

# Exact multilocal renormalization group and applications to disordered problems

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We develop a method, the exact multilocal renormalization group (EMRG) which applies to a broad set of theories. It is based on the systematic multilocal expansion of the Polchinski-Wilson exact renormalization group (ERG) equation together with a scheme to compute correlation functions. Integrating out explicitly the nonlocal interactions, we reduce the ERG equation obeyed by the full interaction functional to a flow equation for a function, its local part. This is done perturbatively around fixed points, but exactly to any given order in the local part. It is thus controlled, at variance with projection methods, e.g., derivative expansions or local potential approximations. Our EMRG method is well-suited to problems such as the pinning of disordered elastic systems, previously described via functional renormalization group (FRG) approach based on a hard cutoff scheme. Since it involves arbitrary cutoff functions, we explicitly verify universality to  $\mathcal{O}(\epsilon=4-D)$ , both of the  $T=0$  FRG equation and of correlations. Extension to finite temperature  $T$  yields the finite size ( $L$ ) susceptibility fluctuations characterizing mesoscopic behavior  $(\Delta\chi)^2 \sim L^\theta/T$ , where  $\theta$  is the energy exponent. Finally, we obtain the universal scaling function to  $\mathcal{O}(\epsilon^{1/3})$  which describes the ground state of a domain wall in a random field confined by a field gradient, compare with exact results and variational method. Explicit two loop exact RG equations are derived and the application to the FRG problem is sketched.

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## I. INTRODUCTION

The idea of writing an exact equation for the scale dependence of the full action functional already appears in the review of Wilson and Kogut [1]. Since it is an equation for a full functional of the fields, its detailed analysis is hindered by technical complications. The much simpler Wilson momentum shell [1] integration method is commonly used for one-loop calculations. Since it does not follow the full functional, subsequent efforts were made to embed it into a better controlled sharp cutoff exact renormalization group (RG) [2,3]. For practical perturbative calculations beyond one loop, field theoretical renormalization methods are more often used since they have proved vastly more efficient. However, the exact RG equations offer the hope to develop *ab initio* calculation relying on no assumption, possibly nonperturbative, from any bare model. In principle, it should be useful to obtain precise results when applied to bare theories for which we have little insight on possible underlying field theoretical description.

In the work of Polchinski [4], the exact RG equation was put on a more precise and aesthetic framework, and used to prove the renormalizability of the  $\phi^4$  theory in four dimensions. The exact renormalization group equations indeed provide formal results or general proofs about symmetries [5]. For practical calculations however, one needs to truncate in some way these highly complicated functional equations. To do so, different procedures have been proposed [6,7], and have been mainly applied to the study of nonperturbative problems [8]. For example the exponents of the  $O(n)$  model in three dimensions were estimated [9] using a choice of truncation. One commonly used projection method is the so-called local potential approximation [10], obtained by a constant background field method neglecting the momentum dependence. Further extensions include additional projections

on higher gradients of the field [11]. Although very interesting, these projection methods are often uncontrolled. More accurate results are expected if more couplings are kept, which is possible with heavy numerical integrations of flow equations. In this respect, exact RG as a tool is now used both in particle and condensed-matter physics. For instance, outstanding problems in strongly correlated electrons such as the Hubbard model in  $D=2$ , have been recently studied by numerically integrating the flow of a large number of vertices [12], using a fermionic version of the Polchinski equation [4].

By contrast, comparatively few works use exact RG method to develop *perturbative* calculations. One example is the computation of the beta function of  $\phi^4$  in  $4-\epsilon$  dimensions to one loop [13], where universality is made particularly explicit through the use of an arbitrary cutoff function. Although obviously more powerful methods are available in that case, there are some problems in condensed-matter physics that appear within reach of perturbative calculations, but for which no coherent field theoretical formulation is available at present. This is the case for the pinning of an elastic system in a random potential, for which a momentum shell RG method has been developed [14]. In this problem, an infinite number of coupling constants becomes relevant for  $D < D_c = 4$ , and one must write a RG equation for a full function  $R(u)$  (the second cumulant of the disorder), hence the name “functional” renormalization group (FRG). As such, it differs from standard field theoretical RG. Thus, to better understand this problem, i.e., to show explicitly universality to one loop and beyond, there is a need for a perturbatively controlled exact RG method, able to admit a full function, the local part  $R(u)$  as a small parameter. Indeed, the field theoretical formulation is fraught with difficulties, in particular because the function  $R(u)$  develops nonanalytic

behavior at finite scale. These issues are discussed in a related work [15].

In this paper, we develop a novel method to solve the Polchinski exact renormalization group equation and use it for explicit calculations. We call this method the exact multilocal renormalization group (EMRG). Writing the action as a sum of multilocal interactions, we note that the Polchinski equation naturally reduces to a hierarchy of equations obeyed by simple functions. This hierarchy can be solved in an expansion in powers of the local part. Indeed, we find that exact integration of the multilocal parts yields a single RG equation for the local part. The method is thus controlled around fixed points where the local part is proportional to a well-defined small parameter (e.g.,  $\epsilon=4-D$ ). The EMRG does not require any arbitrary projection procedure or neglect of operators, as is usually done in derivative expansions or local potential approximations. In addition, we obtain explicit formulas for any correlation functions that allow for practical calculations. Since this is done for arbitrary cutoff functions, it allows an explicit check of universality order by order in the expansion.

The aim of this paper is twofold. On the one hand, we present the general EMRG method to all orders, valid for a large class of theories. We derive the explicit form of the exact RG equation for the local interaction up to third order. On the other hand, we apply this method to several problems, first as a check, to the  $O(n)$  model, and second to the FRG for disordered elastic systems. Explicit calculations and applications in this paper are restricted mainly to one loop. Although briefly mentioned here, applications to two loops will be detailed in a companion paper [16].

Two variants of the method are presented. The most direct one consists in a straight expansion of the action in multilocal terms. The second one consists instead of first absorbing tadpoles into the interaction (so-called Wick ordering), then expanding. Being inequivalent, they provide independent checks of the universal results. The first method yields more complicated equations but can be better suited to some problems, such as the  $T>0$  FRG.

As mentioned above, the method is indeed well suited to the FRG for disordered elastic systems of internal dimension  $D$  since there the full local part is controlled by  $\epsilon=4-D$ . It allows us to show that the one-loop FRG equation, as well as correlation functions, are independent of the cutoff function. In addition, we obtain higher cumulants of the renormalized disorder, which as the second cumulant, are nonanalytic functions. This is necessary to escape the so-called dimensional reduction [18], i.e., the property of the present theory by that all perturbative calculations at  $T=0$  are identical to the same calculation in a trivial Gaussian theory [17] (see Appendix D). This nonanalytic behavior is rounded at finite-temperature  $T$  and we obtain the scaling form of the rounding region. This allows us to compute, for the first time using the FRG method, the susceptibility fluctuations that characterize the glassy behavior of finite-size systems. Finally, we obtain the universal  $\mathcal{O}(\epsilon^{1/3})$  correlation function that describes the ground state of a domain wall in a random field confined by a field gradient, compare with exact results and variational method.

The method presented in this paper also allows us to investigate the theory of disordered elastic systems beyond the lowest order in  $\epsilon$  (one loop). A recent two-loop calculation was presented in [19]. However, since it was performed at  $T=0$ , and for analytic  $R(u)$ , it fails beyond a finite length (Larkin length) and cannot describe universal properties. The application of exact FRG to next order is described in [16]. We sketch here, however, some preliminary results.

The paper is organized as follows. In Sec. II A, we formulate in a concise though pedagogical way the conventional exact RG method. Appendix A and Appendix B provide complements, respectively, about general invariance properties of the correlations and about examples of solvable cases of the Polchinski equation. In Sec. II B, we introduce the multilocal expansion in the local part  $U$ , up to bilocal terms. The ensuing EMRG equations to order  $U^2$  are derived in Sec. II C. The multilocal expansion to arbitrary order and the RG equation to order  $U^3$  are derived in Appendix C. The multilocal expansion of the Wick ordered functional up to second order, and the resulting one-loop RG equation is obtained in Sec. III A. The general multilocal expansion (Appendix C1) and the resulting form to third order is derived in Appendix C2. The explicit two-loop RG equation is obtained in Sec. III 3. As a check of the EMRG method application to the  $O(n)$  model to one loop is presented in Sec. III B. We then turn to applications to disordered elastic systems in Sec. IV (one loop) and Sec. V (two loops). First we recall and generalize in Appendix D the dimensional reduction phenomenon. Then the  $T=0$  FRG equations are established in Sec. IV A and finite-temperature extension are given in Sec. IV B. Finally, the calculations of the scaling function in the random field Ising model is performed in Sec. IV C. We sketch in Appendix E the variational calculation to be compared with the FRG results of Sec. IV C and sketch some preliminary steps of a two-loop FRG in Appendix F.

## II. METHOD

### A. Exact RG procedure

Consider a system whose state is described by a bosonic field  $\phi_x^i = \phi_i(x)$ , where  $x$  denotes position in space, and  $i$  is a general label denoting, e.g., fields indices, spin, replica indices, additional coordinate (e.g., time), etc., (or more generally, any quantity that will not undergo the coarse graining). The system, in the presence of external sources  $J_i^x$ , is described by the partition function:

$$Z(J) = \int_{\phi} e^{J:\phi - S(\phi)} \quad (1)$$

obtained by the integration over the field  $\phi$ , where the action  $S(\phi)$  is a functional of the field  $\phi$ , and  $J:\phi$  denotes here and in the following the full scalar product (e.g.,  $\int_x \sum_i J_i^x \phi_x^i$ , with  $\int_x \equiv \int d^D x$ ). In a problem of equilibrium statistical mechanics,  $S(\phi) = \mathcal{H}(\phi)/T$ , where  $\mathcal{H}(\phi)$  is the Hamiltonian and  $T$  the thermodynamic temperature, the free energy being  $F = -T \ln Z(0)$ . Averages of any observable  $\mathcal{A}(\phi)$  (i.e., functional of  $\phi$ ) are defined by

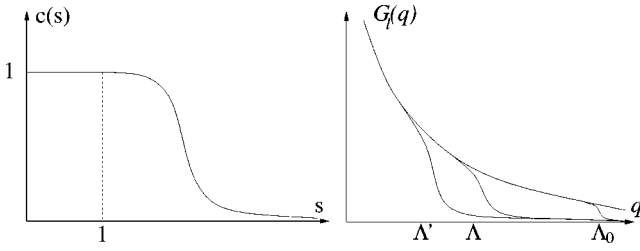


FIG. 1. Cutoff function.

$$\langle \mathcal{A}(\phi) \rangle_S = \frac{\int_{\phi} \mathcal{A}(\phi) e^{-S(\phi)}}{\int_{\phi} e^{-S(\phi)}}. \quad (2)$$

The usual way to compute correlation functions and averages is to perform a perturbation expansion, writing the action as a sum of a quadratic part and a nonlinear part  $\mathcal{V}(\phi)$

$$S(\phi) = \frac{1}{2} \phi : G^{-1} : \phi + \mathcal{V}(\phi), \quad (3)$$

where  $\mathcal{V}(\phi)$  a functional of  $\phi$ , and  $G_{ij}^{xy} = G_{ji}^{yx}$  is a symmetric invertible matrix,  $\phi : G^{-1} : \phi = \sum_{ij} \int_{xy} \phi_x^i (G^{-1})_{ij}^{xy} \phi_y^j$ . In the following, we denote the Gaussian average of any observable by  $[\mathcal{A}(\phi)]_G = \langle \mathcal{A}(\phi) \rangle_{S_G}$  with respect to the quadratic theory  $S_G = (1/2) \phi : G^{-1} : \phi$ . We introduce the generating function of all correlation functions

$$W(J) = \ln[e^{J : \phi - \mathcal{V}(\phi)}]_G. \quad (4)$$

Note that it differs from the usual definition by a  $J$ -independent quantity  $\ln Z(J) = W(J) + (1/2) \text{Tr} \ln G$ . The ultra-violet cutoff, present in physical models, is necessary to yield finite results in the perturbative calculation with respect to  $\mathcal{V}$ . A broad class of soft cutoffs can be implemented on the Gaussian part, giving a vanishing weight to fast fields. For example, a scalar massive theory, rotationally invariant, is regularized in the UV by the following general cutoff function

$$G(q) = \frac{c\left(\frac{q^2}{2\Lambda^2}\right)}{q^2 + m^2}, \quad (5)$$

where  $c(0) = 1$  and  $c(s)$  decreases rapidly to zero for  $s > 1$  as in Fig. 1.

The exact RG method [1,4] consists in varying the cutoff  $\Lambda$  and writing an equation for the function  $\mathcal{V}(\phi)$  so as to conserve exactly the averages of all observables involving only ‘‘slow’’ modes of the field. More precisely, the average of an observable  $\mathcal{A}(\phi)$  depending only on modes  $q < \Lambda'$  of the field  $\phi$  can be computed within any of the theories linked by the equation presented by Polchinski corresponding to a cutoff  $\Lambda > \Lambda'$ . As in any RG procedure, the strategy will be to compute averages of slow observables using the coarse-grained theory of cutoff  $\Lambda' \ll \Lambda_0$ .

To this aim, a set of actions  $S_l$

$$S_l(\phi) = \frac{1}{2} \phi : G_l^{-1} : \phi + \mathcal{V}_l(\phi) \quad (6)$$

is introduced, where the Gaussian part is an arbitrary function  $G_l$  of  $l$  (e.g., corresponding to a cutoff  $\Lambda_l = \Lambda_0 e^{-l}$ ). The initial propagator, corresponding to a cutoff  $\Lambda_0$ , is denoted  $G_{l=0} \equiv G$  and the bare interaction  $\mathcal{V}_{l=0} \equiv \mathcal{V}$ . The correlation functions in  $S_l$  derive from  $W_l(J) = \ln[e^{J : \phi - \mathcal{V}_l(\phi)}]_{G_l} = \ln Z_l(J) - (1/2) \text{Tr} \ln G_l$ .

For any given  $\Lambda'$ , one defines a  $(\Lambda')$ -slow observable to be a functional of  $\phi$  depending only on the  $\phi^q = \phi(q)$  with  $q < \Lambda' = \Lambda_0 e^{-l'}$ . We want to choose the  $l$ -dependent nonlinear part  $\mathcal{V}_l(\phi)$  so that the averages of slow observables remain unchanged. Through differentiation, it is equivalent to ensure that

$$\partial_l W_l(J) \text{ independent of } J \quad (7)$$

for any source  $J$  with  $J^q = 0$  for  $q > \Lambda'$ . Using the general identity

$$\partial_l [\mathcal{A}_l(\phi)]_{G_l} = \frac{1}{2} \text{Tr} \left( \partial_l G_l : \left[ \frac{\delta^2}{\delta \phi \delta \phi} \mathcal{A}_l(\phi) \right]_{G_l} \right) + [\partial_l \mathcal{A}_l(\phi)]_{G_l}, \quad (8)$$

valid for Gaussian averages of any  $l$ -dependent observable  $\mathcal{A}_l(\phi)$  and applying it to  $\mathcal{A}_l(\phi) = e^{J : \phi - \mathcal{V}_l(\phi)}$ , one finds

$$\begin{aligned} \partial_l [e^{J : \phi - \mathcal{V}_l(\phi)}]_{G_l} &= \left\{ \left[ \frac{1}{2} \left( J - \frac{\delta}{\delta \phi} \mathcal{V}_l(\phi) \right) : \partial_l G_l : \left( J - \frac{\delta}{\delta \phi} \mathcal{V}_l(\phi) \right) \right. \right. \\ &\quad \left. \left. - \partial_l \mathcal{V}_l(\phi) - \frac{1}{2} \text{Tr} \partial_l G_l : \frac{\delta^2}{\delta \phi \delta \phi} \mathcal{V}_l(\phi) \right] e^{J : \phi - \mathcal{V}_l(\phi)} \right\}_{G_l}, \end{aligned}$$

where here and in Eq. (8),  $\text{Tr} A : B \equiv \sum_{ij} \int_{xy} A_{ij}^{xy} B_{ji}^{yx}$ . Hence, if  $\mathcal{V}_l(\phi)$  satisfies the Polchinski functional equation

$$\begin{aligned} \partial_l \mathcal{V}_l(\phi) &= -\frac{1}{2} \text{Tr} \left( \partial_l G_l : \frac{\delta^2}{\delta \phi \delta \phi} \mathcal{V}_l(\phi) \right) \\ &\quad + \frac{1}{2} \frac{\delta}{\delta \phi} \mathcal{V}_l(\phi) : \partial_l G_l : \frac{\delta}{\delta \phi} \mathcal{V}_l(\phi), \end{aligned} \quad (9)$$

then the above conservation condition (7) is satisfied. We have used explicitly the condition

$$J : \partial_l G_l = 0, \quad (10)$$

which imposes that the cutoff function verifies  $\partial_l G_l^q = 0$  for  $q < \Lambda'$  and  $l > l'$ . Hence, for the example (5), one has to choose [20] cutoff functions  $c(s)$  such that  $c(s) = 1$  for  $0 \leq s \leq s_0$  with some (arbitrary)  $s_0$ .

The above framework is in fact too restrictive. We can easily lift the restriction on slow modes [and on the form of the cutoff function  $c(s)$ ]. The applications of the Polchinski

equation can be generalized to the computation of *any* observable (not restricted to be “slow”). As shown in Appendix A 1, one can indeed express  $W(J)$  in terms of any of the  $l$ -dependent actions  $S_l(\phi)$ :

$$W(J) = \frac{1}{2} J : (G - G : G_l^{-1} : G) : J + W_l(J : G : G_l^{-1}). \quad (11)$$

In fact we show in Appendix A 2 an even more general method that allows for arbitrary field rescalings.

Differentiating  $W(J)$  once yields  $\langle \phi \rangle_S = G : G_l^{-1} \langle \phi \rangle_{S_l}$ , and once again yields the two-point connected correlation function

$$\langle \phi \phi \rangle_S^c = G + G : G_l^{-1} : (\langle \phi \phi \rangle_{S_l}^c - G_l) : G_l^{-1} : G \quad (12)$$

and so on for higher correlations. When performing a perturbative calculation, the factors  $G : G_l^{-1}$  restore the original propagator for the external lines, whereas internal lines of the graph involve  $G_l$ . Accordingly, with this procedure the function  $c(s)$  can be arbitrary [it is however convenient—to see below—to use  $c'(0) = 0$ ].

Note that if  $G : J = G_l : J$ , one recovers Eq. (7), i.e.,  $W(J) = W_l(J)$  for these slow  $J$ 's as a special case of Eq. (5). In that case, for  $q < \Lambda'$ , Eq. (12) reduces to  $\langle \phi \phi \rangle_S^c = \langle \phi \phi \rangle_{S_l}^c$  as it should.

