Correlation ratchets: Four current reversals and disjunct "windows"

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Multinoise correlation ratchets with a simple sawtooth potential are considered. It is proved that in the case of *symmetric* nonequilibrium three-level Markovian noise the direction and value of the induced current can be controlled by thermal noise. Moreover, it is established that four current reversals (CRs) occur and that for the CRs there exist characteristic disjunct "windows" in temperature and switching rate as control parameters. The necessary and sufficient conditions for the existence of the above effects are given and can be used in particle separation techniques.

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Directed motion of Brownian particles induced by nonequilibrium fluctuations, with no macroscopic driving applied, in spatially periodic structures called ratchets is currently being actively investigated (for a review, see [1]). This field of research was stimulated by cell biology, where the probable mechanism of vesicle transport inside eukaryotic cells was found to be the motion of motor proteins along microtubules modeled as ratchets [1,2]. Beyond that, it was suggested that the ratchet mechanism be used for obtaining efficient separation methods of nanoscale objects, e.g., DNA molecules, proteins, viruses, cells, etc. [3,4]. To date, the feasibility of particle transport by man-made devices has been experimentally demonstrated for several ratchet types [3,5].

It is of importance that two noises acting together can generate a far more organized motion than either of them alone [6], even though the noise sources are statistically independent, and can cause intricate effects, e.g., multiple current reversals (CRs) and multipeaked characteristics [1,7–11]. The fact that CRs lead to a more efficient fluctuation-induced separation of particles makes the models with CRs very promising. It has been shown that the CR effect is attainable in various ways, including modification of the correlation time of nonequilibrium fluctuations, the flatness parameter of the noise [12–15], the power spectrum of the noise source [16], the number of interacting particles per unit cell [17], the mass of the particles [18], the temperature in multinoise cases [7], and the shape of the potential [8,10]. Most of these effects having been established by numerical methods or at the limits of slow and fast noises, there are not many exact results available for correlation ratchets, which would enable one to quantitatively evaluate the values of the noise parameters corresponding to CRs in particular models or to obtain the sufficient and necessary conditions for the existence of CRs [1,10,13–16,19]. It is especially difficult to treat the multinoise case analytically. The advantage of multinoise models involving a thermal noise is that the temperature as the control parameter can be easily varied both in experiments and in potential technological applications.

In this article we consider an overdamped multinoise ratchet where particles move in a one-dimensional spatially periodic piecewise linear potential which has one maximum per period and is characterized by an asymmetry parameter d. The applied noise is additive and consists of thermal noise with temperature D, and a symmetric three-level telegraph process (trichotomous noise) with amplitude a_0 and switching rate ν. Considering the case when the telegraph noise is very flat, we establish on the basis of an exact expression for the current $J=J(d,D,a_0,\nu)$ a number of cooperation effects. (i) The direction of the current can be controlled by means of thermal noise also in the case when it is induced by *symmetric* trichotomous noise. (ii) For certain system parameters there occur four CRs. To our knowledge, more than two CRs have never been reported for correlation ratchets with a simple sawtooth potential. (At the same time, in the case of rocking ratchets, infinitely many CRs may occur [20].) (iii) At large spatial asymmetries the current exhibits characteristic disjunct "windows"(DWs) (see Fig. 1) of temperature and switching rate where the direction of current is opposite to that in the surroundings. (iv) In addition, the necessary and sufficient conditions for the existence of the 4CR and DW effects are given.

Overdamped motion of Brownian particles is described by the dimensionless Langevin equation

$$\frac{dX}{dt}=h(X)+\xi(t)+Z(t), \quad h(x)\equiv-\frac{dV(x)}{dx}, \tag{1}$$

where $V(x)=\tilde{V}(\tilde{x})/\tilde{V}_0$, $\tilde{V}(\tilde{x})$ is a spatial potential with period $\tilde{L}$, and $\tilde{V}_0=\tilde{V}_{\max}-\tilde{V}_{\min}$ is the barrier height. The usual physical variables are indicated by tildes and the space and time coordinates read $X=\tilde{X}/\tilde{L}$ and $t=\tilde{t}\tilde{V}_0/\kappa\tilde{L}^2$ with κ being the friction coefficient.

