{1,2}=\frac{a-2\pm\sqrt{a^2+4\lambda}}{2}. \tag{B13}$$

Matching functions (B10) and (B11) at $x=y$ one obtains the Green's function

$$G(\lambda,x,y)=\frac{y^{1-a}e^y}{\sqrt{a^2+4\lambda}}\frac{\Gamma(-1-k_2)}{\Gamma(k_1-k_2)}[\theta(y-x)G_1(x)G_2(y)+\theta(x-y)G_1(y)G_2(x)], \tag{B14}$$

where $\theta(x)$ is the Heaviside step function.

Expression (B14) allows one to establish analytical properties of G as a function of λ. The function is analytic in the half plane $\mathrm{Re}\,\lambda>0$. There is a branch point at $\lambda=-a^2/4$ with the cut going along the axis $\mathrm{Im}\,\lambda=0$ from the branch point to $-\infty$. In addition to the branch point there also exist poles located at $\lambda=n(n-a)$, where n is an integer number (including zero), such that $n<a/2$. Thus the poles lie between the origin and the branch point. One can easily find the pole contribution to G near $\lambda=0$ corresponding to $n=0$. Using Eq. (B14) we obtain

$$G(\lambda,x,y)=\frac{x^{a-1}\exp(-x)}{\Gamma(a)\lambda}+\cdots, \tag{B15}$$

where dots mean terms regular in λ.

The Green function $\mathcal{G}$ is expressed via its Laplace transform as

$$\mathcal{G}(t,x,y)=\int_{A-i\infty}^{A+i\infty}\frac{d\lambda}{2\pi i}\exp(\lambda t)G(\lambda,x,y), \tag{B16}$$

where the integration contour lies on the right of all singularities of $G(\lambda)$. Shifting the integration contour in Eq. (B16) to the left we find that the first term in the right-hand side of Eq. (B15) produces $\mathcal{P}_0(x)$, while additional contribution is given by

$$\mathcal{G}_1(t,x,y)=\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{d\lambda}{2\pi i}\exp(\lambda t)G(\lambda,x,y), \tag{B17}$$

where $\epsilon>0$ is a small number and $\mathcal{G}_1$ is defined by the relation

$$\mathcal{G}_1(t,x,y)=\mathcal{G}(t,x,y)-\mathcal{P}_0(x). \tag{B18}$$

Shifting the integration contour in Eq. (B17) to the left we encounter the branch point $\lambda=-a^2/4$ if $a<2$ or the pole $\lambda=1-a$ if $a>2$. Therefore at large t the t dependence of $\mathcal{G}_1$ is $\exp(-a^2t/4)$ if $a<2$ and $\exp[(1-a)t]$ if $a>2$. Since

$$\langle f_1[x(t)]f_2[x(0)]\rangle=\langle f_1\rangle\langle f_2\rangle+\int dx\,dy\,\mathcal{P}_0(y)\mathcal{G}_1(t,x,y)f_1(x)f_2(y), \tag{B19}$$

the correlations of x decay exponentially in time.

One can write the asymptote of $\mathcal{G}_1(t,x,y)$ at large t explicitly. Let us first consider $a<2$. Then the integration contour in Eq. (B17) can be deformed into a curve going around the cut starting from $\lambda=-a^2/4$. Calculating the jump on the cut we get the expression

$$\mathcal{G}_1(t,x,y)\approx\frac{2\ln(y/x)c_1+c_1^2}{16t^{3/2}\sqrt{\pi}}x^{a/2-1}y^{-a/2}e^{-a^2t/4}, \tag{B20}$$

valid for small x, y. Here $c_1=2\psi(1)-\psi(-a/2)$ and $\psi(z)$ is the logarithmic derivative of the Γ function. At $a>2$ with the exponential accuracy, the function $\mathcal{G}_1(t,x,y)$ is given by the residue at $\lambda=1-a$,

$$\mathcal{G}_1(t,x,y)\approx\frac{(a-1)F(-1,a-1,x)F(-1,a-1,y)}{\Gamma(a-2)y}\times e^{(1-a)t}x^{a-2}e^{-x}.$$

The finite correlation time of σ follows from $\langle\sigma(t)\sigma(0)\rangle=-\partial_t^2\langle\ln x(t)\ln x(0)\rangle$. The expectation value of $x(t)=\exp[\int_0^t\sigma(t')dt']$ grows exponentially with time at times much larger than the correlation time of σ unless $\int dt\langle\sigma(t)\sigma(0)\rangle=0$. Thus the back reaction stops the growth of x and gives rise to this peculiarity of the statistics of σ.