To compute correlation functions, it is useful to express  $W(J)$  in a perturbation expansion in powers of  $\mathcal{V}_l(\phi)$ , which reads to lowest order

$$W(J) = \frac{1}{2} J : G : J - e^{-(1/2) J : G : G_l^{-1} : G : J} [e^{J : G : G_l^{-1} : \phi} \mathcal{V}_l(\phi)]_{G_l} + \mathcal{O}(\mathcal{V}_l). \quad (13)$$

The Polchinski Eq. (9) can equivalently be written as a functional “diffusion” equation

$$\partial_l e^{-\mathcal{V}_l(\phi)} = -\frac{1}{2} \text{Tr} \partial_l G_l : \frac{\delta^2}{\delta \phi \delta \phi} e^{-\mathcal{V}_l(\phi)}, \quad (14)$$

or in its integrated form

$$e^{-\mathcal{V}_l(\phi)} = [e^{-\mathcal{V}_0(\phi + \phi')}]_{G-G_l}, \quad (15)$$

where the average is over  $\phi'$ , which makes explicit the definition of  $\mathcal{V}_l(\phi)$  as a coarse-grained interaction, i.e., integrated over the “fast part”  $\phi'$  of the field. In fact, the decomposition into slow and fast modes and the definition of coarse-grained observables relies on the property  $[\mathcal{A}(\phi)]_G = \{[\mathcal{A}(\phi + \phi')]\}_{G-G_l}$  of Gaussian averages (see Appendix A 1).

Although, in general, the Polchinski equation is far too complicated to be solved, in some simple cases, one can find exact solutions, e.g., Gaussian models, zero dimensional toy model. Most interestingly, there exist a large class of exact solutions that appear as superpositions of Gaussians. In all these cases, one can explicitly verify an interesting property of the Polchinski equation to generate cusp singularities. This is further discussed in Appendix B and in a forthcoming publication [21].

### B. Multilocal expansion

The Polchinski equation, in addition to being elegant, is conceptually more satisfactory than other RG methods, e.g., Wilson’s shell renormalization, because it is exact and better controlled since, being valid for arbitrary cutoff procedures, it does not suffer from the problems associated with the sharp cutoff [3]. However, this functional equation generates nonlocal operators, which until now, has limited its practical applications. This generation can be seen in terms of Feynman diagrams and compared to Wilson’s shell renormalization, since  $G - G_l$ , which contains a range of wave vectors centered around  $\Lambda_0 e^{-l}$ , plays the role of the on-shell propagator. The term  $\mathcal{V}''$  with a second derivative in Eq. (9) represents tadpoles, while the term  $\mathcal{V}' \mathcal{V}'$  represents diagrams with only one contraction (one particle reducible). These last terms are nonlocal operators. For instance, in  $\phi^4$  theory, it generates the operator  $\phi(x)^3 \partial G^{x-y} \phi(y)^3$ , which is bilocal since it corresponds to a graph where external momenta must be greater than  $\Lambda_l$ . The way Polchinski’s equation reproduces the loop diagrams (i.e., local terms) is that after integration over a slice  $dl$ , a bilocal interaction generated by the second term of Eq. (9) is fed into tadpole diagrams. A fast momentum goes around the corresponding loop, and slow external momenta are allowed. Thus, one needs to integrate the flow and study the feedback of the generated nonlocal operators into local ones.

We now present a method which allows us to perform this program in a controlled way. The following expansion in the number of points (local, bilocal, etc.):

$$\mathcal{V}(\phi) = \int_x U(\phi_x) + \int_{xy} V(\phi_x, \phi_y, x-y) + \dots \quad (16)$$

is valid *a priori* for any translationally invariant functional  $\mathcal{V}(\phi)$  interactions. We discuss here only the first two terms, the general systematics being given in Appendix C. Here,  $U(\phi)$  is a function of the vector  $\phi_l$  and involves the value of the field at one point in space. The bilocal part is a function  $V(\phi, \psi, z)$  of two vectors  $\phi, \psi$  and a space coordinate difference  $z$ . In order that the expansion be well defined, one needs the bilocal interactions to have *no* projection on the local ones. A natural way to define such a projection, inspired from the conventional short-distance expansion, is the exact equality

$$\begin{aligned}
 & \int_{xy} F(\phi_x, \phi_y, x-y) \\
 &= \int_{xy} F(\phi_x, \phi_x, y) + \int_{xy} [F(\phi_{x+y/2}, \phi_{x-y/2}, y) \\
 &\quad - F(\phi_x, \phi_x, y)] \\
 &= \int_x (\bar{P}_1 F)(\phi_x) + \int_{xy} [(1-P_1)F](\phi_x, \phi_y, x-y),
 \end{aligned}$$

where we have introduced the projections

$$(\bar{P}_1 F)(\phi) = \int_y F(\phi, \phi, y), \quad (17)$$

$$(P_1 F)(\phi, \psi, z) = \delta(z) \int_y F(\phi, \psi, y) \quad (18)$$

on the subspaces of local and bilocal interactions respectively. Indeed,  $[(\bar{P}_1(1-P_1)F)(\phi) = \int_x [(1-P_1)F](\phi, \phi, x) = 0$ , i.e.,  $(1-P_1)F$  has zero local part and is thus properly bilocal. Interestingly, this definition implies that the function  $V(\phi, \psi, z)$  appearing in the proper bilocal operator of Eq. (16) also satisfy the stronger property  $\int_z V(\phi, \psi, z) = 0$  for any  $\phi, \psi$ . Note that with no loss of generality,  $V(\phi, \psi, z) = V(\psi, \phi, -z)$ . Here, in addition, we will consider parity invariant theories [ $V(\phi, \psi, z) = V(\phi, \psi, -z)$  too].

For theories where the initial interaction  $U$  is local and is formally treated as a ‘‘small’’ quantity  $U$  (e.g., the  $\phi^4$  theory in  $D=4-\epsilon$  where  $U \sim \epsilon$ ), it is natural to consider that the bilocal term will be of higher order  $\mathcal{O}(U^2)$ . In fact, this property results from the Polchinski equation since the term that creates bilocal interactions from local ones is  $\mathcal{O}(U^2)$  (the first part  $\mathcal{V}$  does not increase the degree of nonlocality of  $\mathcal{V}$ ). This property that solutions of the Polchinski equation can be organized in powers of  $U$  depending on their locality holds to arbitrary orders [ $p$ -local operators are  $\mathcal{O}(U^p)$ ] as is discussed below and shown in Appendix C. Thus, to lowest nontrivial order  $\mathcal{O}(U^2)$ , the flow equations involve local and bilocal parts. Their schematic structure is

$$\partial U = U'' + \bar{P}_1(V'' + U'U') + \mathcal{O}(UV) + \mathcal{O}(V^2), \quad (19)$$

$$\partial V = (1-P_1)(V'' + U'U') + \mathcal{O}(UV) + \mathcal{O}(V^2), \quad (20)$$

where we have written the subdominant terms that will be neglected in the following.

A simplification occurs if we choose, as done in this section,  $(\partial_l G_l)_{ij}(q=0) = 0$ . Indeed, the term  $\bar{P}_1 U' U'$  vanishes since

$$\int_x (\partial_l G_l^x)_{ij} \partial_i U(\phi) \partial_j U(\phi) = 0. \quad (21)$$

Let us write now Eqs. (19) and (20) in an explicit form:

$$\begin{aligned}
 \partial_l U_l(\phi) &= -\frac{1}{2} \partial G_{ij}^{x=0} \partial_i \partial_j U_l(\phi) \\
 &\quad - \int_x \partial G_{ij}^x \partial_i^1 \partial_j^2 V_l(\phi, \phi, x) \partial_l V_l(\phi, \psi, x) \\
 &= \frac{1}{2} \partial G_{ij}^x \partial_i U_l(\phi) \partial_j U_l(\psi) \\
 &\quad - \frac{1}{2} \partial G_{ij}^{x=0} (\partial_i^1 \partial_j^1 + \partial_i^2 \partial_j^2) V_l(\phi, \psi, x) \\
 &\quad - \partial_i^1 \partial_j^2 \left( \partial G_{ij}^x V_l(\phi, \psi, x) \right. \\
 &\quad \left. - \delta(x) \int_y \partial G_{ij}^y V_l(\phi, \psi, y) \right),
 \end{aligned}$$

where  $\partial G$  stands for  $\partial_l G_l$  and  $\partial_i^1 A(\phi, \psi)$  [ $\partial_i^2 A(\phi, \psi)$ ] for  $\partial/\partial\phi_i A(\phi, \psi)$  [ $\partial/\partial\psi_i A(\phi, \psi)$ ].

### C. Solution to the lowest-order RG equations

To solve Eqs. (19) and (20), one switches to Fourier space (in the field):

$$U^K = \int d\phi e^{-iK \cdot \phi} U(\phi) \quad (22)$$

$$V^{KP_x} = \int d\phi d\psi e^{-iK \cdot \phi - iP \cdot \psi} V(\phi, \psi, x), \quad (23)$$

where  $K \cdot \phi \equiv \sum_i K_i \phi^i$ . It turns out that the equation for  $V_l$  can be integrated explicitly as a retarded function of  $U_l$ :

$$V_l^{KP_x} = \frac{1}{2} \left( F_l^{KP_x} - \delta(x) \int_y F_l^{KP_y} \right), \quad (24)$$

$$F_l^{KP_x} = - \int_0^l dl' (K \cdot \partial G_{l'}^x \cdot P) U_{l'}^K U_{l'}^P e^{(1/2)K \cdot (G_l^{x=0} - G_{l'}^{x=0}) \cdot K + (1/2)P \cdot (G_l^{x=0} - G_{l'}^{x=0}) \cdot P + K \cdot (G_l^x - G_{l'}^x) \cdot P}, \quad (25)$$

since we have chosen  $V_{l=0}^{KP_x} = 0$ . One can then reinject this result in Eq. (19) and obtain a closed RG equation for  $U_l(\phi)$ :

$$\begin{aligned} \partial_l U_l(\phi) = & -\frac{1}{2} \partial G_{ij}^{x=0} \partial_i \partial_j U_l(\phi) - \frac{1}{2} \int_{KP} e^{i(K+P) \cdot \phi} \\ & \times \int_x K \cdot (\partial G_l^x - \partial G_l^{x=0}) \cdot P \int_0^l dl' K \cdot \partial G_{l'}^x \cdot P e^{(1/2)K \cdot (G_l^{x=0} - G_{l'}^{x=0}) \cdot K + (1/2)P \cdot (G_l^{x=0} - G_{l'}^{x=0}) \cdot P + K \cdot (G_l^x - G_{l'}^x) \cdot P} U_{l'}^K U_{l'}^P. \end{aligned} \quad (26)$$

This is the *exact* renormalization equation for an arbitrary local interaction  $U(\phi)$  to  $\mathcal{O}(U^2)$ . Note that the second-order term is retarded in  $l$ , since as discussed above, local terms are generated only after integration.

More generally, this procedure can be carried out to any order in  $U$  using the hierarchical structure of the flow equations for  $p$ -local interactions (see Appendix C). It is found that the general structure for the flow of the local part is

$$\begin{aligned} \partial_l U_l(\phi) = & \beta[U_{l' < l}(\phi')](\phi) \\ = & \sum_{n \geq 1} \int_{l_1 < l_2 < \dots < l_n < l} \mathcal{K}_{l, l_1 \dots l_{n-1}}^n \left[ \frac{\partial}{\partial \phi_1}, \dots, \frac{\partial}{\partial \phi_n} \right] \\ & \times U_{l_1}(\phi_1) \dots U_{l_n}(\phi_n) \Big|_{\phi_p = \phi}, \end{aligned} \quad (27)$$

and in (26) we achieved the calculation of the  $\beta$  function to second order in  $U$ . Once the solution of Eq. (27) is known up to  $\mathcal{O}(U^p)$ , all the  $p'$ -local flowing interactions for  $p' \leq p$  are also known by injecting the solution for  $U_l$ . For example, for  $p=2$ , the bilocal part is obtained from Eqs. (24) and (25) injecting the solution to Eq. (26).

To a given order in  $U$  one can also perform a loop expansion by expanding the exponentials of propagators that appear in Eq. (26) [see Appendix C4 for the  $\mathcal{O}(U^3)$  equation]. To order  $U^2$  and one loop it reads

$$\partial_l U_l(\phi) = J_l^0 \partial_{ii} U_l(\phi) + \int_0^l dl' J_{ll'}^D \partial_{ij} U_{l'}(\phi) \partial_{ij} U_{l'}(\phi), \quad (28)$$

$$J_l^0 = -\frac{1}{2} \int_q \partial G_l^q,$$

$$J_{ll'}^D = -\frac{1}{2} \int_q \partial G_l^q \partial G_{l'}^q,$$

where here and in the following  $\partial_{ij} = \partial_{\phi_i} \partial_{\phi_j}$ .

In order to compute the correlation functions for small wave-vectors  $q$ , the strategy of the RG consists in performing a perturbative calculation in the theory renormalized up to  $l' = \ln(\Lambda_0/q)$ . In the favorable cases, the interaction  $\mathcal{V}_l$  flows, from a small initial-initial interaction  $\mathcal{V}_0$ , to ‘‘fixed points’’ in functional space (up to appropriate rescalings) controlled by a small parameter (such as the offset from the critical dimension). Once the asymptotic large  $l$  behavior of  $U_l(\phi)$  is known, one uses the invariance property of  $W(J)$  [see Eq. (11) and Appendix C5 where this is done in detail]

to compute the observables. To lowest order in  $U$ , it is sufficient to keep only the local part in Eq. (13) which yields to  $\mathcal{O}(U_l)$

$$W(J) = \frac{1}{2} J : G : J - \int_x \int_K U_l^K e^{-(1/2)K \cdot G^{x=0} \cdot K} e^{iK \cdot (G:J)^x}.$$

Thus, one has for the two-point function  $\langle \phi_q^i \phi_{q'}^j \rangle_S^c = \delta(q + q') \mathbf{C}_{ij}^q$  with

$$\begin{aligned} \mathbf{C}_{ij}^q = & G_{ij}^q + \int_K (K \cdot G^q)_i (K \cdot G^q)_j \hat{U}_l^K, \\ \hat{U}_l^K = & U_l^K e^{-(1/2)K \cdot G_l^{x=0} \cdot K}, \end{aligned}$$

and more generally the  $n$ -point function ( $n \neq 2$ ):

$$\langle \phi_{q_1}^{j_1} \dots \phi_{q_n}^{j_n} \rangle_S^c = -\delta_{\sum_i q_i} \left( \prod_i G_{j_i k_i}^{q_i} \partial_{k_i} \right) \hat{U}_l(0), \quad (29)$$

with  $\delta_q \equiv (2\pi)^D \delta(q)$ . To compute, e.g., the two-point correlation function at wave-vector  $q$ , one carries perturbation theory in  $U_{l'}$  at a large scale  $l'$  and it is convenient to choose  $l' = \ln \Lambda/q$ . To first order in  $U_{l'}(\phi)$ , one has

$$\begin{aligned} \mathbf{C}_{ij}^q = & G_{ij}^q + \int_K (K \cdot G^q)_i (K \cdot G^q)_j U_{l'=\ln \Lambda/q}^K \\ & \times e^{-(1/2)K \cdot G_{l'=\ln \Lambda/q}^{x=0} \cdot K}. \end{aligned} \quad (30)$$

Of course, since  $W(J)$  is by construction independent of  $l$  the result should not depend on the choice of  $l'$ . Using the RG flow equation, it can be checked order by order in perturbation in powers of  $U_l$  that this is the case.

In the above computation of the two-point function, the natural vertex that appear is not  $U_l(\phi)$  but  $\hat{U}(\phi) = \int_K e^{iK \cdot \phi} \hat{U}_l^K$ ; it is thus interesting to study directly its flow equation.

### III. REMOVING OF TADPOLES AND APPLICATION TO $\phi^4$

#### A. Modified Polchinski equation

It is useful for some applications, and in particular to simplify higher-order calculations, to get rid of the linear term in the Polchinski equation. This can be achieved exactly by introducing the following functional:

$$\frac{d}{dl} \bullet = \frac{1}{2} \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \text{---} \end{array}$$

FIG. 2. Graphical representation of the modified Polchinski equation. The point represents any vertex, the broken line is a propagator on shell  $\partial G_l$  and the full line is a  $G_l$ .

$$\hat{\mathcal{V}}_l(\phi) = e^{(1/2)(\delta/\delta\phi):G_l:(\delta/\delta\phi)} \mathcal{V}_l(\phi). \quad (31)$$

Inserted in Eq. (9) one finds that it satisfies

$$\begin{aligned} \partial_l \hat{\mathcal{V}}_l(\phi) &= \frac{1}{2} e^{(\delta/\delta\phi_1):G_l:(\delta/\delta\phi_2)} \frac{\delta}{\delta\phi_1} : \partial G_l : \frac{\delta}{\delta\phi_2} \\ &\times \hat{\mathcal{V}}_l(\phi_1)|_{\phi_1=\phi} \hat{\mathcal{V}}_l(\phi_2)|_{\phi_2=\phi}. \end{aligned} \quad (32)$$

The graphical representation of this equation is drawn in Fig. 2. Since this equation does not contain a linear term, its solution does not contain tadpolelike diagrams. This functional has thus several advantages: first it enters directly the computation of any observable, second its flow is simpler than the one of the bare vertices. Finally, in the context of quantum field theory it has the direct meaning of being the normally ordered vertices.

We now perform a multilocal expansion, similar to the one introduced in the previous section, but on the functional  $\hat{\mathcal{V}}_l(\phi)$  as:

$$\hat{\mathcal{V}}_l(\phi) = \int_x \hat{U}_l(\phi_x) + \int_{xy} \hat{\mathcal{V}}_l(\phi_x, \phi_y, x-y) + \dots \quad (33)$$

The modified Polchinski Eq. (32) can be solved order by order in  $\hat{U}_l(\phi)$ . The general analysis is performed in Appendix C1. Here, we give only the result to order  $\hat{U}_l^2$ , which reads

$$\partial_l \hat{U}_l(\phi) = \frac{1}{2} \int_x e^{\partial^1 G_l^x \partial^2} \partial^1 \partial G_l^x \partial^2 \hat{U}_l(\phi_1)|_{\phi_1=\phi} \hat{U}_l(\phi_2)|_{\phi_2=\phi}, \quad (34)$$

where  $\partial^1 = \partial_{\phi_1}$ . An interesting property of this equation is that it is now local in  $l$ .

Expansion in the number of loops  $k$ , restricted to order  $\hat{U}_l^2$ , is thus straightforward:

$$\partial_l \hat{U}_l(\phi) = \frac{1}{2} \sum_{k=1}^{+\infty} \frac{1}{k!} I_l^k \partial_{i_1, \dots, i_{k+1}} \hat{U}_l(\phi) \partial_{i_1, \dots, i_{k+1}} \hat{U}_l(\phi), \quad (35)$$

$$I_l^k = \int_x (G_l^x)^k \partial G_l^x = \frac{1}{k+1} \partial_l \int_x (G_l^x)^{k+1}. \quad (36)$$

Note that the multilocal expansion for  $\hat{\mathcal{V}}$  and  $\mathcal{V}$  are not identical, i.e., they do not produce order-by-order equations that can be transformed back into each other. However, they should yield the same result at the end when calculating

universal fixed point quantities. This property will be tested and used in the rest of the paper.