The thermal noise satisfies $\langle\xi(t)\rangle=0$ and $\langle\xi(t_1)\xi(t_2)\rangle=2D\delta(t_1-t_2)$ with $D=k_BT/\tilde{V}_0$. For brevity, in what follows we shall call D just the temperature. As concerns the random force $Z(t)$, we assume it to be a zero-mean trichotomous Markovian stochastic process [14,15,21], which consists of jumps among three values $z=\{a_0,0,-a_0\},a_0>0$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities $P_s(a_0)=P_s(-a_0)=q$ and $P_s(0)=(1-2q)$. In a stationary state the fluctuation process satisfies $\langle Z(t)\rangle=0$ and $\langle Z(t+\tau)Z(t)\rangle=2qa_0^2\exp(-\nu\tau)$, where $a_0=\tilde{L}\tilde{a}_0/\tilde{V}_0$ and the

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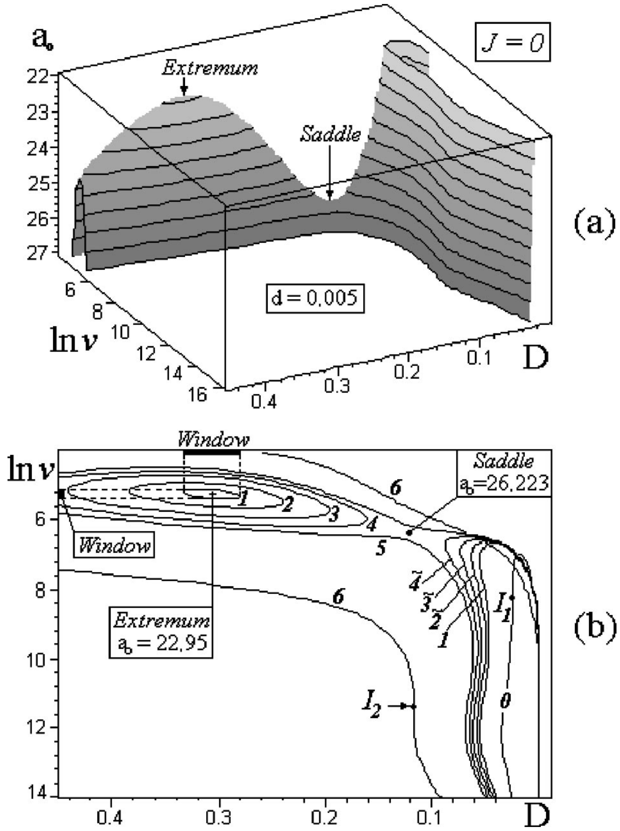

FIG. 1. (a) The surface of CRs $J(D,a_0,\nu)=0$ for a fixed asymmetry parameter $d=0.005$. (b) The projection of the surface onto the plane (D,ν). The level curves correspond to the following values of the noise amplitude: (0) $a_0=14.356$, (1) $a_0=23.000$, (2) $a_0=23.500$, $(\tilde{2})$ $a_0=23.875$, (3) $a_0=24.500$, $(\tilde{3})$ $a_0=24.750$, (4) $a_0=25.500$, $(\tilde{4})$ $a_0=25.625$, (5) $a_0=26.500$, (6) $a_0=42.294$.

switching rate $\nu=\kappa\tilde{L}^2\tilde{\nu}/\tilde{V}_0$ is the reciprocal of the noise correlation time $\tau_c=1/\nu$, i.e., $Z(t)$ is a symmetric zero-mean exponentially correlated noise. The trichotomous process is a particular case of the kangaroo process [13] with flatness parameter $\varphi=\langle Z^4(t)\rangle/\langle Z^2(t)\rangle^2=1/(2q)$.