[1] A. Groisman and V. Steinberg, Nature (London) **405**, 53 (2000).

[2] P. S. Virk, AIChE J. **21**, 625 (1975).

[3] W. McComb, *The Physics of Fluid Turbulence* (Claredon, Oxford, 1990).

[4] A. Gyr and H.-W. Bewersdorf, *Drag Reduction in Turbulent Flows by Additives* (Kluwer, London, 1995).

[5] S. R. Quake, H. Babcock, and S. Chu, Nature (London) **388**, 151 (1997); T. T. Perkins, S. R. Quake, D. E. Smith, and S. Chu, Science **264**, 822 (1994).

[6] J. W. Hatfield and S. R. Quake, Phys. Rev. Lett. **82**, 3548 (1999).

[7] J. L. Lumley, J. Polym. Sci., Part D: Macromol. Rev. **7**, 263 (1973).

[8] J. L. Lumley, Symp. Math. **9**, 315 (1972).

[9] E. Balkovsky, A. Fouxon, and V. Lebedev, Phys. Rev. Lett. **84**, 4765 (2000).

[10] H. Furstenberg, Trans. Am. Math. Soc. **108**, 377 (1963).

[11] Ya. B. Zeldovich, A. A. Ruzmaikin, S. A. Molchanov, and D. D. Sokolov, J. Fluid Mech. **144**, 1 (1984).

[12] M. Tabor and P.-G. de Gennes, Europhys. Lett. **2**, 519 (1986); P.-G. de Gennes, Physica A **140**, 9 (1986).

[13] E. J. Hinch, Phys. Fluids **20**, S22 (1977).

[14] C. Kittel, *Elementary Statistical Physics* (Wiley, New York, 1967).

[15] R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, 2nd ed. (Wiley, New York, 1987), Vol. 2.

[16] M. Chertkov, Phys. Rev. Lett. **84**, 4761 (2000).

[17] S. B. Pope, Annu. Rev. Fluid Mech. **26**, 23 (1994).

[18] E. Balkovsky and A. Fouxon, Phys. Rev. E **60**, 4164 (1999).

[19] I. Goldhirsch, P-L. Sulem, and S. A. Orszag, Physica D **27**, 311 (1987).

[20] L. D. Landau and E. M. Lifshitz, *Statistical Physics, Part 1* (Pergamon Press, New York, 1980).

[21] M. Chertkov, G. Falkovich, I. Kolokolov, and M. Vergassola, Phys. Rev. Lett. **83**, 4065 (1999).

[22] R. Ellis, *Entropy, Large Deviations and Statistical Mechanics* (Springer-Verlag, Berlin, 1985).

[23] M. Chertkov, I. Kolokolov, and M. Vergassola, Phys. Rev. E **56**, 5483 (1997).

[24] M. Chertkov, Phys. Fluids **10**, 3017 (1998).

[25] U. Frisch, *Turbulence: the Legacy of A. N. Kolmogorov* (Cambridge University Press, New York, 1995) Chap. 8.

[26] R. Kraichnan, Phys. Fluids **8**, 1385 (1965).

[27] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, England, 1978).

[28] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 9 (1941).

[29] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, England, 1967).

[30] J. Kronjäger, Diploma Thesis, Marburg, 2001.

[31] K. R. Sreenivasan and C. M. White, J. Fluid Mech. **409**, 149 (2000).

[32] J. L. Lumley, Annu. Rev. Fluid Mech. **1**, 367 (1969).

[33] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach, Glasgow, 1990), Vol 3.