We also give the expression of the bilocal term, as one must check that it effectively reaches a fixed point for consistency. It reads as a function of the local term:

$$\begin{aligned} \hat{\mathcal{V}}_l(\phi_1, \phi_2, q) &= \frac{1}{2} \int_x (e^{iqx} - 1) \int_0^l dl' e^{\partial^1 G_{l'}^x \partial^2} \partial^1 \partial G_{l'}^x \partial^2 \\ &\times \hat{U}_{l'}(\phi_1) \hat{U}_{l'}(\phi_2). \end{aligned}$$

### B. Modified RG equation to one loop and application to $O(n)$ model

Expanding the exponential of the propagator in Eq. (34) yields a loop expansion of the beta function to order  $U^2$ . To one loop this gives

$$\partial_l \hat{U}_l(\phi) = I_l^D \partial_{ij} \hat{U}_l(\phi) \partial_{ij} \hat{U}_l(\phi), \quad (37)$$

$$I_l^D = \frac{1}{2} \int_q \partial G_l^q G_l^q. \quad (38)$$

As a check of our EMRG method, let us apply it to the  $O(n)$  model in  $D=4-\epsilon$  with the polynomial interaction:

$$U_l(\phi) = \frac{1}{2} g_{2,l} \phi^2 + \frac{1}{4!} g_{4,l} (\phi^2)^2 + \frac{1}{6!} g_{6,l} (\phi^2)^3 + \dots, \quad (39)$$

the dimensionless variables being  $\tilde{g}_{n,l} = g_{n,l} \Lambda_l^{-D+n(D-2)/2}$ . The RG can be performed either using  $U_l(\phi)$  as in the previous section, or in terms of  $\hat{U}_l(\phi)$ . Let us follow the second method for completeness. One has

$$\begin{aligned} \hat{U}_l(\phi) &= e^{(1/2)G_l^x \partial^2} U_l(\phi) = \frac{1}{2} a_{2,l} \phi^2 + \frac{1}{4!} a_{4,l} (\phi^2)^2 \\ &+ \frac{1}{6!} a_{6,l} (\phi^2)^3 + \dots, \end{aligned} \quad (40)$$

and, similarly [22]  $\tilde{a}_{n,l} = a_{n,l} \Lambda_l^{-D+n(D-2)/2}$ , thus

$$\tilde{a}_{2,l} = a_{2,l} \Lambda_l^{-2}, \quad (41)$$

$$\tilde{a}_{4,l} = a_{4,l} \Lambda_l^{-\epsilon}, \quad (42)$$

$$\tilde{a}_{6,l} = a_{6,l} \Lambda_l^{2-2\epsilon}. \quad (43)$$

From  $\partial_{ij} \hat{U}_l(\phi) = a_{2,l} \delta_{ij} + (a_{4,l}/3!)(\delta_{ij} \phi^2 + 2\phi_i \phi_j) + (a_{6,l}/5!)[\delta_{ij}(\phi^2)^2 + 4\phi_i \phi_j \phi^2] + \dots$ , one obtains the RG equations:

$$\partial \tilde{a}_{2,l} = 2\tilde{a}_{2,l} + I^D \frac{2}{3} (n+2) \tilde{a}_{4,l} \tilde{a}_{2,l} + \mathcal{O}(\tilde{a}_{4,l}^2), \quad (44)$$

$$\partial \tilde{a}_{4,l} = \epsilon \tilde{a}_{4,l} + I^D \frac{2}{3} (n+8) \tilde{a}_{4,l}^2 + \mathcal{O}(\tilde{a}_{4,l}^3), \quad (45)$$

$$\partial \tilde{a}_{6,l} = (-2 + 2\epsilon) \tilde{a}_{6,l} + \mathcal{O}(\tilde{a}_{4,l}^3), \quad (46)$$

in terms of the single integral  $I^D$ , which has a universal value in  $D=4$ :

$$I^D = \Lambda_l^\epsilon I_l^D = \Lambda_l^\epsilon \frac{1}{2} \int_q \frac{c' \left( \frac{q^2}{2\Lambda_l} \right)}{\Lambda_l^2} \frac{c \left( \frac{q^2}{2\Lambda_l} \right)}{q^2} \quad (47)$$

$$= \frac{1}{2} S_D \int_{s>0} (2s)^{-\epsilon/2} c'(s) c(s) = -\frac{S_4}{4} + \mathcal{O}(\epsilon), \quad (48)$$

with  $S_4 = 1/(8\pi^2)$  and here, and in the following,  $S_D$  is the unit sphere area divided by  $(2\pi)^D$ . Thus,  $\tilde{a}_{4,l}$  flows to the fixed value:

$$\tilde{a}_4^* = \frac{3}{(n+8)} 16\pi^2 \epsilon, \quad (49)$$

which is universal to this order. We have indicated terms in the above RG equations that arise at two loops and yield fixed point for  $\tilde{a}_{2,l}$  and  $\tilde{a}_{6,l}$  with

$$\tilde{a}_2^* = \mathcal{O}(\epsilon^2), \quad \tilde{a}_6^* = \mathcal{O}(\epsilon^3), \quad (50)$$

and more generally,  $\tilde{a}_{2n}^* = \mathcal{O}(\epsilon^n)$  for  $n \geq 3$ . The derivation of this is simple but goes beyond this paper [16]. This fixed point is unstable in the direction  $\tilde{a}_2$  and stable in all other directions. The stability eigenvalues to  $\mathcal{O}(\epsilon)$  read

$$\lambda_2 = 2 - \frac{n+2}{n+8} \epsilon, \quad (51)$$

$$\lambda_4 = -\epsilon, \quad (52)$$

$$\lambda_{2n} = -2(n-2) + \epsilon(n-1) \quad \text{for } n \geq 3. \quad (53)$$

The critical manifold of the  $O(n)$  model corresponds to  $\tilde{a}_2 = \tilde{a}_2^*$ . This corresponds indeed to the massless case, since the self-energy

$$\Sigma^q = \hat{U}^n(0) + \mathcal{O}(\hat{U}^2) = \Lambda_l^2 \tilde{a}_{2,l} \quad (54)$$

vanishes to lowest order on the critical manifold. The instability eigenvalue at the fixed point gives the critical exponent  $\gamma$

$$\gamma = \frac{2}{\lambda_2} = 1 + \frac{(n+2)}{2(n+8)} \epsilon + \mathcal{O}(\epsilon^2), \quad (55)$$

thus recovering the standard result. One also gets  $\omega = -\epsilon$  and  $\eta = \mathcal{O}(\epsilon^2)$ .

Note that to first order in  $\epsilon$  there is no  $q$  dependence of  $\Sigma^q$  and no wave-function renormalization. This, and the calculation of  $\eta$  can be incorporated in the method to two loops,

but this goes beyond the present paper. Such a calculation, using a slightly modified version of the present method, will be presented in [23].

Having checked the method on a well-known example, we turn to far-less chartered territory.

## IV. APPLICATION TO DISORDERED ELASTIC SYSTEMS

### A. FRG equations and universality to one loop

#### 1. The model

Let us consider an elastic system of internal dimension  $D$  embedded in a disordered medium. It is described by a single-component displacement field  $u^x$  ( $x$  is the internal variable), which is either the height function for an interface problem, or the continuous deformation field for periodic systems. Applications of such models to physical systems have been discussed in recent reviews to which we refer the reader [35,34]. The aim of this section is to apply our EMRG method to this class of problems and show how one computes correlation functions and prove universality. The model usually studied is defined by the energy where it reads

$$H(u) = \int_x \left( \frac{c}{2} |\nabla u^x|^2 - W(x, u^x) + \frac{m^2}{2} |u^x|^2 \right), \quad (56)$$

where a short-distance cutoff (e.g., in momentum space) is implicit. The elastic constant is set to  $c=1$  here, and the mass to  $m=0$ , its effect will be studied in Sec. IV C. The disordered potential  $-W(x, u)$  is a random variable that has the following properties: (i)  $W(x, u)=0$ ; (ii) the potential at different  $x$  are uncorrelated; (iii) the distribution of  $W(x, u)$  is translationally invariant in  $u$  space. Its cumulants read

$$\overline{W(x_1, u_1), \dots, W(x_N, u_N)^c} = \delta(x_1 - x_2), \dots, \delta(x_1 - x_N) S^{(N)}(u_1, \dots, u_N), \quad (57)$$

where the symmetric functions  $S^{(N)}$  satisfy  $S^{(N)}(u_1, \dots, u_N) = S^{(N)}(u_1 + u, \dots, u_N + u)$  for any  $u$ . In particular, the second cumulant is denoted by  $R(u - u') = S^{(2)}(u, u')$ . Two cases are usually considered when studying a domain-wall interface in disordered spin models: random-field disorder that yields long-range correlations  $R(u) \sim |u|$  at large  $u$  (see [24] and Sec. IV C below) and random-bond disorder yielding short-range correlations  $R(u) \rightarrow 0$  at large  $u$ . When studying a charge-density wave or a vortex lattice, one is led to consider a periodic disorder [27,28]. In that case the cumulants have the periodicity of the system considered (a wavelength for a density wave and a lattice constant for a vortex lattice). We assume parity symmetry  $S^{(N)}(-u_1, \dots, -u_N) = S^{(N)}(u_1, \dots, u_N)$ . This problem is usually studied by introducing  $n$  replicas  $\phi_a^x$ ,  $a = 1, \dots, n$ , of  $u^x$  and by averaging over the disorder. It yields the action

$$\mathcal{S}(\phi) = \int_x \left[ \frac{1}{2T} \sum_a |\nabla \phi_a^x|^2 - \sum_{N \geq 2} \frac{1}{N! T^N} \sum_{a_1, \dots, a_N} S^{(N)}(\phi_{x}^{a_1}, \dots, \phi_{x}^{a_N}) \right]. \quad (58)$$



Thus the bare  $\mathcal{S}$  has the general form (3) with

$$G_{ab}^q = \frac{T}{q^2} c\left(\frac{q^2}{2\Lambda^2}\right) \delta_{ab},$$

$$\mathcal{V}(\phi) = \int_x U(\phi_x),$$

$$U(\phi) = -\frac{1}{2T^2} \sum_{ab} R(\phi^a - \phi^b) - \frac{1}{3!T^3} \sum_{abc} S^{(3)}(\phi^a, \phi^b, \phi^c) + \dots$$

We will consider any cutoff function  $c(s)$  such that  $c(0) = 1$  with no loss of generality and with  $c'(0) = 0$  for convenience (see below).

Direct perturbation theory can be performed on this model [18,24]. One can show that it has a well-defined  $T=0$  limit (see Appendix D). Furthermore, at  $T=0$  this perturbation theory—at least at the naive level—is in fact trivial, i.e., the disorder average of any observable is identical to its value in linear random force model, as shown in the Appendix D. Within the exact RG one can in fact escape this well-known dimensional reduction phenomenon, since, as we will see below, the flowing disorder becomes nonanalytic. As shown below, it yields nontrivial results for correlations.

## 2. RG analysis

We now, use the exact RG method introduced above. For now, we use the RG equations based on the multilocal expansion of  $\mathcal{V}$  while the other method in terms of  $\hat{\mathcal{V}}$  (explained in Sec. III A and Appendix C 1) will be used in Sec. IV C. The method with  $\mathcal{V}$  turns out to be more convenient to analyze finite  $T$  effects. The  $l$  dependence of the Gaussian part is implemented by the choice

$$(G_l)_{ab}^q = T(\bar{G}_l)_{ab}^q = \frac{T}{q^2} c\left(\frac{q^2}{2\Lambda_l^2}\right) \delta_{ab}, \quad (59)$$

where  $\Lambda_l = \Lambda e^{-l}$ . This choice is particularly convenient here since there is no correction to any order to the connected quadratic part (statistical tilt symmetry [25]). The flowing interaction functional  $\mathcal{V}_l(\phi)$  remains translationally and parity invariant in  $x$  space. Since translation invariance in the  $u$  space is conserved, its local part  $U_l(\phi)$  remains of the form of the bare  $U_0$ . In order to obtain fixed points it is convenient to define a rescaled dimensionless temperature  $\tilde{T}_l = T\Lambda_l^{D-2}$  and rescaled functions

$$\begin{aligned} \tilde{U}_l(\phi) &= \Lambda_l^{-D} U_l(\phi) \\ &= -\frac{1}{2\tilde{T}_l^2} \sum_{ab} \tilde{R}_l(\phi^a - \phi^b) \\ &\quad - \frac{1}{3!\tilde{T}_l^3} \sum_{abc} \tilde{S}^{(3)}(\phi^a, \phi^b, \phi^c) + \dots, \end{aligned}$$

$$\tilde{R}_l(u) = \Lambda_l^{D-4} R_l(u),$$

$$\tilde{S}_l(u_1, \dots, u_N) = \Lambda_l^{DN-2N-D} S_l^{(N)}(u_1, \dots, u_N).$$

The general RG Eq. (27) for  $U_l(\phi)$  implies a set of flow equations for the rescaled cumulants  $\tilde{R}_l(u)$ ,  $\tilde{S}_l^{(N)}(u_1, \dots, u_N)$  (since the former is in fact a set of equations for functions of a  $n$ -dimensional vector  $\phi$  for any  $n$ ). The rescalings above have been chosen such that these rescaled functions flow to fixed functions denoted  $\tilde{R}^*(u)$ ,  $\tilde{S}^{(N)*}(u_1, \dots, u_N)$  independent of  $T$ .

An important property of the theory (58) is that it admits a well-defined  $T=0$  limit, at least at the perturbative level. This can be seen either by examination of the diagrammatics (all negative powers of  $T$  in the perturbative calculation of observables are in factor of a positive power of  $n$ , see Appendix D), or in the  $T=0$  dynamics [15]. Similarly, there exists a well-defined  $T=0$  limit of the set of flow equations for the cumulants. For small  $\epsilon = 4 - D$ , this complicated set of coupled equations can be organized in powers of  $\epsilon$ . Specifically, one finds that  $\tilde{R}^* = \mathcal{O}(\epsilon)$  while  $\tilde{S}^{(N)*} = \mathcal{O}(\epsilon^N)$  for  $N \geq 3$ . This can be seen on the schematic structure (ordered in  $U$  and  $T$ ) of the flow equations obeyed by the rescaled  $\tilde{R}_l$ ,  $\tilde{S}_l^{(N)}$ , which can be read off from Appendix C4:

$$\partial U = TU + T^2 U^2 e^T + T^3 U^3 e^T + \dots \quad (60)$$

The two first terms reproduce Eq. (26) since  $\partial G \propto T$ , while the third term mimics the  $\mathcal{O}(U^3)$  in the  $\beta$  function. Its three  $U$  vertices must be linked by at least three propagators because of the constraint of locality. Substituting symbolically  $\tilde{U} = \tilde{R}/T^2 + \tilde{S}/T^3$ , where we restrict to the two lowest cumulants, one finds

$$\partial \tilde{R} = \epsilon \tilde{R} + \tilde{S}|_2 + \tilde{R}^2|_2 + \tilde{R} \tilde{S} e^T/T|_2 + \tilde{R}^3 e^T/T|_2,$$

$$\partial \tilde{S} = (6 - 2D) \tilde{S} + \tilde{R} \tilde{S}|_3 + \tilde{R}^3|_3 + \tilde{S}^2 e^T/T|_3 + \tilde{R}^2 \tilde{S} e^T/T|_3,$$

where we have denoted projections on two and three replica parts. All terms containing  $1/T$  vanish after these projections since a well-defined  $T=0$  limit exists. We have discarded terms, such as  $\partial_l \tilde{S} = T \tilde{R}^2|_3$ , which (formally) vanish at  $T=0$ .

One sees immediately on these equations that the fixed  $\tilde{R}^* = \mathcal{O}(\epsilon)$  while  $\tilde{S}^* = \mathcal{O}(\epsilon^3)$ . This can be generalized by noting that the lowest-order (in  $\epsilon$ ) correction to  $S^{(N)}$  is of the form  $\tilde{R}^N|_N$ , thus,  $\tilde{S}^{(N)*} = \mathcal{O}(\epsilon^N)$ . To  $\mathcal{O}(\epsilon^2)$  at  $T=0$  we thus need

$$\partial \tilde{R} = \epsilon \tilde{R} + \tilde{R}^2|_2 + \tilde{S}|_2 + \tilde{R}^3 e^T/T|_2, \quad (61)$$

$$\partial \tilde{S} = (6 - 2D) \tilde{S} + \tilde{R}^3|_3. \quad (62)$$

In this paper, we simply perform the  $\mathcal{O}(\epsilon)$  calculation to which we now turn, for which consideration of two replica terms is sufficient.

We perform the analysis in the  $T=0$  limit as explained above. The propagator can be expressed in terms of dimensionless quantities as  $G_l^x = \tilde{T}_l \int_q [c(q^2/2)/q^2] e^{iq\Lambda_l x}$ . At finite

$T$ , the exponentials of propagators in Eq. (26) would reduce to one asymptotically at large  $l$ . This is also true in the  $T=0$  limit for any  $l$ . It is thus *a priori* unnecessary to include higher number of loops within order  $U^2$ .

Denoting by  $\tilde{M}_l(\phi)$  the two-replica term contained in the local operator  $U$

$$\tilde{M}_l(\phi) = -\frac{1}{2} \sum_{ab} \tilde{R}_l(\phi^a - \phi^b), \quad (63)$$

the flow equation to one-loop Eq. (28) (using the change of variables  $l' \rightarrow l-l'$  yields

$$\begin{aligned} \partial_l \tilde{M}_l(\phi) &= (4-D) \tilde{M}_l(\phi) \\ &\quad - \frac{1}{2} \int_0^l dl' \mathcal{K}_{l'} \sum_{ab} [\partial_{ab} \tilde{M}_{l-l'}(\phi)]^2|_{2-rep}, \end{aligned} \quad (64)$$

where the kernel responsible for the retarded nature of the flow is

$$\begin{aligned} \mathcal{K}_{l'} &= 4J_{l,l-l'}^D \Lambda_{l-l'}^{2\epsilon} \Lambda_l^{-\epsilon} \\ &= 2e^{-(6-D)l'} \int_q c' \left( \frac{q^2}{2} \right) c' \left( \frac{q^2}{2} e^{-2l'} \right). \end{aligned} \quad (65)$$

Since  $c'(u)$  is typically peaked around  $u \sim 1$  and decreases fast at infinity, one sees in Eq. (65) that the range of the kernel  $\mathcal{K}_l$  is also of order one and can be made as small as desired by choosing narrow enough cutoff functions. The above RG Eq. (64) involves computing the contraction

$$\begin{aligned} &\sum_{ab} [\partial_a \partial_b \tilde{M}(\phi)]^2 \\ &= \sum_{ab} [\tilde{R}''(\phi^a - \phi^b)^2 - 2\tilde{R}''(0) \tilde{R}''(\phi^a - \phi^b)] \\ &\quad + \sum_{abc} \tilde{R}''(\phi^a - \phi^c) \tilde{R}''(\phi^c - \phi^b), \end{aligned}$$

where we have used  $\partial_a \partial_b \tilde{M}(\phi) = \delta_{ab} \sum_c \tilde{R}''(\phi^a - \phi^c) - \tilde{R}''(\phi^a - \phi^b)$ . The last sum being a three replica term, it does not enter the equation for  $\tilde{R}$  (it is a correction to  $S$  proportional to  $T$ ), which reads

$$\begin{aligned} \partial_l \tilde{R}_l(u) &= \epsilon \tilde{R}_l(u) + \int_0^l dl' \mathcal{K}_{l'} \left( \frac{1}{2} \tilde{R}_{l-l'}''(u)^2 \right. \\ &\quad \left. - \tilde{R}_{l-l'}''(0) \tilde{R}_{l-l'}''(u) \right). \end{aligned} \quad (66)$$

Let us first study the case of periodic elastic systems, with  $\tilde{R}_l(\phi)$  periodic of period one. Taking the large  $l$  limit we find the fixed-point equation:

$$0 = \epsilon \tilde{R}^*(u) + \left( \int_0^{+\infty} dl' \mathcal{K}_{l'} \right) \left( \frac{1}{2} \tilde{R}^{*''}(u)^2 - \tilde{R}^{*''}(0) \tilde{R}^{*''}(u) \right). \quad (67)$$

It is now easy to see that the factor  $\mathcal{K} = \int_0^\infty dl' \mathcal{K}_{l'}$  in Eq. (67), which *a priori* depends on the dimension of space and of the whole arbitrary cutoff function  $c(s)$ , becomes universal in  $D=4$ . Indeed,

$$\begin{aligned} \mathcal{K} &= 2S_D 2^{-\epsilon/2} \int_0^\infty ds s^{-\epsilon} c'(s) \int_0^s dt t^\epsilon c'(t) \\ &= 2S_4 \int_0^\infty ds c'(s)(c(s)-1) + O(\epsilon) = S_4 + O(\epsilon), \end{aligned} \quad (68)$$

where we used the new variables  $s = q^2/2$ , then  $t = s e^{-2l}$ , kept only the lowest order in  $\epsilon$ , and used  $c(0) = 1$ . We denote by  $S_D$  the surface of the unit sphere in  $D$  dimensions divided by  $(2\pi)^D$ . Thus, to lowest order in  $\epsilon$ , the FRG equation *does not depend on the cutoff procedure*. It coincides with the fixed-point equation obtained [14,26] from Wilson's momentum shell renormalization.