The master equation corresponding to Eq. (1) reads

$$\frac{\partial}{\partial t}P_n(x,t)=-\Gamma P_n(x,t)+\sum_m U_{nm}P_m(x,t), \qquad (2)$$

where $\Gamma=\partial_x[h(x)+z_n-D\partial_x]$ and $P_n(x,t)$ is the probability density for the combined process (x,z_n,t); $n,m=1,2,3$; $z_1\equiv-a_0$, $z_2\equiv 0$, $z_3\equiv a_0$, and $U_{ik}=\nu[q+(1-3q)\delta_{i2}(\delta_{1k}+\delta_{2k}+\delta_{3k})-\delta_{ik}]$. The stationary current $J=\Sigma_n j_n(x)$ is then evaluated via the current densities $j_n(x)=[h(x)+z_n-Dd_x]P_n^s(x)$, where $P_n^s(x)$ is the stationary probability density for the state (x,z_n). To find the stationary probability density in the x space $P(x)=\Sigma_n P_n^s(x)$ and the stationary particle current $J=\text{const}$, we can derive from Eq. (2) a fifth-order ordinary differential equation, which has a unique solution if we impose on it the conditions of periodicity $P(x)=P(x+1)$ and normalization $\int_0^1 P(x)dx=1$ over the rescaled period interval $L=1$ of the ratchet potential $V(x)$.

To derive an exact formula for J, we assume that the potential $V(x)=V(x-1)$ in Eq. (1) is piecewise linear (sawtoothlike) and its asymmetry is determined by a parameter $d\in(0,1)$, with $V(x)$ being symmetric when $d=1/2$. The force corresponding to our potential is $h(x):=1/d$ for $x\in(0,d)$ and $h(x):=-1/(1-d)$ for $x\in(d,1)$. Under these assumptions, a complex exact formula as a quotient of two 11th-order determinants can be derived for the probability current J. To obtain a more manageable explicit formula, one can assume that the flatness parameter is large, $\varphi\gg 1$, and expand the current as $J=qJ^{(1)}+q^2J^{(2)}+\cdots$. An exact but still complex formula for the leading order term $qJ^{(1)}$ was derived by one of the present authors in Ref. [15] where its asymptotic limits were also studied. As it is quite difficult to carry on the survey by analytic methods, we will here apply numerical calculations to analyze the intermediate regimes. (In what follows we shall write J for $J^{(1)}$.)

Next we will discuss the 4CR effect. Figure 1(b) exhibits the zeros of the current $J=J(D,\nu)=0$ for $d=0.005$ at different values of a_0 where the level curves $a_0=\text{const}$ may be considered as functions $D=D(\nu;d,a_0)$ of ν with d and a_0 being parameters. Here two types of level curve are distinguishable. These are, on the one hand, the connected ones and, on the other, the ones comprising two components, viz., a closed curve and a two-branched ''fork.'' Both branches of the fork approach zero as ν grows. Regarding the branch on the right, if $a_0<d^{-1}(1-d)^{-1}$, then D becomes zero only at the limit $\nu\to\infty$, whereas if $a_0>d^{-1}(1-d)^{-1}$, then D has a zero at a finite ν. In what follows we are concerned with the branch of the fork on the left. The four CRs vs ν effect exists if and only if the function $D=D(\nu;d,a_0)$ has a local minimum. By gradually varying a_0 and d we will obtain the surprising result that the region of existence of the 4CR effect shrinks to a four-point C, which has the following coordinates: $d_C\approx 0.044\,75$, $a_C\approx 6.737\,98$, $D_C\approx 0.169\,21$ and $\nu_C\approx 755$. The values d_C and D_C are the upper values of the parameters for the 4CR effect to occur while a_C is the relevant lower value, i.e., the 4CR effect is possible if $d\in(0,d_C)$, $a_0\in(a_C,\infty)$, and $D\in(0,D_C)$ (see Fig. 2).