In Appendix C 3, we also mention the result of a two-loop calculation of the beta function in our exact renormalization framework.

The solution to Eq. (67) is known to be the one-periodic function defined by

$$\tilde{R}^*(u) = \frac{\epsilon}{72S_4} \left( \frac{1}{36} - u^2(1-u)^2 \right) \quad (70)$$

for  $0 < u < 1$ . This fixed point function is nonanalytic, which is an important and unusual feature. It was argued in [14] that this nonanalyticity appears at a finite scale. This scale  $R_c = e^{l_c}/\Lambda$  can be identified with the Larkin length at which metastability and glassiness appears. Taking the fourth derivative at  $u=0$  of Eq. (66) yields a closed retarded equation for  $\tilde{R}_l^{iv}(0)$

$$\partial_l \tilde{R}_l^{iv}(0) = \epsilon \tilde{R}_l^{iv}(0) + 3 \int_0^l dl' \mathcal{K}_{l'} \tilde{R}_{l-l'}^{iv}(0)^2. \quad (71)$$

In the limit of narrow cutoffs, the equation becomes local and  $\tilde{R}_l^{iv}(0)$  diverges at a finite scale. One can show that this feature persists in the nonlocal equation.

The case of an interface (i.e., a directed polymer for  $D=1$ ) corresponds to another fixed point where one must re-scale the function  $\tilde{R}_l(u)$  as follows:

$$\tilde{R}_l(u) = e^{4\zeta l} r_l(u e^{-\zeta l}), \quad (72)$$

and we must now determine  $\zeta = \mathcal{O}(\epsilon)$  such that  $r_l(v)$  converges to a fixed point  $r^*(v)$ . Inserting Eq. (72) into Eq. (66) yields

$$\begin{aligned} \partial_l r_l(v) &= (\epsilon - 4\zeta) r_l(v) + \zeta v r_l'(v) \\ &+ \int_0^l dl' \mathbf{K}_{l'} e^{-4\zeta l'} \left( \frac{1}{2} r_{l-l'}''(v e^{\zeta l'})^2 \right. \\ &\left. - r_{l-l'}''(0) r_{l-l'}''(v e^{\zeta l'}) \right). \end{aligned} \quad (73)$$

Although the kernel has been modified, this does not affect the results for the fixed point to lowest order in  $\epsilon$ . The fixed point equation reads

$$\begin{aligned} 0 &= (\epsilon - 4\zeta) r^*(v) + \zeta v r^{*'}(v) \\ &+ S_4 \left( \frac{1}{2} r^{*''}(v)^2 - r^{*''}(0) r^{*''}(v) \right), \end{aligned} \quad (74)$$

and is thus universal, independent of  $c(s)$ . This shows that  $\zeta$ , which, as shown below, is the roughness exponent, is universal to one loop. It will be studied below for the random-field case and in the case of short-range disorder it is thus equal to  $\mathcal{O}(\epsilon)$  to the values given in [14].

### 3. Correlation function

Let us now compute the two-point correlation function at  $T=0$  using Eq. (12). To lowest order in  $\epsilon$ , it is sufficient to use the first-order formula (30). The bare Gaussian part  $G^q$  vanishes at  $T=0$ . We thus get

$$\begin{aligned} C_{ab}^q &= -\frac{T^2}{q^4} c \left( \frac{q^2}{2\Lambda^2} \right)^2 \partial_a \partial_b U_l(\phi) |_{\phi=0} \\ &= -\frac{R_{l=\ln(q/\Lambda)}''(0)}{q^4} \\ &= -\frac{\bar{R}^{*''}(0)}{q^D} = \frac{\epsilon}{S_4} \frac{1}{36} q^{-D}, \end{aligned} \quad (75)$$

where we used that  $\bar{R}_l$  converges to the fixed point  $\bar{R}^*$  and small  $q$  such that  $c(q^2/2\Lambda^2) = 1$ . In real space, it yields logarithmic growth of the displacements with a universal prefactor

$$\overline{\langle (u_x - u_0)^2 \rangle} = \frac{\epsilon}{18} \ln |\Lambda x|. \quad (76)$$

In the case of short-range disorder (e.g., random bond for Ising interface) one gets instead

$$C_{ab}^q = -\frac{R_{l=\ln(q/\Lambda)}''(0)}{q^4} = -e^{2\zeta l} \Lambda_l^\epsilon \frac{r^{*''}(0)}{q^D} \sim \frac{1}{q^{D+2\zeta}}. \quad (77)$$

This yields  $\overline{\langle (u_x - u_0)^2 \rangle} \sim |x|^{2\zeta}$  with a universal roughness exponent:

$$\zeta = 0.208\,298\,04\epsilon \quad (78)$$

to lowest order [14,15] as well as a nonuniversal amplitude [since the FRG fixed-point Eq. (74) is invariant under  $r^*(v) \rightarrow \lambda^4 r^*(v/\lambda)$  and, contrarily to the periodic case, nothing here fixed the scale].

We can now investigate in more detail the structure of the asymptotic flow of the various higher-order interactions (three replica terms and higher, as well as bilocal interaction and more). Although this is beyond the scope of this paper, such an analysis is in principle necessary for consistency, i.e., to ensure the existence of a global fixed point (for all interactions) and the validity of the result to  $\mathcal{O}(\epsilon)$ . We sketch it here for the periodic case  $\zeta=0$ , generalizations to interfaces being simple.

We start with estimating the higher cumulants of the renormalized disorder, i.e., the higher replica components of the local interaction  $U_l$ . To lowest order in  $R$ , the correction to  $S_l^{(N)}$  is proportional to  $R_l^N$  and takes the schematic form

$$\delta \bar{S}^{(N)} = -(2N + D - DN) \bar{S}^{(N)} + \bar{R}^{*N}. \quad (79)$$

Graphically, the diagram is made of one loop. We dropped the numerous higher-order terms in  $\epsilon$  coming from contractions of various other cumulants than  $R$ . One finds that the fixed point  $\bar{S}^{(N)*}$  takes the following form to lowest order in  $\epsilon$ :

$$\begin{aligned} &\sum_{a_1, \dots, a_N} \bar{S}^{(N)*}(\phi^{a_1}, \dots, \phi^{a_N}) \\ &= c_{N,D} \left( \text{Tr}(W^N) - \sum_{a_1, \dots, a_N, b} \bar{R}_{ba_1}^{*''}, \dots, \bar{R}_{ba_N}^{*''} \right), \end{aligned}$$

where  $W_{ab}(\phi) = \delta_{ab} \sum_c \bar{R}_{ac}^{*''} - \bar{R}_{ab}^{*''}$ ,  $\bar{R}_{ab}^{*''}$  denotes  $\bar{R}^{*''}(\phi_a - \phi_b)$ , and  $c_{N,D}$  is some number depending on the cutoff procedure. The last term in the trace has been subtracted since the product of the  $N$   $\delta$ 's is a  $N+1$  replica term. For instance, the third cumulant is of order  $\mathcal{O}(\epsilon^3)$  and reads

$$\begin{aligned} &\bar{S}^{(3)*}(u_1, u_2, u_3) \\ &= c_{3,D} \text{Sym}_{u_1, u_2, u_3} [\bar{R}^{*''}(u_2 - u_3) \bar{R}^{*''}(u_3 - u_1) \\ &\quad \times \bar{R}^{*''}(u_1 - u_2) - 3 \bar{R}^{*''}(u_1 - u_2) \\ &\quad \times \bar{R}^{*''}(u_1 - u_3)^2 + 3 \bar{R}^{*''}(0) \bar{R}^{*''}(u_2 - u_1) \\ &\quad \times \bar{R}^{*''}(u_3 - u_1)], \end{aligned} \quad (80)$$

where  $c_{3,4}$  is computed in [16] and reads

$$c_{3,4} = \frac{S_4}{12} \int_0^\infty ds \frac{[1 - c(s)]^3}{s^2}. \quad (81)$$

Now we check that the bilocal part has a well-defined fixed point. Its expression is given by Eq. (24), where at  $T=0$ , the exponentials should be expanded at most to first order. The zeroth-order term yields three replica terms, while the first-order term yields two replica terms (as well as a

correction proportional to  $T$  to three replica terms that we can discard at  $T=0$ ). Thus we get

$$V_l(\phi_1, \phi_2, q) = -\frac{\Lambda_l^\epsilon}{2T^2} \sum_{ab} \tilde{V}_l^{(2,2)}\left(\frac{q}{\Lambda_l}\right) - \frac{\Lambda_l^{2\epsilon-2}}{6T^3} \sum_{abc} \tilde{V}_l^{(2,3)}\left(\frac{q}{\Lambda_l}\right),$$

where we have explicitly separated two and three replica terms, respectively,

$$\tilde{V}^{*(2,2)}(\tilde{q}) = f(\tilde{q}) \sum_{ab} \tilde{R}_{ab}^{*n1} \tilde{R}_{ab}^{*n2}, \quad (82)$$

$$\tilde{V}^{*(2,3)}(\tilde{q}) = -3 \frac{c(\tilde{q}^2/2) - 1}{\tilde{q}^2} \sum_{abc} \tilde{R}_{ab}^{*r1} \tilde{R}_{ac}^{*r2}, \quad (83)$$

where we have defined

$$f(\tilde{q}) = \frac{1}{2} \int_k \left\{ \frac{1}{k^2(k+\tilde{q})^2} \left[ c\left(\frac{k^2}{2}\right) - 1 \right] \left[ c\left(\frac{(k+\tilde{q})^2}{2}\right) - 1 \right] - \text{idem } q=0 \right\}, \quad (84)$$

where  $\tilde{R}_{ab}^{r1}$  stands for  $\tilde{R}'(\phi_a^1 - \phi_b^1)$ , etc. We have discarded a term proportional to  $\tilde{R}^{**}(0) \sum_{ab} (\tilde{R}_{ab}^{*n1} + \tilde{R}_{ab}^{*n2})$  since it yields vanishing contribution to the action. The bilocal term thus has a scale invariant fixed form of order  $\epsilon^2$  and is a well-defined function of  $\tilde{q} = q/\Lambda_l$  with no divergences. This shows consistency of our procedure to this order.

More generally, we conjecture, and verify here to lowest order, that there is a fixed asymptotic form for all multilocal interactions  $V^{(p)}$  that can be explicitly written as a sum of properly rescaled multireplica terms as

$$V_l^{(p)}(\phi_1, \dots, \phi_p, x_1, \dots, x_p) = \Lambda_l^{Dp} \sum_{c \geq 2} \frac{\Lambda_l^{c(2-D)}}{c! T^c} \times \sum_{a_1, \dots, a_c} \tilde{V}_l^{(p,c)}(\{\phi_\alpha^{a_i}\}_{i=1, \dots, p}, \Lambda_l x_1, \dots, \Lambda_l x_p), \quad (85)$$

where the number of replicas  $c$  corresponds graphically to the number of connected components. The consistency of the method demands that the  $\tilde{V}_l$  flow to well-defined fixed points, perturbative in  $\epsilon$ . It is indeed natural to conjecture that in this theory there is no wave-function renormalization.

We can now come back to the calculation of the correlation function. Although for convenience we have computed  $C_{ab}^q$  from the theory with  $l = \ln(q/\Lambda)$ , this is unnecessary. As discussed above, the existence of a fixed point with well-

defined functions of  $\tilde{q} = q/\Lambda_l$  implies that the  $\epsilon$  expansion of  $C_{ab}^q$  at fixed  $q \ll \Lambda$  should be of the form

$$C_{ab}^q = \frac{1}{q^4} \Lambda_l^\epsilon \tilde{\Sigma}(\tilde{q}), \quad (86)$$

where the dimensionless self-energy  $\tilde{\Sigma}(\tilde{q})$  depends only on  $\tilde{q}$  and  $\epsilon$ . Since  $C_{ab}^q$  is  $l$  independent, it implies, taking the derivative, that  $\tilde{\Sigma}(\tilde{q}) = C_\epsilon \tilde{q}^\epsilon$  and thus

$$C_{ab}^q = C_\epsilon q^{-D}, \quad (87)$$

$$C_\epsilon = \frac{2\pi^2}{9} \epsilon, \quad (88)$$

to lowest order in  $\epsilon$ , as in [27]. The form (87) should be valid to all orders in  $\epsilon$  if the hypothesis about the fixed point formulated above Eq. (85) are satisfied.

## B. FRG at finite temperature

### 1. Renormalization equations

Since in Eq. (26) the terms in the exponentials containing the temperature go to zero as  $\Lambda_l^{D-2}$  one can first study the effect of temperature, compared to  $T=0$  by looking at the linear term. Up to order  $TR$  and  $R^2$  (i.e., to one loop) the RG equation thus reads

$$\partial_l \tilde{R}_l(u) = \epsilon \tilde{R}_l(u) + \hat{T}_l \tilde{R}_l''(u) + \int_0^l dl' \mathcal{K}_{l'} \left( \frac{1}{2} \tilde{R}_{l-l'}''(u)^2 - \tilde{R}_{l-l'}''(0) \tilde{R}_{l-l'}''(u) \right), \quad (89)$$

with

$$\hat{T}_l = -\partial G_l^{x=0} = -T \int_q \Lambda_l^{-2} c' \left( \frac{q^2}{2\Lambda_l^2} \right) = 2TS_4 \Lambda_l^2 \int_{s>0} c(s) + \mathcal{O}(\epsilon).$$

In the case of the sharp cutoff, this equation has been studied in [29,30]. It was found that at fixed  $u$ ,  $\tilde{R}_l(u)$  converges to  $\tilde{R}^*(u)$  but that temperature rounds the cusp of the  $T=0$  solution in a boundary layer of size  $u \sim \hat{T}_l$ . As in [30] we look for a solution of the form

$$\tilde{R}_l(0) - \tilde{R}_l(u) = -\tilde{R}_l''(0) \frac{u^2}{2} - \mathcal{K} \frac{\hat{T}_l^3}{\epsilon^2 \kappa^2} H_l \left( \frac{u \epsilon \kappa}{\hat{T}_l} \right). \quad (90)$$

Here,  $\kappa$  (of order  $\epsilon^0$ ) is defined by  $\epsilon \kappa = \tilde{R}^{*m}(0^+)$  and thus  $H^{*iv}(0) = -1$ . One has also  $H_l(0) = H_l'(0) = 0$ . Injecting Eq. (90) into Eq. (89) and identifying the leading order in  $\hat{T}_l$ , one gets

$$\frac{x^2}{2} = H^{*n}(x) + \frac{1}{2} \int_{l>0} K_l e^{4l} H^{*n}(x e^{-2l})^2, \quad (91)$$

with  $\int_{l>0} K_l = 1$ . This equation can be solved iteratively in  $n$  for  $H^*(x) = \sum_{n>0} a_n x^{2n}/(2n)!$ . One has  $a_1 = 1$ ,  $a_2 = -3$  but higher  $a_n$ 's are nonuniversal. The large  $x$  behavior of  $H^*$  is universal and given by  $H^*(x) \sim x$ . In the case of the sharp cutoff one recovers [30]

$$H^{*n}(x) = \sqrt{1+x^2} - 1. \quad (92)$$

This result should be further examined by consideration of consistency within higher-loop corrections, which goes beyond this paper.

The most important result of the section is that the following relation between the finite-temperature solution and the  $T=0$  solution:

$$\lim_{l \rightarrow +\infty} \hat{T}_l \tilde{R}_l^{\text{iv}}(0) = C \bar{R}_l^{*n}(0^+)^2, \quad (93)$$

where  $C = S_4$  holds irrespective of the cutoff function, and thus is determined by the  $T=0$  fixed point. This property will be used below.

## 2. Calculation of universal susceptibility fluctuations

It was noted recently that a signature of glassy behavior in a disordered system was the large sample-to-sample fluctuations of the response to external perturbations [31,33,34]. These are described by the following susceptibility:

$$\chi = \frac{1}{T} \frac{1}{L^D} \int_{xy} (\langle \partial_\alpha u_x \partial_\alpha u_y \rangle - \langle \partial_\alpha u_x \rangle \langle \partial_\alpha u_y \rangle) \quad (94)$$

in a finite system of size  $L$ , which measures the response in a given sample to a field coupling to  $\nabla u$  (e.g., the tilt or compression a flux lattice, or the compression response). The  $\langle X \rangle$  denotes the thermal averages in a given sample. These have been studied in connection with mesoscopic behavior of disordered systems [36]. Here we have considered only the trace of the response tensor (extension being straightforward). To perform the calculation in the replicated theory we define

$$C_{ab}^{\alpha\beta} = \frac{1}{T} \frac{1}{L^D} \int_{xy} \langle \partial_\alpha u_x^a \partial_\beta u_y^b \rangle, \quad (95)$$

$$C_{abcd} = \frac{1}{T^2} \frac{1}{L^{2D}} \int_{xyzt} \langle \nabla u_x^a \cdot \nabla u_y^b \nabla u_z^c \cdot \nabla u_t^d \rangle. \quad (96)$$

We now compute, respectively, the first and second moment of the sample-to-sample fluctuations of the susceptibility. They read

$$\bar{\chi} = C_{aa} - C_{ab} = 1, \quad (97)$$

$$\overline{\chi^2} = C_{aabb} + C_{abcd} - 2C_{aabc}, \quad (98)$$

where  $\bar{\chi}$  denote disorder averages and  $a, b, c, d$  take values all distinct from each other. Note the well-known property [25] that the average susceptibility is identical to the susceptibility of the pure system. We now compute  $C_{abcd}$  to lowest order in  $\epsilon$ . Only zero and one loop graphs involving, respectively,  $R_l''(0)$  and  $R_l'''(0)$  contribute. Interestingly, due to the quadratic nature of the term proportional to  $R_l''(0)$  the zero loop graphs cancel in  $\overline{\chi^2}$ , as can be easily seen since  $C_{abcd} = C_{ab}^2 + 2\sum_{\alpha\beta} (C_{ab}^{\alpha\beta})^2$  for any Gaussian theory (performing the Wick contractions). One is left with

$$C_{abcd} = (n \delta_{abcd} - (\delta_{bcd} + \delta_{cda} + \delta_{dab} + \delta_{abc}) \\ + (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc})) A R_l'''(0),$$

where  $A = L^{-2D} \int_w (\int_{xy} \nabla G^{x-w} \cdot \nabla G^{y-w})^2$ , and thus

$$\overline{(\Delta\chi)^2} \equiv \overline{\chi^2} - \bar{\chi}^2 = A R_l^{\text{iv}}(0) \sim R_l^{\text{iv}}(0) C L^{4-D} \quad (99)$$

for a system of finite-size  $L$  [see also [34] for a similar result in straight perturbation theory]. Note that one can equivalently study the perturbation of an infinite system (i.e.,  $L \rightarrow \infty$  first) by a periodic external field of wave-vector  $q_{\text{ext}}$ . In that case,  $A = q_{\text{ext}}^{D-4}$ . Thanks to the exact RG equations at finite  $T$  and substituting  $l = \ln L$ , we obtain the mesoscopic susceptibility fluctuations at low temperature as

$$\overline{(\Delta\chi)^2} = C' \frac{L^\theta}{T}, \quad (100)$$

where  $\theta = D - 2 + 2\zeta$  is the energy fluctuation exponent and  $C' = \mathcal{O}(\epsilon^2)$  for a periodic system ( $\zeta = 0$ ) and  $C' = \mathcal{O}(\epsilon^{4/3}) \sigma^{2/3}$  for an interface in random-field disorder (see Sec. IV C).

This result, derived here through exact FRG calculation, is consistent with the droplet picture [37]. Indeed, the second moment of the susceptibility fluctuations is dominated by the rare configurations of disorder (of probability  $p_{\text{deg}} \ll 1$ ) with two almost degenerate [i.e., within  $\mathcal{O}(T)$  in energy] ground states as follows:

$$\overline{(\Delta\chi)^2} \sim p_{\text{deg}} (\delta\chi_{\text{typ}})^2, \quad (101)$$

where  $p_{\text{deg}} \sim T/L^\theta$  and the typical fluctuation is  $\delta\chi_{\text{typ}} \sim T^{-1} L^{-D} L^{2D-2+2\zeta}$  from Eq. (94). One thus recovers the above result since  $\theta = D - 2 + 2\zeta$ .

## C. Interface in a biased random field and toy model

In this section we study the model (56) in the presence of a mass term  $m > 0$ , which confines the fluctuations of the displacement  $u_x$ . We consider two cases (i) random-field disorder (ii) periodic disorder. A physical realization of (i) consists in a domain wall separating the  $\pm$  phases in a ferromagnet, submitted to a random-magnetic field, as modeled, e.g., by the random-field Ising model. The magnetic energy of the interface, assumed without overhangs and represented in Fig. 3, is

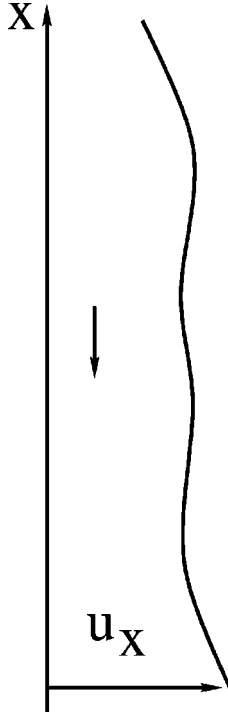


FIG. 3. Interface between two domains with opposite magnetization in a ferromagnet.

$$\mathcal{E}(u) = 2 \int d^D x \int_0^{u_x} du' h(u', x). \quad (102)$$

Thus, the effect of the mass term corresponds to applying an additional field gradient  $h(u, x) \rightarrow h(u, x) + m^2 u/2$ . Note that this field gradient can either stabilize ( $m^2 > 0$ ) or destabilize ( $m^2 < 0$ ) the domain wall. We will study the approach to the critical value  $m^2 \rightarrow 0^+$ . The case (ii) is of interest when studying the competition between disorder and, e.g., a periodic potential. In the phase where the periodic potential is relevant, it is natural to approximate it by replacing it by an harmonic well (see, e.g., [38]).

Here we examine only ground-state properties (zero temperature). We show that the disorder-induced fluctuations of the displacement  $u_x$  is described, as  $m \rightarrow 0$  by a universal scaling function of the form

$$\overline{u_q u_{-q}} = m^{-\alpha} F[cq^2/m^2], \quad (103)$$

which we determine to lowest order in  $\epsilon = 4 - D$ . Note that  $c$  can be measured from the thermal connected correlation, which is unchanged by disorder for models like Eq. (56) which possess the statistical tilt symmetry.

### 1. RG equations in the presence of a mass

In this section, it is more convenient to use the RG equation resulting from the multilocal expansion on  $\hat{\mathcal{V}}$ , which is local in  $l$  to this order. Since we are studying  $T=0$ , we set  $\hat{R}_l = R_l$  in the following. The RG equation reads

$$\partial_l R_l(u) = J_l \left( \frac{1}{2} R_l''(u)^2 - R_l''(0) R_l''(u) \right), \quad (104)$$

$$J_l = -2 \int_q \partial \bar{G}_l^q \bar{G}_l^q. \quad (105)$$

Using the action  $S_l$ , the  $T=0$  correlation function reads to lowest order in  $R$

$$\overline{u_q u_{-q}} = -R_l''(0) \left( \frac{c(q^2/2\Lambda^2)}{c q^2 + m^2} \right)^2. \quad (106)$$

It is easy to transform Eq. (104) into the RG equation in the absence of a mass. Using the change of variable

$$R_l(u) = c^2 \Lambda_{t(l)}^{\epsilon-4\zeta} \bar{R}_{t(l)}(u \Lambda_{t(l)}^\zeta), \quad (107)$$

one finds that  $\bar{R}_l$  satisfies the  $m=0$  flow equation

$$\begin{aligned} \partial_l \bar{R}_l(x) = & (\epsilon - 4\zeta) \bar{R}_l(x) + \zeta x \bar{R}_l'(x) \\ & + S_D \left( \frac{1}{2} \bar{R}_l''(x)^2 - \bar{R}_l''(0) \bar{R}_l''(x) \right), \end{aligned}$$

provided that  $t(l)$  satisfies

$$\Lambda_{t(l)}^{-\epsilon} \frac{dt}{dl} = J_l \frac{c^2}{S_D}, \quad (108)$$

which is integrated into

$$\frac{e^{\epsilon t} - 1}{\epsilon} = \int_0^\infty q^{D-1} dq \frac{c(q^2/2)^2 - c(q^2 e^{2l}/2)^2}{(q^2 + \alpha)^2}, \quad (109)$$

with  $\alpha = (m^2/c\Lambda^2)$ . The function  $t(l)$  is increasing and bounded. Its limit  $t(+\infty) = t_\infty$  for  $D < 4$  is given by

$$\frac{e^{\epsilon t_\infty} - 1}{\epsilon} = \int_0^\infty q^{D-1} dq \frac{c(q^2/2)^2}{(q^2 + \alpha)^2}, \quad (110)$$

and it diverges for  $m \rightarrow 0$  as

$$e^{\epsilon t_\infty} \sim \epsilon \int_0^\infty q^{D-1} dq \frac{1}{(q^2 + \alpha)^2} \sim \alpha^{-\epsilon/2} \left( 1 - \frac{\epsilon}{2} \right) \frac{(\epsilon\pi/2)}{\sin(\epsilon\pi/2)}. \quad (111)$$

In  $D=4$  it diverges as

$$t_\infty \sim \frac{1}{2} \ln \left( \frac{1}{\alpha} \right). \quad (112)$$

We now distinguish the two cases.

### 2. Random field

In that case, the correlations of the potential are  $\overline{[W(r, u) - W(r', u')]^2} = -2 \delta^D(r - r') R(u - u')$  with [from Eq. (102)],  $R(u) \sim -\sigma|u|$  at large  $u$ . In the massless case, it is known that the FRG to one loop reproduces the purely

dimensional result  $u_0 u_r \sim \sigma^{2/3} c^{-4/3} r^{2(4-D)/3}$  with a roughness exponent  $\zeta = (4-D)/3$ . From this we expect, in the massive case, the small  $m$  behavior:

$$u \sim \sigma^{1/3} m^{-\epsilon/3} c^{-D/6}.$$

It is known [14] that one must fix  $\zeta = \epsilon/3$  to obtain a reasonable fixed point. From the above equation, the (reduced) correlator of the force  $\Delta_t(x) = -\tilde{R}''_t(x) S_D$  then satisfies

$$\partial_t \Delta_t(x) = \frac{\epsilon}{3} [x \Delta_t(x)]' - \frac{1}{2} [\Delta_t(x) - \Delta_t(0)]^2,$$

and flows for  $t \rightarrow \infty$ , to a fixed point  $\Delta_\infty(x)$  given in terms of a function  $y(x)$  [14] implicitly defined as [30]

$$\begin{cases} \Delta_\infty(x) = \Delta_\infty(0) y \left( x \sqrt{\frac{\epsilon}{3 \Delta_\infty(0)}} \right), \\ \Delta_\infty(0) = \left( \frac{\epsilon}{24 \gamma^2} \right)^{1/3} \left( \int_{-\infty}^{+\infty} dx \Delta_0(x) \right)^{2/3}, \\ \frac{x^2}{2} = y(x) - 1 - \ln y(x), \end{cases}$$

with  $\gamma = \int_0^1 dy \sqrt{y-1} - \ln y = 0.5482228893$ . Note that  $\int_{-\infty}^{+\infty} dx \Delta_t(x)$  is  $t$  independent and thus equal to  $\int_{-\infty}^{+\infty} dx \Delta_0(x) = -(S_D/c^2) f R_0'' = 2\sigma(S_D/c^2)$ .

Putting this together with Eqs. (106),(111) yields the result

$$\overline{u_q u_{-q}} = \sigma^{2/3} m^{-D-2\zeta} c^{D/6} F_D [c q^2 / m^2],$$

$$F_D[x] = C_D \frac{1}{(1+x)^2} + \text{h.o.t.}, \quad (113)$$

$$C_D = \left( \frac{(4\pi)^{D/2}}{6\Gamma\left(\frac{\epsilon}{2}\right)\gamma^2} \right)^{1/3},$$

with  $\zeta = \epsilon/3$ . Note that the universal scaling function must behave at large  $x$  as  $F_D[x] \sim x^{\zeta-D/2}$ , and is determined here only to order 0 in  $\epsilon$ .

From this, one also finds the local fluctuation

$$\overline{(u_x)^2} = \epsilon^{-2/3} 6^{-1/3} \left( \frac{\epsilon \Gamma\left(\frac{\epsilon}{2}\right)}{(4\pi)^{D/2} \gamma} \right)^{2/3} \sigma^{2/3} m^{-2\zeta} c^{-D/3}, \quad (114)$$

which is also universal. The fact that this quantity is dominated by large-scale fluctuations can be seen from the convergence of the integral ( $\zeta > 0$ ).

The calculation can also be performed exactly in  $D=4$ . One finds

$$\overline{u_q u_{-q}} = \left( \ln \frac{1}{m} \right)^{-1/3} \left( \frac{4\pi^2}{3\gamma^2} \right)^{1/3} \frac{\sigma^{2/3} c^{2/3}}{m^4 \left( 1 + \frac{c q^2}{m^2} \right)^2}.$$

The values we find for  $C_D$ , which vanishes as  $C_D \sim 3.5246\epsilon^{2/3}$  as  $D \rightarrow 4^-$  are as follows  $C_3 \approx 2.40653$ ,  $C_2 \approx 1.91006$ ,  $C_1 \approx 1.30416$ ,  $C_0 \approx 0.82157$ , remarkably close to the exact result in  $D=0$  [28]  $C_0 = 1.05423856519$ .

It is also useful to compare these results with the Gaussian variational method with replica symmetry breaking. Extending the calculation of [31] to the nonzero mass case, which is done in Appendix E, we find the same form as Eq. (113) with

$$F_D[x] = C'_D \left[ \frac{1}{(1+x)^2} + \frac{\theta(2-\theta)}{4(1+x)} \int_1^{+\infty} \frac{dy}{y^{1+\theta/2}} \frac{y-1}{y+x} \right], \quad (115)$$

$$C'_D/C_D = \left( \frac{12}{\pi} \gamma^2 \right)^{1/3} = 1.04708, \quad (116)$$

and  $\theta = (6-\epsilon)/3$  and  $\zeta = \epsilon/3$ . Since as  $\epsilon \rightarrow 0$  for fixed  $x$  the second integral is subdominant, the leading order in  $\epsilon$  are identical, and the amplitude of the RSB solution compared to the FRG solution is  $C'_D/C_D$  as  $\epsilon \rightarrow 0$ .

### 3. Periodic case

In the case of a periodic system with period  $a$ , one gets a fixed-point function with  $\zeta = 0$  which reads

$$\Delta_\infty(u) = \frac{\epsilon}{6} \left( \frac{a^2}{6} - u(a-u) \right). \quad (117)$$

It yields

$$\overline{u_q u_{-q}} = a^2 c^{D/2} m^{-D} \frac{(4\pi)^{D/2}}{36\Gamma\left[\frac{\epsilon}{2}\right]} \frac{1}{\left(1 + \frac{c q^2}{m^2}\right)^{D/2}} \quad (118)$$

$$\overline{u_r^2} \sim \frac{\epsilon}{36} a^2 \ln\left(\frac{1}{m}\right), \quad (119)$$

and in  $D=4$ ,

$$\overline{u_q u_{-q}} = \frac{2\pi^2}{9} \frac{a^2 c^2}{m^4 \ln \frac{1}{m}} \frac{1}{\left(1 + \frac{c q^2}{m^2}\right)^2}. \quad (120)$$

## V. TOWARDS THE TWO-LOOP FRG

The exact EMRG method allows to compute quantities beyond the lowest order in  $\epsilon$ . It can be carried either at  $T \rightarrow 0$  for fixed system size ( $T=0$  limit) or at finite  $T$ . Solving the exact RG equation at  $T=0$  requires us to follow nonanalytic functions. This is a difficult question, e.g., distinguishing the various cumulants in the local part demands a special

procedure that we have developed. This is discussed in [16,15].

The resulting two-loop FRG equation at  $T=0$  derived in [16] is given in the Appendix F. This equation differs from the one obtained in [19] as it contains a new ‘‘anomalous’’ term of the form  $\lambda R'''(0^+)^2 R''(u)$ . We find that the coefficient of this term is universal with  $\lambda = 1/2$ . Interestingly, this value is consistent with the renormalizability arguments given in [15]. The approach of [15] and the present one at  $T=0$  yield consistent results and allow us to compute the roughness exponent  $\zeta$  to order  $O(\epsilon^2)$  (see Appendix F).

The difficulty which was faced *a priori* was (i) possible ambiguities when distinguishing the various cumulants in the local part and (ii) a nonuniversal  $\lambda \neq 1/2$  value that would imply that fixed-point solutions would exhibit the property of a nonzero value of  $R'(0^+)$ , referred to as a supercusp since it is a stronger nonanalyticity than the one-loop one [ $R'''(0^+) \neq 0$ ]. This feature would be unpleasant as it naively yields (by perturbative expansion) additional divergences as  $T^{-1/2}$ , a sign of possible fractional dependence in  $\epsilon$  (a related phenomenon is discussed in [26]). In fact, as our study indicates, such ambiguities can be avoided [16] and the FRG equation does not yield to a supercusp (see Appendix F).

At finite temperature these possible problems do not arise since the singularity is smoothed within a boundary layer (at one loop see Sec. IV B). In Appendix F, we have used the exact RG flow to third order (given in Sec. III 4) and obtain the two-loop exact FRG equation for the second cumulant  $R_l(u)$  at  $T>0$ . At large  $l$ , the effective temperature  $\hat{T}_l \rightarrow 0$  and one recovers an ‘‘effective’’ zero-temperature equation. The analysis of this limit, and its relation to the structure of the boundary layers at high orders, is complex and further examined in [16].

## VI. CONCLUSION

In this paper, we have introduced a systematic method, the EMRG, which turns the exact, though abstract, RG functional equation of Wilson-Polchinski into a tool for concrete perturbative calculations to any number of loops using arbitrary cutoff functions for a broad class of models. The strategy was to explicitly integrate out all nonlocal interactions, which can be expressed in terms of the local part alone, order by order in the local part. In the process, we have preserved the exactness and the controlled nature of the original Wilson-Polchinski equation. Indeed, no approximation was made, and the resulting RG equation for the local part, as well as the expressions for the nonlocal ones and for the correlation functions, are formally exact order by order in an expansion in the local part. This expansion will be useful for theories where the local part is small, i.e., when it is controlled by a small parameter (e.g., the shift from the upper critical dimension) and when the RG equation admits a perturbative fixed point in this parameter. We have considered here theories with a bare local interaction and a fixed point for the local part, e.g., as in the  $O(n)$  model, but the method is more general and can be extended to theories where the bilocal part of the interaction serves as the small parameter, e.g., for self-avoiding manifolds [39]. In a sense, the exact

multilocal RG in the operational form presented here directly translates the ideas of Wilson and provides explicit checks of universality.

In addition to presenting the method formally to all orders, we have derived the *explicit* RG equation for the local part up to third order. Further expanding in the number of loops, we have explicitly given the coefficients up to two loops and third order. Two distinct, although equivalent, methods have been presented, depending on whether one considers the Wick ordered functional or not. Each method has its advantages: the Wick ordered method yields apparently simpler (less nonlocal) RG equations, but it is not always the most adequate (e.g., for the finite  $T$  one-loop FRG analysis). Although the present paper contains all the material necessary for two loop applications (for the FRG) we have preferred to defer giving the detailed calculations and results to a companion paper [16].

We have thus considered here mostly one-loop applications. The first one was a simple check to recover the one-loop exponents of the  $O(n)$  model. The second application was to the theory that describes elastic systems in random potentials. It was previously analyzed through simpler Wilson momentum shell integration [14] but the rather unusual nature of the theory ( $n \rightarrow 0$  limit, nonanalyticity) made it important to verify explicitly that the results are universal. Also universality in disordered systems is rather less established than in pure systems, especially in the  $T=0$  limit where it is known to fail in some cases. Thus, we first derived the  $T=0$  one-loop RG equation for the second cumulant function  $R(u)$  in arbitrary cutoff scheme and found that its coefficients are universal to this order. This yields the universality to  $O(\epsilon)$  of the roughness exponent  $\zeta$  of pinned interfaces. In the periodic case, we also explicitly verified that the correlation function contains a universal amplitude. Similarly, we computed the scaling function of the ground-state deformations of a confined interface in a random field and found a universal result. This quantity can be experimentally measured in disordered magnets in the presence of a small additional field gradient.