The necessary conditions for the existence of the 4CR effect are shown in Fig. 2(a) by the shaded regions in the planes (d,D) and (d,a_0), and in Fig. 2(b) by the slightly shaded region in the plane (D,a_0), while the intensely shadowed narrow wedge-shaped areas in Fig. 2(b) fix the values of d, D, and a_0 that are necessary and sufficient for the existence of the 4CR effect. To illustrate the geometrical meaning of the end points of the wedges in the phase space, let us consider the case $d=0.005$. The boundary points I_1 and I_2 of the necessary regions in Fig. 2 correspond to the values of a_0 at which the local minimum of the function $D(\nu;d,a_0)$ disappears (see Fig. 1) and the current J has one threefold and one single zero. Actual ascertaining of the 4CRs is greatly simplified by the fact that a wedge-shaped object representing the necessary and sufficient condition is situated in the vicinity of the line segment drawn through the end points of the ''wedge.'' Let us note that at large spatial asymmetries ($d<0.003$) the slope of the line segments can be obtained from the formula $\triangle D/\triangle a_0\approx 5d/8$. In the region

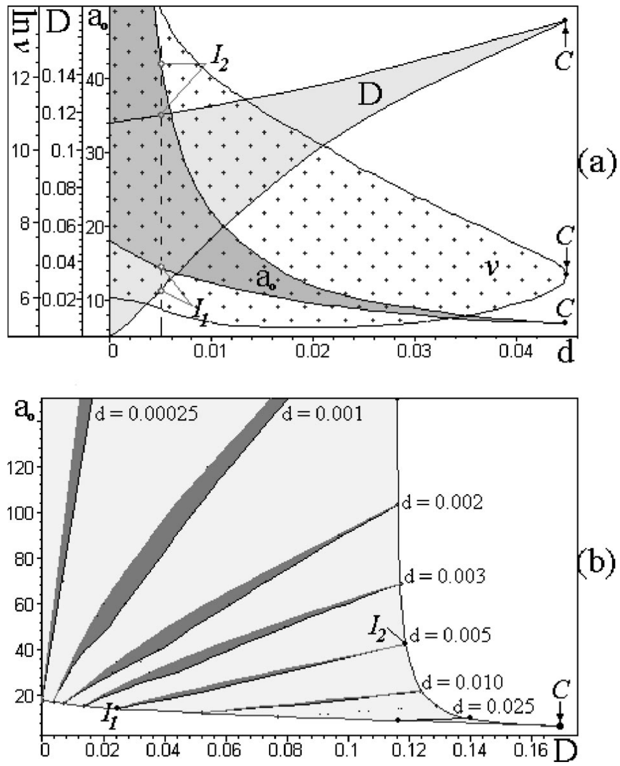

FIG. 2. The four current reversals effect: (a) the necessary and (b) the necessary and sufficient conditions. The dotted region in (a) displays the possible range of the four zeros of the current $J(\nu)$ at different values of d and demonstrates that the range has a lower bound $\nu_{\min}=181.27<\nu_C$.

under consideration, at the lower end of the wedge-shaped objects, the values of a_0 and D being relatively small, the last formula can equivalently be written as $8D\tau_c \approx 5da_0\tau_c$. This is a remarkable formula that throws some light on the physics of the 4CR effect; namely, it relates the characteristic distances of thermal diffusion $\sqrt{D\tau_c}$, the asymmetry of the