Although temperature is formally irrelevant (the dimensionless temperature flows to zero) it is well known to be ‘‘dangerously’’ so. Our exact FRG at  $T>0$  shows that although the ‘‘boundary layer,’’ i.e., the detailed asymptotic form of the cumulant  $R(u)$  for  $u \sim T_l$  is nonuniversal, some of its features are universal, and in particular, we were able to extract from it the universal divergence of the mesoscopic fluctuations of the susceptibility  $\Delta\chi$ . The divergence of this quantity, which is dominated by rare, almost degenerate, low-energy configurations, is an accepted unambiguous measure of ‘‘glassiness’’ in a disordered system and is measured in experiments, e.g., in microsize vortex systems.

Some of the peculiar features of the theory of pinned elastic systems have been also discussed. We have found it useful to give a detailed diagrammatic proof of the triviality of naive perturbation theory, as we have not seen it explicitly in the literature (though more general statements about dimensional reduction appear in a number of other works). We have discussed how the nonanalytic nature of the theory yields nontrivial results.



Pinning of disordered media thus provided us here with one example of a problem where exact renormalization is needed to get insight, as no field theoretical description is yet available. The reason for it is that one must follow in principle a rather complicated object, the full probability distributions or the disorder, or equivalently the whole series of cumulants. The method seems thus promising for other problems with similar features, such as random sine Gordon models [40]. It is interesting to note that while presenting the FRG method, Fisher pointed out (Ref. 12 in [14]) that the momentum-shell RG “suffers from pathologies due to the sharp cutoff,” and that the cusp in  $R(u)$  “requires a careful analysis of the full renormalization group.” Through the use of the exact EMRG method presented here, we provide a simple way to integrate explicitly and exactly what is left aside in the traditional RG and check that no damage comes from neglected operators. We are thus able to control the approximations of former approaches and to perform new calculations. Furthermore, thanks to our general framework expressed in terms of *any* cutoff function, the universality of the results is checked.

Let us close by noting that the application of the EMRG method via the multilocal solution to the exact RG equation seems promising also to study other disordered problems, or even give a new perspective on simpler pure problems. For instance, one could apply it to wetting problems taking into account the nonlinear part, or to the roughening problems to improve on previous analysis using uncontrolled projections methods [41].

The multilocal expansion allows also interesting extensions to theories with bilocal bare action, such as polymers, mutually interacting or with disorder. Finally, it is also worth studying more closely the set of exact solutions to the Polchinski equation presented in this paper (Appendix B). Some of these extensions will be explored in future publications.

## APPENDIX A: INVARIANCE PROPERTIES OF THE GENERATING FUNCTIONAL AND RENORMALIZATION EQUATION

In this appendix, we give a concise derivation of the exact invariance properties of the generating functional of correlation functions under coarse graining. These properties provide the basis for developing exact renormalization procedures of the Polchinski type. In the second part, we generalize the framework to include additional field transformations, such as rescaling.

### 1. Invariance under coarse graining

We use only the two following properties of Gaussian averages. The notations are the same as in the body of the paper. First, transformation under a change of variable  $\phi \rightarrow \phi + \psi$  for any field  $\psi$  in the functional integration over  $\phi$  yields

$$[\mathcal{A}(\phi)]_G = e^{-(1/2)\psi:G^{-1}:\psi} [e^{-\psi:G^{-1}:\phi} \mathcal{A}(\phi + \psi)]_G \quad (\text{A1})$$

for any functional  $\mathcal{A}(\phi)$ . We will also use the composition property

$$\{[\mathcal{A}(\phi_1 + \phi_2)]_{G_2}\}_{G_1} = [\mathcal{A}(\phi)]_{G_1 + G_2}, \quad (\text{A2})$$

where in the left-hand side (l.h.s.) the average over  $\phi_i$  is performed using Gaussian correlations  $G_i$ .

Using successively the shifts  $\phi_1 \rightarrow \phi_1 - G_2 : J$  and  $\phi_2 \rightarrow \phi_2 + G_2 : J$  yields the fundamental relation

$$\begin{aligned} [e^{J:\phi - \mathcal{V}(\phi)}]_{G_0} &= \{[e^{J:(\phi_1 + \phi_2) - \mathcal{V}(\phi_1 + \phi_2)}]_{G_2}\}_{G_1} \\ &= e^{-(1/2)J:(G_2 + G_2:G_1^{-1}:G_2):J} \\ &\quad \times \{e^{J:(1 + G_2:G_1^{-1}):\phi_1} [e^{-\mathcal{V}(\phi_1 + \phi_2)}]_{G_2}\}_{G_1}, \end{aligned}$$

where we denoted  $G_2 = G_0 - G_1$ . Thus, if  $\mathcal{V}_1$  is coarse-grained transformed of the interaction  $\mathcal{V}$ , defined by

$$e^{-\mathcal{V}_1(\phi_1)} = [e^{-\mathcal{V}(\phi_1 + \phi_2)}]_{G_0 - G_1}, \quad (\text{A3})$$

then one has

$$\begin{aligned} [e^{J:\phi - \mathcal{V}(\phi)}]_{G_0} &= e^{(1/2)J:(G_0 - G_0:G_1^{-1}:G_0):J} \\ &\quad \times [e^{J:G_0:G_1^{-1}:\phi_1} e^{-\mathcal{V}_1(\phi_1)}]_{G_1}. \end{aligned} \quad (\text{A4})$$

We now use this property of Gaussian integrals as follows. One defines a family of actions  $\mathcal{S}_G(\phi)$  and their associated generating functional  $W_G(J)$

$$\mathcal{S}_G(\phi) = \frac{1}{2} \phi:G^{-1}:\phi + \mathcal{V}_G(\phi),$$

$$W_G(J) = \text{Ln}[e^{J:\phi - \mathcal{V}_G(\phi)}]_G.$$

They are indexed by the matrix  $G$  and we choose them to be related by the coarse-graining operation (A3) where  $G$  plays the role of  $G_1$ , namely:

$$e^{-\mathcal{V}_G(\phi)} = [e^{-\mathcal{V}_{G_0}(\phi + \psi_2)}]_{G_0 - G}, \quad (\text{A5})$$

or, equivalently in a differential form, the  $\mathcal{V}_G$  satisfy the “RG equation”

$$\frac{\delta}{\delta G} e^{-\mathcal{V}_G(\phi)} = -\frac{1}{2} \frac{\delta^2}{\delta \phi \delta \phi} e^{-\mathcal{V}_G(\phi)} \quad (\text{A6})$$

obtained by differentiating Eq. (A5) with respect to  $G$ . The coarse-graining Eq. (A6), read along a given path  $l \rightarrow G_l$ , is the Polchinski equation in its “diffusive” form (14).

It is easy to see from Eq. (A4) that this choice of a family  $\mathcal{V}_G$  implies the property

$$\tilde{W}_G(J:G':G^{-1}), \text{ independent of } G, \quad (\text{A7})$$

where we have defined the interaction part  $\tilde{W}_G(J) = W_G(J) - (1/2)J:G:J$ . It allows us to relate correlations within any member of the family  $\mathcal{S}_G$ , i.e., under coarse graining.

## 2. Generalization including rescaling and change in the Gaussian part

The previous properties can be extended to a larger set of transformations that include simultaneous (i) coarse graining, (ii) linear transformation of the fields, and (iii) redefinition of the Gaussian part (such as needed to absorb its possible renormalization). It is based on the following properties of Gaussian integrals: One defines

$$W_{G,\mathcal{V}}(J) = \ln[e^{J:\phi-\mathcal{V}(\phi)}]_G. \quad (\text{A8})$$

The first property corresponds to performing an arbitrary linear transformation on the field

$$W_{G,\mathcal{V}}(J) = W_{M^{-1}:G:M^{-1},\mathcal{V}(M:\phi)}(J:M) \quad (\text{A9})$$

valid for any  $G,\mathcal{V},J,M$ . The second property is simply the identity obtained when redistributing the Gaussian part

$$W_{G,\mathcal{V}}(J) = -\frac{1}{2}\text{Tr}\ln(1+H:G) \\ + W_{(G^{-1}+H)^{-1},\mathcal{V}(\phi)-(1/2)\phi:H:\phi}(J)$$

valid for any  $G,\mathcal{V},J,H$ .

First, let  $\mathcal{V}_G(\phi)$  satisfy the RG Eq. (A6). Then from the previous section we know that  $W_{G,\mathcal{V}_G}(J:G':G^{-1}) - (1/2)J:G':G^{-1}:G':J$  is independent of  $G$  for any  $J,G'$ . Setting  $G=G_l$  and  $G'=G_0$  one gets the Polchinski equation and one can compute  $W(J)$ . One now defines

$$\mathcal{V}_{G,M}(\phi) = \mathcal{V}_G(M:\phi) + \frac{1}{2}\phi:(M:G^{-1}:M-G^{-1}):\phi.$$

Using the above properties, one has that

$$W_{G,\mathcal{V}_{G,M}}(J:G':G^{-1}:M) - \frac{1}{2}J:G':G^{-1}:G':J \\ - \frac{1}{2}\text{Tr}\ln(1+M^{-1}:G^{-1}:M^{-1}:G) \quad (\text{A10})$$

is independent of  $G$  and  $M$ , for any  $G',J$ .

We have used the two invariances choosing  $H = M^{-1}:G^{-1}:M^{-1} - G^{-1}$  leading to the intermediate formula

$$W_{G,\mathcal{V}}(J) = -\frac{1}{2}\text{Tr}\ln(1+M^{-1}:G^{-1}:M^{-1}:G) \\ + W_{G,\mathcal{V}(M:\phi)+(1/2)\phi:(M:G^{-1}:M-G^{-1}):\phi}(J:M).$$

Defining a new family of functional indexed by  $l$  as

$$\mathcal{V}_l = \mathcal{V}_{G_l,M_l}, \quad (\text{A11})$$

and symmetric matrices  $G,M$  one finds that the functional  $\mathcal{V}_l$  now satisfies a new RG equation

$$\partial_l \mathcal{V} + \phi:M^{-1}\partial M:\frac{\delta \mathcal{V}}{\delta \phi} = \phi:K:\phi \quad (\text{A12})$$

$$-\frac{1}{2}\text{Tr}\left(M^{-1}\partial G M^{-1}:\left[\frac{\delta^2 \mathcal{V}}{\delta \phi \delta \phi} - \frac{\delta \mathcal{V}}{\delta \phi} \frac{\delta \mathcal{V}}{\delta \phi}\right]\right), \quad (\text{A13})$$

with  $K = -M^{-1}\partial M G^{-1} + (1/2)(M\partial G^{-1}M - \partial G^{-1})$ , which contains additional terms. Once this equation is solved, the correlations can be related within any of the corresponding  $\mathcal{S}_l$  theories using the above invariance property (A10) of  $W_{G,\mathcal{V}_{G,M}}(J)$ .

## APPENDIX B: GENERAL PROPERTIES AND EXACT SOLUTIONS OF THE POLCHINSKI EQUATION

Let us first mention a few general properties of Eqs. (9),(14). For the class of cutoff functions (5) used in practice, the diffusion tensor in Eq. (14) is positive  $c'(s) \leq 0$  (but not definite positive since there exists modes with  $\partial_l G_l^q = 0$ ). There are some exactly formally conserved quantities, such as  $\int \phi e^{-\mathcal{V}_l(\phi)}$  and  $[e^{-\mathcal{V}_l(\phi)}]_{G_l}$ . Since (14) is a diffusion equation, it satisfies a  $H$ -theorem of increase of the ‘‘entropy’’  $\mathcal{S}_l = \int \phi \mathcal{V}_l(\phi) e^{-\mathcal{V}_l(\phi)}$ , which flows as  $\partial_l \mathcal{S}_l = -\frac{1}{2} \int \phi (\delta/\delta \phi) e^{-\mathcal{V}_l(\phi)} : \partial_l G_l : (\delta/\delta \phi) e^{-\mathcal{V}_l(\phi)} \geq 0$  and is compatible with the fact that RG trajectories do not have limit cycles. Finally, since (14) is a linear equation, if we now a set of solutions  $\mathcal{V}_l^\alpha(\phi)$ , then any superposition such as

$$\mathcal{V}_l(\phi) = -\ln \sum_\alpha c^\alpha e^{-\mathcal{V}_l^\alpha(\phi)} \quad (\text{B1})$$

is also a solution.

This can now be used to construct nontrivial exact solutions to the Polchinski equation. The simplest family of exact solutions is of course the quadratic potential, for which one finds the solutions

$$\mathcal{V}_l(\phi) = \frac{1}{2}(\phi - \psi):M_l:(\phi - \psi) - \frac{1}{2}\text{Tr}\ln M_l, \quad (\text{B2})$$

$$M_l = (M_0^{-1} + G_0 - G_l)^{-1}, \quad (\text{B3})$$

where  $\psi$  is an  $l$ -independent field, with  $\langle \phi \rangle_S = (1 + M^{-1}:G^{-1})^{-1}:\psi$ .

A much less trivial family of *exact solutions* of the Polchinski equation is obtained by superposition of Gaussians, i.e., of quadratic potentials. It reads

$$\mathcal{V}_l(\phi) = -\ln \sum_\alpha c^\alpha e^{-1/2(\phi - \psi^\alpha):M_l^\alpha:(\phi - \psi^\alpha) + (1/2)\text{Tr}\ln M_l^\alpha}, \quad (\text{B4})$$

with arbitrary constant coefficients  $c^\alpha$  and each  $M_l^\alpha$  satisfies Eq. (B3). This is somewhat reminiscent of a decomposition into ‘‘pure states’’ and is clearly of interest to describe low-temperature states in pure models (in phases with broken symmetry) or in disordered models and glasses (with many metastable states). It is an interesting question to ask, quite generally, whether this family of solutions can in some cases be an *attractive* manifold in a larger functional space, or

whether one can carry perturbation around this subspace. These and related issues will be discussed in a future publication [21].

A generic property of these solutions to the Polchinski equation is to generate cusp singularities separating the ‘‘pure states.’’ This can be seen directly above since negative curvatures tend to increase in absolute value [see Eq. (B3)] and is presumably a very general mechanism. It can also be seen on the simple example of the zero-dimensional toy model. There, the field  $\phi$  is a real number and

$$Z = \int_{-\infty}^{\infty} d\phi e^{-(\phi^2/2G) - V(\phi)}, \quad (\text{B5})$$

where  $V$  is an arbitrary function. One can introduce  $G_l = G - l$ , and  $V_l(\phi)$  for this model, verifying

$$\partial_l V_l(\phi) = \frac{1}{2} [V_l''(\phi) - V_l'(\phi)^2], \quad (\text{B6})$$

with initial condition  $V_0(\phi) = V(\phi)$ , and one can integrate up to  $l = G$ . One has  $e^{-V_l(\phi)} = [e^{-V(\phi+\psi)}]_l$ . The evolution of  $V_l(\phi)$  is that the curvature  $M_l = V_l''(0)$ , which would obey Eq. (B3)  $M_l = M_0/(1 + M_0 l)$  for a quadratic hill or well, diverges at a finite  $l$  for maxima and decreases as  $1/l$  for minima of  $V(\phi)$ . Thus, the landscape  $V_l(\phi)$  develops cusps, encoding for discontinuities in the force  $-V'(\phi)$ .

In the case of a periodic landscape, the natural superposition of Gaussian solutions is the *Villain potential*  $V(\phi) = -\ln \sum_n c e^{-(\phi-n)^2/(2l)}$ . In these sine Gordon-type potential, as well as in the  $2D$   $XY$  model, it is a well-known property that the renormalized potential converges towards the Villain form at low temperature as found in [42] from the Migdal Kadanoff RG (see also more recently [41]). The detailed behavior of the RG flow can be studied in a more controlled way using the method presented of this paper [21].

### APPENDIX C: MULTILOCAL EXPANSION AND HIGHER-ORDER RG EQUATION

In this appendix we derive the systematic multilocal expansion and obtain the RG equation to higher orders. We give a detailed presentation for the functional  $\hat{V}_l(\phi)$ , which is simpler, and give explicitly the corresponding RG equation to order  $\hat{V}_l^3$  and up to two loops. Then we simply sketch the result for the same procedure applied to the functional  $\mathcal{V}_l(\phi)$ , which is more involved and will be presented in [16].

#### 1. Multilocal expansion for $\hat{V}$

The tadpole-free functional  $\hat{V}_l(\phi)$ , defined in Eq. (31), can be written as a sum of multilocal interactions

$$\hat{V}_l(\phi) = \sum_{p>0} \int_{x_1, \dots, x_p} V^{(p)}(\phi_{x_1}, \dots, \phi_{x_p}, x_1, \dots, x_p). \quad (\text{C1})$$

Note that we are not even assuming here translational invariance. The translationally invariant case discussed in Sec. II B

can be recovered by setting  $V_l^{(1)}(\phi_1, x_1) = \hat{U}_l(\phi_1)$ ,  $V_l^{(2)}(\phi_1, \phi_2, x_1, x_2) = \hat{V}_l(\phi_1, \phi_2, x_1 - x_2)$ , etc. Since we want to impose that each  $V^{(p)}$ ,  $p > 1$ , has zero local part (this is sufficient for our purpose), we define [extending Eqs. (17), (18)], respectively, the projection operator  $\bar{P}_1$ , which projects a  $p$ -local interaction on a local one, and the projection operator  $P_1$ , which transforms a  $p$ -local interaction into another  $p$ -local interaction as

$$(\bar{P}_1 A)(\phi, t) = \int_{x_1, \dots, x_p} \delta\left(t - \frac{x_1 + \dots + x_p}{p}\right) \times A(\phi, \dots, \phi, x_1, \dots, x_p) \quad (\text{C2})$$

$$(P_1 A)(\phi_1, \dots, \phi_p, x_1, \dots, x_p) = \delta(x_1 - x_2) \dots \delta(x_1 - x_p) \times \int_{y_1, \dots, y_p} \delta\left(x_1 - \frac{y_1 + \dots + y_p}{p}\right) \times A(\phi_1, \dots, \phi_p, y_1 \dots y_p). \quad (\text{C3})$$

The property

$$\begin{aligned} & \int_{x_1, \dots, x_p} (P_1 A)(\phi_{x_1}, \dots, \phi_{x_p}, x_1, \dots, x_p) \\ &= \int_t (\bar{P}_1 A)(\phi_t, t) \\ &= \int_t A(\phi_t, \dots, \phi_t, t, \dots, t) \end{aligned} \quad (\text{C4})$$

ensures that one can choose the  $V^{(p)}$ ,  $p > 1$  in the decomposition (C1) to have no local part, i.e.,

$$P_1 V_l^{(p)} = 0 \quad \bar{P}_1 V_l^{(p)} = 0 \quad (\text{C5})$$

for any  $l$  by applying  $P_1$  and  $1 - P_1$  act on both sides of the Polchinski equation.

Since the modified Polchinski Eq. (32) concatenates two operators, it is then easy to see that if the  $V_l^{(p)}$  satisfy the following set of equations:

$$\begin{aligned} \partial_l V^{(1)}(\phi, t) &= \frac{1}{2} \sum_{p>0} \sum_{q=1}^{p-1} \int_{x_1, \dots, x_p} \delta\left(t - \frac{x_1 + \dots + x_p}{p}\right) \\ &\times e^{\partial^1, \dots, q G_l \partial^{q+1}, \dots, p} \partial^1, \dots, q \partial G_l \partial^{q+1}, \dots, p \\ &\times V^{(q)}(\phi_{x_1}, \dots, \phi_{x_q}, x_1, \dots, x_q) \\ &\times V^{(p-q)}(\phi_{x_{q+1}}, \dots, \phi_{x_p}, x_{q+1}, \dots, x_p) \Big|_{\phi_i = \phi} \end{aligned} \quad (\text{C6})$$

$$\begin{aligned}
 & \times \partial_l V^{(p)}(\phi_{x_1}, \dots, \phi_{x_q}, x_1, \dots, x_q) \\
 & = \frac{1}{2} \mathbf{S}(1 - P_1) \sum_{q=1}^{p-1} e^{\partial^1, \dots, q G_l \partial^{q+1}, \dots, p} \\
 & \times \partial^1, \dots, q \partial G_l \partial^{q+1}, \dots, p \\
 & \times V^{(q)}(\phi_{x_1}, \dots, \phi_{x_q}, x_1, \dots, x_q) \\
 & \times V^{(p-q)}(\phi_{x_{q+1}}, \dots, \phi_{x_p}, x_{q+1}, \dots, x_p) \text{ for } p > 1
 \end{aligned} \tag{C7}$$

then Eq. (32) is obeyed by  $\hat{Y}_l(\phi)$ . Since we prefer to work with symmetric functions we have defined the symmetrization operator:

$$\begin{aligned}
 & \mathbf{SB}^{(p)}(\phi_1, \dots, \phi_p, x_1, \dots, x_p) \\
 & = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} B^{(p)}[\phi_{\sigma(1)}, \dots, \phi_{\sigma(p)}, x_{\sigma(1)}, \dots, x_{\sigma(p)}],
 \end{aligned} \tag{C8}$$

we have also defined the following shorthand notations (with  $\partial_i^\alpha = \partial_{\phi_i^\alpha}$ ):

$$\partial^1, \dots, q \partial G \partial^{q+1}, \dots, p = \sum_{\alpha=1}^q \sum_{\beta=q+1}^p \partial G_{ij}^{x_\alpha x_\beta} \partial_i^\alpha \partial_j^\beta, \tag{C9}$$

and similarly  $\partial G \rightarrow G$ .

It is easy to see that if  $V^{(1)}$  is considered formally as ‘‘small’’ in some sense (e.g., controlled by a small parameter such as  $\epsilon$ ) then one can integrate exactly these equations order by order in  $V^{(1)}$  and check that  $V^{(p)} = O[V^{(1)p}]$ . More precisely, to a given order one can exactly integrate the equations for higher point functions and reduce to a single equation for  $V^{(1)}$ . This is the procedure that we now follow. The structure to the lowest-order  $O[V^{(1)2}]$  is simply a closed equation for  $V^{(1)}$  of the schematic form

$$\partial_l V^{(1)} = \bar{P}_l[V^{(1)} * V^{(1)} + O(V^{(1)3})]. \tag{C10}$$

To next-order  $O[V^{(1)3}]$ , one needs to solve the coupled set

$$\partial_l V^{(1)} = \bar{P}_l[V^{(1)} * V^{(1)} + V^{(1)} * V^{(2)}] + O[V^{(1)4}], \tag{C11}$$

$$\partial_l V^{(2)} = (1 - P_1)[V^{(1)} * V^{(1)}]. \tag{C12}$$

The second equation is explicitly integrated, which yields  $V^{(2)}[V^{(1)}]$ , which is then substituted in the first equation, producing a closed equation for  $V^{(1)}$ . This procedure can be extended to any order in  $V^{(1)}$ . We now give the explicit calculation.

## 2. RG equation up to order $V^{(1)3}$

To (lowest) order  $V^{(1)2}$ , the beta function is local in  $l$  as the modified Polchinski equation itself and reads

$$\begin{aligned}
 \partial_l V^{(1)}(\phi, t) & = \frac{1}{2} \int_{x_1 x_2} \delta\left(t - \frac{x_1 + x_2}{2}\right) \\
 & \times e^{\partial^1 G_l \partial^2} \partial^1 \partial G_l \partial^2 V^{(1)}(\phi_1, x_1) \Big|_{\phi_1 = \phi} \\
 & \times V^{(1)}(\phi_2, x_2) \Big|_{\phi_2 = \phi}
 \end{aligned} \tag{C13}$$

up to terms of order  $\mathcal{O}[V^{(1)3}]$ .

To next-order  $V^{(1)3}$ , as explained above, one first computes the bilocal operator as a function of  $V^{(1)}$ . Its flow equation to the necessary order reads

$$\begin{aligned}
 & \partial_l V_l^{(2)}(\phi_1 \phi_2, x_1 x_2) \\
 & = \frac{1}{2} e^{\partial^1 G_l \partial^2} \partial^1 \partial G_l \partial^2 V^{(1)}(\phi_1, x_1) V^{(1)}(\phi_2, x_2) \\
 & - \delta(x_1 - x_2) \int_{y_1 y_2} \delta\left(x_1 - \frac{y_1 + y_2}{2}\right) \\
 & \times e^{\partial^1 G_l \partial^2} \partial^1 \partial G_l \partial^2 V^{(1)}(\phi_1, y_1) V^{(1)}(\phi_2, y_2),
 \end{aligned} \tag{C14}$$

up to  $\mathcal{O}[V^{(1)3}]$  terms. Integrating  $\partial_\mu V_\mu^{(2)}(\phi_1 \phi_2, x_1 x_2)$  using Eq. (C14) from 0 to  $l$  and substituting the result into the equation for  $V^{(1)}$  one finds the RG equation of the local part of the interaction to order  $V^{(1)3}$ :

$$\begin{aligned}
 \partial_l V^{(1)}(\phi, t) & = \frac{1}{2} \int_{x_1 x_2} \delta\left(t - \frac{x_1 + x_2}{2}\right) \\
 & \times e^{\partial^1 G_l \partial^2} \partial^1 \partial G_l \partial^2 V^{(1)}(\phi_1, x_1) V^{(1)}(\phi_2, x_2) \\
 & + \frac{1}{2} \int_{x_1 x_2 x_3} \delta\left(t - \frac{x_1 + x_2 + x_3}{3}\right) \\
 & \times e^{\partial^{12} G_l \partial^3} \partial^{12} \partial G_l \partial^3 \left( \int_0^l d\mu \right.
 \end{aligned} \tag{C15}$$

$$\begin{aligned}
 & \times \left[ e^{\partial^1 G_\mu \partial^2} \partial^1 \partial G_\mu \partial^2 V_\mu^{(1)}(\phi_1, x_1) \Big|_{\phi_1 = \phi} \right. \\
 & \times V_\mu^{(1)}(\phi_2, x_2) \Big|_{\phi_2 = \phi} - \delta(x_1 - x_2) \\
 & \times \int_{y_1 y_2} \delta\left(x_1 - \frac{y_1 + y_2}{2}\right) e^{\partial^1 G_\mu \partial^2} \partial^1 \partial G_\mu \partial^2 \\
 & \times V_\mu^{(1)}(\phi_1, y_1) \Big|_{\phi_1 = \phi} V_\mu^{(1)}(\phi_2, y_2) \Big|_{\phi_2 = \phi} \left. \right] \\
 & \times V_l^{(1)}(\phi_3, x_3) \Big|_{\phi_3 = \phi}
 \end{aligned} \tag{C16}$$

up to  $\mathcal{O}[V^{(1)4}]$  terms.

## 3. Translation invariant theory and loop expansion

In a spatially translational invariant theory, the local interaction does not depend explicitly on the space variable  $t$ ,

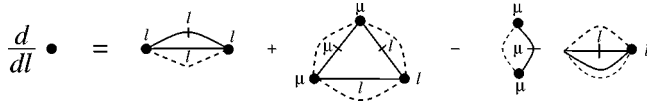


FIG. 4. Graphical representation of the expansion of the modified RG equation in powers of the local part up to  $\mathcal{O}[V^{(1)3}]$ . The points represent the vertices  $\hat{U}_l$ , the broken solid lines are propagators on shell  $\partial G$ , the full solid lines are propagators  $G$ , the indices  $l$  or  $\mu$  are indicated at each vertex and at each bond. While the solid lines are necessarily present, additional dashed lines appear in an arbitrary number when performing the loop expansion.

$V^{(1)}(\phi, t) = \hat{U}_l(\phi)$ . The above formulas, when expanding the exponentials in a loop expansion, possess a representation in terms of Feynman graphs as indicated in Fig. 4. Interestingly, all one-particle reducible graphs vanish due to the property  $\partial_l G_l^{q=0} = 0$  [ $c'(0) = 0$ ]. In addition, since each graph to order  $\hat{U}^3$  possesses a counterpart with a minus sign which is the product of two (factorized) graphs with independent sets of loop integrations, this automatically cancels all such (factorized) graphs.

The RG equation, at any given order in  $\hat{U}_l$ , can be further expanded in the number of loops by expanding the exponentials in Eq. (C15). Let us give the specific result for the case of a diagonal Gaussian part  $G_{l,ij}^q = \delta_{ij} G_l^q$ , the generalization being straightforward. To order  $\hat{U}_l^3$  and up to two loops we obtain from Eq. (C15) the RG equation for  $\hat{U}_l(\phi)$  as:

$$\begin{aligned} \partial_l \hat{U}_l(\phi) = & I_l^D D_l(\phi) + I_l^F F_l(\phi) + \int_0^l d\mu [I_{l\mu}^T T_{l\mu}(\phi) \\ & + I_{l\mu}^A A_{l\mu}(\phi) + I_{l\mu}^{A'} A'_{l\mu}(\phi)], \end{aligned} \quad (\text{C17})$$

up to  $\mathcal{O}(\hat{U}_l^4)$  terms, where the contraction graphs are

$$D_l(\phi) = \partial_{ij} \hat{U}_l(\phi) \partial_{ij} \hat{U}_l(\phi), \quad (\text{C18})$$

$$F_l(\phi) = \partial_{ijm} \hat{U}_l(\phi) \partial_{ijm} \hat{U}_l(\phi), \quad (\text{C19})$$

$$T_{l\mu}(\phi) = \partial_{ij} \hat{U}_l(\phi) \partial_{jm} \hat{U}_\mu(\phi) \partial_{mi} \hat{U}_\mu(\phi), \quad (\text{C20})$$

$$A_{l\mu}(\phi) = \partial_{ij} \hat{U}_l(\phi) \partial_{imn} \hat{U}_\mu(\phi) \partial_{jmn} \hat{U}_\mu(\phi), \quad (\text{C21})$$

$$A'_{l\mu}(\phi) = \partial_{ij} \hat{U}_\mu(\phi) \partial_{imn} \hat{U}_\mu(\phi) \partial_{jmn} \hat{U}_l(\phi), \quad (\text{C22})$$

and the momenta graphs read

$$I_l^D = \frac{1}{2} \int_q \partial G_l^q G_l^q, \quad (\text{C23})$$

$$I_l^F = \frac{1}{4} \int_{q_1 q_2 q_3} \delta_{q_1+q_2+q_3} \partial G_l^{q_1} G_l^{q_2} G_l^{q_3}, \quad (\text{C24})$$

$$I_{l\mu}^T = \int_q G_l^q \partial G_l^q \partial G_\mu^q, \quad (\text{C25})$$

$$I_{l\mu}^A = \int_{q_1 q_2 q_3} (\delta_{q_1+q_2+q_3} - \delta_{q_2+q_3}) G_l^{q_1} \partial G_l^{q_1} G_\mu^{q_2} \partial G_\mu^{q_3}, \quad (\text{C26})$$

$$\begin{aligned} I_{l\mu}^{A'} = & \int_{q_1 q_2 q_3} \delta_{q_1+q_2+q_3} \left( \frac{1}{2} \partial G_l^{q_1} G_l^{q_2} G_l^{q_3} \partial G_\mu^{q_1} \right. \\ & \left. + \partial G_l^{q_2} G_l^{q_3} G_l^{q_1} \partial G_\mu^{q_1} \right). \end{aligned} \quad (\text{C27})$$

Note that to two loops the RG flow is generically nonlocal in  $l$ . The values of the above integrals will be computed in [16].

#### 4. RG equation for $U_l(\phi)$

The systematic expansion in multilocal interactions can also be performed directly on the functional  $\mathcal{V}(\phi)$ . The procedure parallels the previous section and its details are given in [16]. Here we give only the result for a translationally invariant theory, for the RG flow of the function  $U_l(\phi)$  in the translationally invariant case to order  $U_l^3$ . We define

$$\begin{aligned} \mathcal{P}_{l\mu}(x_1, \dots, x_p, \partial_{\phi_1}, \dots, \partial_{\phi_3}) = & \partial^{1, \dots, p} \partial G \partial^{1, \dots, p} \\ = & \sum_{\alpha, \beta=1}^p \partial G_{ij}^{x_\alpha x_\beta} \partial_i^\alpha \partial_j^\beta, \end{aligned} \quad (\text{C28})$$

and similarly  $\partial G \rightarrow G$ , as well as the operator

$$\begin{aligned} \mathcal{K}_{l\mu}(x_1, x_2, x_3, \partial_{\phi_1}, \partial_{\phi_2}, \partial_{\phi_3}) \\ = & \partial^{123} (\partial G_l - \partial G_l^{x=0}) \partial^{123} e^{- (1/2) \partial^{123} G_{l\mu} \partial^{123}} \partial^{12} \partial G_\mu \partial^3. \end{aligned} \quad (\text{C29})$$

It reads

$$\begin{aligned}
 \partial U_l(\phi) = & -\frac{1}{2}(\partial G_l)_{\alpha\beta}^{x=0} \partial_\alpha \partial_\beta U_l(\phi) - \frac{1}{4} \int_{x_1 x_2} \delta\left(\frac{x_1+x_2}{2}\right) \partial^{12}(\partial G_l - \partial G_l^{x=0}) \partial^{12} \int_0^l d\mu e^{-(1/2)\partial^{12} G_{l\mu} \partial^{12}} \partial^1 \partial G_\mu \\
 & \times \partial^2 U_\mu(\phi_1)|_{\phi_1=\phi} U_\mu(\phi_2)|_{\phi_2=\phi} - \frac{1}{4} \int_{x_1 x_2 x_3} \delta\left(\frac{x_1+x_2+x_3}{3}\right) \\
 & \times \int_0^l d\mu \left[ \mathcal{K}_{l\mu}(x_1, x_2, x_3, \partial_{\phi_1}, \partial_{\phi_2}, \partial_{\phi_3}) - \mathcal{K}_{l\mu}\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \partial_{\phi_1}, \partial_{\phi_2}, \partial_{\phi_3}\right) \right] \\
 & \times \int_0^\mu dv e^{-(1/2)\partial^{12} G_{\mu\nu} \partial^{12}} \partial^1 \partial G_\nu \partial^2 U_\nu(\phi_1)|_{\phi_1=\phi} U_\nu(\phi_2)|_{\phi_2=\phi} U_\mu(\phi_3)|_{\phi_3=\phi}. \tag{C30}
 \end{aligned}$$

Again, the 1PR diagrams are eliminated by construction since they have one point  $x_i$  on which one can integrate freely, producing a  $\partial G^{q=0}$ , which vanishes by construction. The loop expansion of this formula will be detailed in [16].

### 5. Computation of the correlation functions via RG

The invariance of the generating functional  $W(J)$  of the (connected) correlation functions with respect to  $l$  is now used as a tool for computing the correlation functions of the initial model  $S_0$ . The expansion of  $W(J)$  in powers of the running interaction  $\mathcal{V}_l(\phi)$  reads formally to all orders

$$W(J) = \frac{1}{2} J : G : J + \sum_{m=1}^{+\infty} \kappa_m, \tag{C31}$$

where we have defined

$$\mu_n = [e^{J : G : G_l^{-1} : \phi} (\mathcal{V}_l(\phi))^n]_{G_l}, \tag{C32}$$

$$Y = e^{-(1/2)J : G : G_l^{-1} : G : J}, \tag{C33}$$

$$\sum_{m=1}^{+\infty} \kappa_m x^m = \ln \left( 1 + Y \sum_{n=1}^{+\infty} \frac{(-x)^n}{n!} \mu_n \right). \tag{C34}$$

Up to second order, this expansion reduces to:

$$\begin{aligned}
 W(J) = & \frac{1}{2} J : G : J - e^{-(1/2)J : G : G_l^{-1} : G : J} [e^{J : G : G_l^{-1} : \phi} \mathcal{V}_l(\phi)]_{G_l} \\
 & + \frac{1}{2} e^{-(1/2)J : G : G_l^{-1} : G : J} [e^{J : G : G_l^{-1} : \phi} \mathcal{V}_l(\phi)^2]_{G_l} \\
 & - \frac{1}{2} (e^{-(1/2)J : G : G_l^{-1} : G : J} [e^{J : G : G_l^{-1} : \phi} \mathcal{V}_l(\phi)]_{G_l})^2 \\
 & + O(\mathcal{V}_l^3). \tag{C35}
 \end{aligned}$$

The interaction functional  $\hat{\mathcal{V}}_l(\phi)$  defined by Eq. (31) naturally appears in the expansion of  $W(J)$ . Using the properties (A1) and  $[\mathcal{A}(\phi)]_G = e^{(1/2)(\delta/\delta\phi) : G : (\delta/\delta\phi)} \mathcal{A}(0)$ , one obtains:

$$\begin{aligned}
 W(J) = & \frac{1}{2} J : G : J - \hat{\mathcal{V}}_l(G : J) + \frac{1}{2} [\widehat{\mathcal{V}}_l^2(G : J) - \hat{\mathcal{V}}_l^2(G : J)] \\
 & + O(\mathcal{V}_l^3), \tag{C36}
 \end{aligned}$$

where  $\widehat{\mathcal{V}}_l^2(\psi) = e^{(\delta/\delta\phi_1) : G_l : (\delta/\delta\phi_2)} \hat{\mathcal{V}}_l(\phi_1) \hat{\mathcal{V}}_l(\phi_2)|_{\phi_1=\phi_2=\psi}$ . On this expression, it becomes obvious that  $W(J)$  is indeed  $l$  independent (order by order), as a consequence of the RG equation for  $\hat{\mathcal{V}}_l(\phi)$ . As is clear from these formulas, all external legs of correlation functions will carry the propagator  $G$  while all internal legs will carry  $G_l$ .