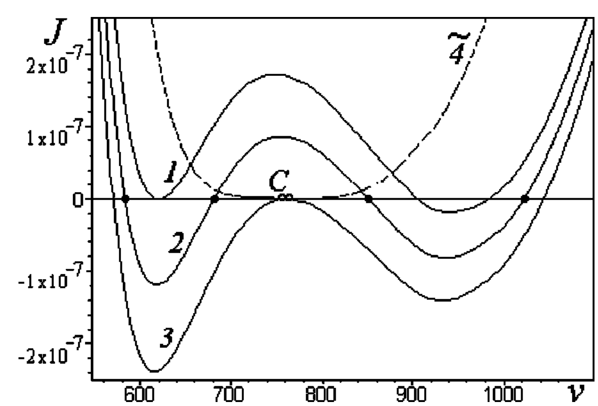

FIG. 3. Four current reversals vs switching rate. For curves (1)−(3) $d=0.044\,00$, $a_0=6.794\,00$ and the temperatures are (1) $D=D_1=0.167\,717\,420$, (2) $D=D_2=0.167\,716\,650$, and (3) $D=D_3=0.167\,715\,875$. Curve 2 has four single zeros. Curves 1 and 3 have two single zeros and one twofold zero. Curve $\tilde{4}$ is the critical curve $J=J(\nu;d_C,D_C,a_C)$; it has a fourfold zero at $\nu=\nu_C$.

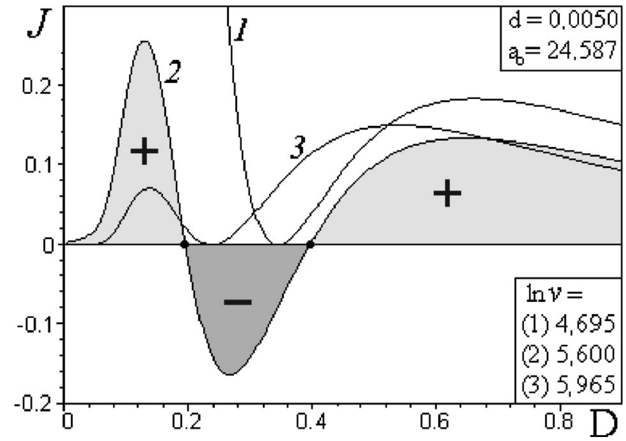

FIG. 4. Current vs temperature: two current reversals in the region of the disjunct ''windows'' on the plane (D,ν).

deterministic potential d, and the trichotomous noise $a_0\tau_c$, and demonstrates that all three agents mentioned act in unison to generate the 4CR effect.

Some curves $J=J(\nu)$ illustrating the 4CR effect near the critical point C are shown in Fig. 3.

Next we will examine the DWs. As already mentioned, at large spatial asymmetries the current exhibits characteristic disjunct zones of temperature and switching rate on the borders of which CRs occur. That is, for certain values of (d,a_0) there exist closed curves in the plane (D,ν) on which CRs take place (see Fig. 1). The closed curves encircle the regions where the direction of the current is negative, whereas in the surrounding regions the current direction is positive (see also Fig. 4).

The DWs exist if and only if the surface $J(D,a_0,\nu)=0$ (with a fixed asymmetry parameter $d=$const) has a local extremum and a saddle point [see Fig. 1(a)]. By varying a_0 and d step by step, we will obtain all the values of d and a_0 for which the DWs exist: to every point within the shaded region in Fig. 5 corresponds one closed curve on which $J(D,\nu)=0$. The region of existence of the DWs shrinks to a critical four-point S, where the saddle and extremum points

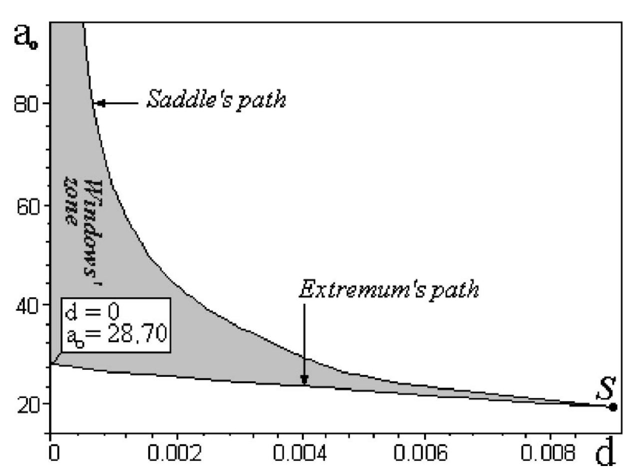

FIG. 5. The necessary and sufficient condition for the existence of the disjunct ''windows'' for current reversals.

merge, and which has the following coordinates: $d_S \approx 0.009$, $a_S \approx 19.40$, $D_S \approx 0.250$, and $\ln \nu_S \approx 5.25$. The values d_S and a_S are the upper and lower values of the parameters for the DWs to occur, i.e., the effect is possible if $d \in (0,d_S)$ and $a_0 \in (a_S,\infty)$ (see Fig. 5). The upper and lower boundaries of the shaded area in Fig. 5 can be approximated, respectively, by the polynomials $a_0 = 1.16\times 10^{-8}/d^3 - 4.725\times 10^{-5}/d^2 + 0.087/d + 10.4$ valid for $d \in (0.0005,0.009)$ and $a_0 = -10.33\times 10^2 d + 28.70$ for $d \in (0,0.009)$, which simplify the actual determination of DWs.

To illustrate the use of DWs, let us take the case of separation of particles with friction κ^*. Having six free parameters $\tilde{L}$, $\tilde{V}_0$, $\tilde{\nu}$, $\tilde{a}_0$, d, and T, let us first fix $d = 0.005$ and $a_0 = 23$. By this we will obtain the closed curve 1 in Fig. 1(b), which determines a ν window $\nu \in (170,216)$ and a D window $D \in (0.281,0.334)$. Within these windows we find the values of ν and D at which the absolute value of current is maximal, viz., $\nu^* = \tilde{L}\tilde{\nu}\kappa^*/\tilde{V}_0 \approx 190$ and $D^* \approx 0.305$. Using the limits of the ν window, we will finally find that only the particles with friction $\kappa \in (0.895\kappa^*, 1.137\kappa^*)$ move in the negative direction; the rest move in the positive direction. The above range of κ can be made narrower by varying a_0 or d; however, this is accompanied by a decrease in the absolute value of the current. Let us digress to mention that DWs exist at temperatures that are relevant for cell biology [2]. While one can realize particle separation by 2CRs without applying DWs, the DW method possesses certain advantages, e.g., it allows one to obtain a sharp extremum of $J(\nu)$ with a relatively large absolute value (see Fig. 4).

To comment on the formation of the hypersurface of CRs, let us note that at large flatness $\varphi \geqslant 2$ the trichotomous noise $Z(t)$ in Eq. (1) can be presented as the sum of two statistically independent zero-mean asymmetric dichotomous noises, $Z(t) = Z_1(t) + Z_2(t)$, where $\nu_1 = \nu_2 = \nu$, $z_1 \in \{a_0 - a_2, -a_2\}$, $z_2 \in \{a_2 - a_0, a_2\}$, and $0 < a_2 < a_0$. In this case the probability $q = a_2(a_0 - a_2)/a_0^2$, from which it follows that in the limit of large flatness $q \to 0$ the dichotomous noises $Z_i(t)$ $(i=1,2)$ must be of large asymmetry $a_2 \ll a_0$. Replacing in Eq. (1) $Z(t)$ by the dichotomous noises $Z_i(t)$, we obtain an equation that has been thoroughly studied in Ref. [7]. The asymmetry parameters of noise Θ and of potential k used therein can be expressed as $\Theta_1 \approx a_0$, $\Theta_2 \approx -a_0$, and $k = 1/2 - d$. Reference [7] shows that there exist ranges of temperature where the subsystems $i=1$ and $i=2$ generate currents with opposite signs. Since the absolute values of these currents depend heavily and nonmonotonically on the remaining parameters of the system, nonlinear addition of the currents results in the complicated structure of the hypersurface of CRs in our model (see the current inversion tailoring method in Ref. [1]).

In conclusion, it is remarkable that at large asymmetries of the spatial potential and at large flatness of the trichotomous noise new cooperation effects occur between the statistically independent white and colored noises, namely, four current reversals and disjunct ''windows'' in the control parameters. We believe that the latter will be useful for particle separation techniques.

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