We must now distinguish between the two methods that consist of performing the multilocal expansion on  $\mathcal{V}_l(\phi)$ ,  $\hat{\mathcal{V}}_l(\phi)$ , respectively. Before doing so, we give a formula, in Fourier representation, which is valid in both cases:

$$\begin{aligned}
 W(J) = & \frac{1}{2} J : G : J - \int_{xK} \hat{U}_l^K e^{iK \cdot (G : J)^x} \\
 & - \int_{xyKP} e^{iK \cdot (G : J)^x + iP \cdot (G : J)^y} \left( \hat{\mathcal{V}}_l^{KP}(x-y) \right. \\
 & \left. - \frac{1}{2} (e^{-K \cdot G_l^{xy} \cdot P} - 1) \hat{U}_l^K \hat{U}_l^P \right)
 \end{aligned}$$

$$\hat{U}_l^K = U_l^K e^{-(1/2)K \cdot G_l^{x=0} \cdot K}$$

$$\hat{\mathcal{V}}_l^{KP}(x) = V_l^{K,P}(x) e^{-(1/2)K \cdot G_l^{x=0} \cdot K - (1/2)P \cdot G_l^{x=0} \cdot P - K \cdot G_l^x \cdot P},$$

the way to compute the functions  $\hat{U}_l$  and  $\hat{\mathcal{V}}_l$  being however different in each case. Inserting the corresponding formula for  $\hat{\mathcal{V}}_l$  as a function of  $\hat{U}_l$  yields expressions in terms of  $\hat{U}_l$  only, which we now give in each case (for variety, we also alternate between the—equivalent—field and Fourier representations).

#### a. Method with $\hat{\mathcal{V}}_l(\phi)$

We start with the formalism using the multilocal expansion of  $\hat{\mathcal{V}}_l$ . One finds

$$\begin{aligned}
W(J) = & \frac{1}{2} J : G : J - \int_x \hat{U}_l[(G : J)^x] \\
& \left. \int_z e^{-K \cdot G_l^z \cdot P + K \cdot (G_l^z - G_l^0) \cdot P} \right] + O(U^3). \quad (C40) \\
& + \frac{1}{2} \int_{xy} \int_{l' > 0} \left[ \delta_{ll'} (e^{\partial_1 G_l^{x-y} \partial_2} - 1) \right. \\
& \left. - \theta_{ll'} \partial_{l'} \left[ e^{\partial_1 G_l^{x-y} \partial^2} - \delta(x-y) \int_z e^{\partial_1 G_l^z \partial_2} \right] \right. \\
& \left. \times \hat{U}_{l'}(\phi_1) \Big|_{\phi_1 = (G : J)^x} \hat{U}_{l'}(\phi_2) \Big|_{\phi_2 = (G : J)^y} \right], \quad (C37)
\end{aligned}$$

up to  $O(U^3)$  terms. We denote  $\delta_{ll'} = \delta(l-l')$  and  $\theta_{ll'} = \theta(l-l')$ . From this formula, one can compute all connected correlations to  $O(U^2)$ . Let us give the self energy, defined as usual from the two-point function  $C = G + \delta C$  as

$$\begin{aligned}
\Sigma = & C^{-1} - G^{-1} = -G^{-1} \delta C G^{-1} + G^{-1} \delta C G^{-1} \delta C G^{-1} \\
& + O(\delta C^3). \quad (C38)
\end{aligned}$$

It reads

$$\begin{aligned}
\Sigma_{ij}^{q=0} = & \partial_i \partial_j \hat{U}_l(0) - \int_x [(\partial_i^1 \partial_j^1 + \partial_i^1 \partial_j^2)(e^{\partial_1 G_l^x \partial^2} - 1) \\
& - \partial_i^1 \partial_j^2 \partial^1 G^x \partial^2] \hat{U}_l(\phi_1) \hat{U}_l(\phi_2) \Big|_{\phi_i=0}, \\
\Sigma^q - \Sigma^{q=0} = & \int_x (e^{iqx} - 1) \Sigma^x, \quad (C39) \\
\Sigma_{ij}^x = & -\partial_i^1 \partial_j^2 \int_{l' > 0} [\delta_{ll'} (e^{\partial_1 G_l^x \partial^2} - \partial^1 G^x \partial^2 - 1) \\
& - \theta_{ll'} \partial^1 \partial G_l^x \partial^2 e^{\partial_1 G_l^x \partial^2}] \hat{U}_{l'}(\phi_1) \hat{U}_{l'}(\phi_2) \Big|_{\phi_i=0}.
\end{aligned}$$

Note that it involves a term with a  $G$  propagator.

### b. Method with $\mathcal{V}_l(\phi)$

Inserting the multilocal expansion of  $\mathcal{V}_l$ , Eq. (C35) transforms into an expansion in powers of the local interaction  $U_l(\phi)$ :

$$\begin{aligned}
W(J) = & \frac{1}{2} J : G : J - \int_x \int_K \hat{U}_l^K e^{iK \cdot G_x : J} \\
& + \frac{1}{2} \int_{xy} \int_{KP} e^{iK \cdot G_x : J + iP \cdot G_y : J} \\
& \times \left[ \hat{U}_l^K \hat{U}_l^P (e^{-K \cdot G_l^{x-y} \cdot P} - 1) \right. \\
& \left. - \int_0^l dl' \hat{U}_{l'}^K \hat{U}_{l'}^P \partial_{l'} \left( e^{-K \cdot G_{l'}^{x-y} \cdot P} - \delta(x-y) \right) \right]
\end{aligned}$$

Using the RG equation for  $U_l$ , it is easily checked again that this expression is  $l$ -independent order by order. From Eq. (C37), one can compute the self energy  $\Sigma^q$  of the theory. One gets to order  $U_l^2$ :

$$\begin{aligned}
\Sigma_{ij}^{q=0} = & - \int_K K_i K_j U_l^K e^{-(1/2)KG_l^{x=0}K} + \int_{KP} [K_i K_j A_l^{KP}(q=0) \\
& + K_i P_j B_l^{KP}(q=0)], \quad (C41)
\end{aligned}$$

$$\Sigma^q - \Sigma^{q=0} = \int_{KP} K_i P_j [B_l^{KP}(q) - B_l^{KP}(q=0)],$$

where

$$\begin{aligned}
A_l^{KP}(q) = & \int_{l' > 0} U_{l'}^K U_{l'}^P e^{-(1/2)(KG_{l'}^{x=0}K + PG_{l'}^{x=0}P)} \\
& \times \left\{ \delta_{l-l'} \int_x e^{iqx} (e^{-KG_l^x P} - 1) \right. \quad (C42)
\end{aligned}$$

$$\left. + \theta_{l-l'} \int_x [e^{iqx} - e^{K \cdot (G_l^x - G_l^0) \cdot P}] K \partial G_{l'}^x P e^{-KG_{l'}^x P} \right\} \quad (C43)$$

$$\begin{aligned}
B_l^{KP}(q) = & \int_{l' > 0} U_{l'}^K U_{l'}^P e^{-(1/2)(KG_{l'}^{x=0}K + PG_{l'}^{x=0}P)} \\
& \times \left\{ \delta_{l-l'} \int_x e^{iqx} (e^{-KG_l^x P} + KG_0^x P - 1) \right. \\
& \left. + \theta_{l-l'} \int_x [e^{iqx} - e^{K \cdot (G_l^x - G_l^0) \cdot P}] K \partial G_{l'}^x P e^{-KG_{l'}^x P} \right\}. \quad (C44)
\end{aligned}$$

## APPENDIX D: DIMENSIONAL REDUCTION FROM GRAPHS

### 1. Perturbation theory

In this appendix, we sketch diagrammatically how the perturbation expansion in  $R$  of the average of any observable  $A[u]$  at  $T=0$  is the same as the one that would be obtained in the Gaussian theory corresponding to a simple random force.

Precisely, the actions

$$\mathcal{S}[u] = \frac{1}{2T} \sum_a \int_{xy} u_x^a (\bar{G}^{-1})^{xy} u_y^a - \frac{1}{2T^2} \sum_{ab} \int_x R(u_x^a - u_x^b), \quad (D1)$$

and

$$\mathcal{S}_{\text{nl}}[u] = \frac{1}{-2R''(0)} \sum_{ab} \int_{xy} u_x^a ((\bar{G} * \bar{G})^{-1})^{xy} u_y^b \quad (D2)$$

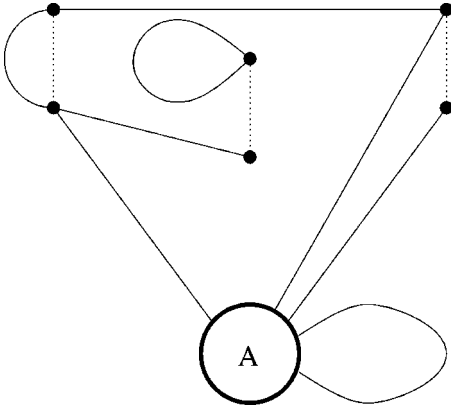


FIG. 5. Typical graph contributing to the computation of the average of an observable  $A$  to third order ( $p=3$ ) in perturbation theory. Note here that  $v=2p=6$ ,  $t=1$ ,  $k=3$ ,  $e=4$ ,  $c=3$ , and  $l=1$ .

where  $(\bar{G} * \bar{G})^x = \int_z \bar{G}^{x-z} \bar{G}^z$  yield the same results when computing the average of any functional  $A[u]$  of the replicated field at  $T=0$ , e.g.,  $A[u] = \prod_i u_{x_i}^{a_i}$  (see Fig. 5).

To this aim, one must first show that the perturbation expansion within Eq. (D1) is well defined at  $T=0$ . We use a diagrammatics with propagator

$$\langle u_x^a u_y^b \rangle = T \delta_{ab} \bar{G}^{xy}, \quad (\text{D3})$$

which conserves the replica index, and vertex

$$-\frac{1}{2T^2} \sum_{ab} \int_x R(u_x^a - u_x^b) \quad (\text{D4})$$

associated to *one* point in space but involving a summation over *two* replica indices. Thus, we choose to split the vertex into two subvertices corresponding to each replica index.

For any graph occurring in the computation of  $\langle A[u] R^p \rangle_S^c$ , let us denote by  $k$  the number of lines connecting  $A$  to the vertices (involving the extraction of  $k$  legs from  $A$ :  $\partial_{u_{x_1}^{a_1}} \dots \partial_{u_{x_k}^{a_k}} A[u]$ ). Let  $\mathcal{K}$  be the graph obtained by considering only the splitted vertices, the propagators between them, forgetting the observable and the  $k$  lines attached to it. The graph  $\mathcal{K}$  has  $v=2p$  subvertices. Contrarily to the initial graph with unsplit vertices,  $\mathcal{K}$  is not necessarily connected and is made of  $c$  connected components. To each one corresponds a replica index. If one of them is not connected to the observable, i.e., if it does not inherit from a replica index contained in  $A$ , then the summation over this index is free, giving a factor  $n$ . Hence, each connected component has to be linked to the observable in order to survive the  $n \rightarrow 0$  limit, which yields  $k \geq c$ .

Collecting the factors of  $T$  in front of the initial graph ( $\mathcal{K}$ ,  $A$ , the  $k$  propagators, and possibly  $t$  tadpoles on  $A$ ), the power of  $T$  is  $t+e+k-v$  where  $e$  is the number of propagators in  $\mathcal{K}$ . Euler relation in  $\mathcal{K}$  reads  $v+l=c+e$  where  $l$  is the number of loops in  $\mathcal{K}$ . But since  $k \geq c$  and  $l \geq 0, t \geq 0$ , one

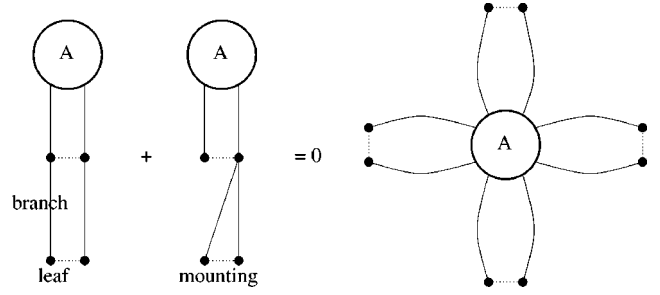


FIG. 6. Vanishing contribution to the average of  $A$ : the second graph is obtained by mounting a branch to the first graph. The only remaining graphs are simple “flowers.”

obtains that each graph is in factor of a non-negative power of  $T$ . The existence of the  $T=0$  perturbation theory is thus confirmed.

The graphs that remain at  $T=0$  have  $t=0$ ,  $l=0$ ,  $k=c$ , which means the following properties: (i) their subgraph  $\mathcal{K}$  has no loop, each of its connected component is a tree, (ii) there is no tadpole on  $A$ , (iii)  $A$  is linked to each connected component of  $\mathcal{K}$  by one unique propagator. This result is easily extended to a non-Gaussian disorder, which possess higher cumulants of the general form (57).

The second part of the argument uses the property of translation invariance in  $u$  space of the disorder distribution, on which the first part does not rely. Since each connected component of  $\mathcal{K}$  is linked to  $A$ , let us call *root* the point to which it is attached. This provides a natural orientation to the *branches* of the trees from root to *leaves*. If any point of  $\mathcal{K}$  possess at least a branch going to the direction of the leaves, the graph obtained by mounting this branch to the companion point (which belongs to the same  $R$  before splitting) has the opposite value. One can convince oneself that such graphs can be grouped by mutually canceling pairs. Thus, the only graphs that survive to this mounting operation look like *flowers*, with  $A$  at the center and petals  $R$  made of two propagators (see Fig. 6).

The generalization including higher cumulants is straightforward, but yields a non-Gaussian theory. The corresponding equivalent action for computing observables is

$$\begin{aligned} \mathcal{S}_{\text{eff}}[u] = & \frac{1}{2T} \sum_a \int_{xy} u_x^a (\bar{G}^{-1})^{xy} u_y^a \\ & - \sum_{N \geq 2} \frac{1}{N! T^N} \partial_{1, \dots, N} \mathcal{S}^{(N)}(0 \dots 0) \int_x \left( \sum_a u_x^a \right)^N. \end{aligned}$$

Even if this is not obvious on its expression,  $\mathcal{S}_{\text{eff}}[u]$  possesses statistical tilt symmetry as can be checked thanks to  $n \rightarrow 0$ .

## 2. Corrections to $R$

The computation of the effective action  $\Gamma$  (1PI) involves corrections to the various cumulants of the disorder. At  $T=0$ , the graphs correcting a  $N$  replica term ( $N$ th cumulant) is made of  $N$  connected components, so that there exists a free sum over  $N$  replica indices. The power of  $T$  in front of such a cumulant has to be  $-N$ . The graphs correcting  $R$  with  $p$



$R$ 's are made of two connected components ( $c=2$ ), the power of  $T$  is  $e-v$  where  $e$  is the number of propagators and  $v=2p$ . Euler relation yields  $e-v=l-c\geq -2$  with equality for  $l=0$ . Hence, such graphs are made of two trees.

Furthermore, the graphs such that the two points of a splitted  $R$  are connected to the same connected component and such that one of them is connected to at least two branches vanish. This can be seen by mounting one of these two branches on the companion point. Hence, if two points of a  $R$  belong to the same connected component, then each one is connected to a unique branch. As a corollary, the two points of an  $R$  cannot be connected to each other by a branch (since it would be impossible to connect this  $R$  to the rest of the diagram thanks to the argument above).

This considerably reduces the form of the possible corrections to  $R$ . These corrections obey in particular

$$\delta R = (\epsilon - 4\zeta)R + \zeta u R' + \sum_{p>0} \left(\frac{d}{du}\right)^{4(p-1)} R^p, \quad (\text{D5})$$

where the last term symbolically only means that the  $p$ th order term contains  $4(p-1)$  derivatives [not that it is a total  $4(p-1)$  derivative]. We allowed for a field rescaling with exponent  $\zeta$ . To order  $R^3$  the arguments above allow only for the following corrections

$$\begin{aligned} \delta R = & (\epsilon - 4\zeta)R + \zeta u R' + K[\frac{1}{2}R''^2 - R''R''(0)] \\ & + A[R'' - R''(0)]R''' + C[R'' - R''(0)]^2 R^{iv}, \end{aligned}$$

with some constants  $K$ ,  $A$ , and  $C$ , and valid only for an analytic  $R(u)$ . In the periodic case ( $\zeta=0$ ), the fixed point equation is easily solved since there exists to any order in  $\epsilon$  a fixed point function of the form

$$R^*(u) = a + bu(1-u) + c[u(1-u)]^2, \quad (\text{D6})$$

where  $a, b, c$  can be computed in series of  $\epsilon$ , once the coefficients of the fixed point equation are known. This is further examined in Appendix F.

## APPENDIX E: VARIATIONAL CALCULATION

Here we sketch the derivation of the scaling function for the confined interface using the replica variational method, extending the explicit solution of Ref. [31] to a nonzero mass. We use all notations of Ref. [31] and Ref. [27].

The disorder correlator for the random-field problem studied here corresponds to the case  $\gamma=1/2$  and  $g=\sigma$  for the parameters of Ref. [31]. Applying the variational ansatz for  $N=1$  components yields the function  $\tilde{f}(x) = \hat{g}\sqrt{r_f^2 + x}$ , which describes the correlations, and  $\hat{g} = \sigma\sqrt{2/\pi}$ . We have artificially extended the correlator to small scales, so as to obtain a well-defined  $T=0$  limit. The large scale results however are independent of the small scale details in the limit of small  $m$ . The variational equation reads

$$[r_f^2 + 2TB(u)]^{3/2} = \hat{g}j_D \{[\sigma(u) + m^2]\}^{-\epsilon/2}, \quad (\text{E1})$$

with  $j_D = \int_k (k^2 + 1)^{-2} = \Gamma(2-d/2)/(4\pi)(d/2)$  and the equation for the breakpoint (see Ref. [27]) is:  $[r_f^2 + 2TB(u_c)]^{3/2} = \hat{g}j_D(\Sigma_c + m^2)^{-\epsilon/2}$ . This yields the solution:

$$[\sigma](u) + m^2 = Au^{2\theta} = \Sigma_c(u/u_c)^{2\theta} u^* < u < u_c, \quad (\text{E2})$$

$$\sigma(u) = \sigma(0) = A^{\theta/2} \frac{2}{2-\theta} m^{2-\theta} u < u^*, \quad (\text{E3})$$

and  $[\sigma](u) = \Sigma_c$  for  $u > u_c$ . Here  $\theta = (6-\epsilon)/3$  is the free energy fluctuation exponent and  $A = [(\epsilon/6T)\hat{g}^{2/3}j_D^{-1/3}]^{2/\theta}$ . This solution allows us to compute the small  $q$  behavior of the correlation (for small  $m \sim q$ ) as

$$\begin{aligned} \frac{1}{u_q u_{-q}} = & \frac{T}{q^2 + m^2} \left( 1 + \int_{u_c(m^2/\Sigma_c)^{\theta/2}} \frac{du}{u^2} \frac{\Sigma_c(u/u_c)^{(2/\theta)} - m^2}{q^2 + \Sigma_c(u/u_c)^{(2/\theta)}} \right. \\ & \left. + \sigma(0) \frac{1}{q^2 + m^2} \right), \end{aligned}$$

which yields the large scale result (116) given in the text. In  $D=0$  one recovers the result of [32]

$$\frac{1}{u^2} = \frac{3}{(4\pi)^{1/3}} m^{-8/3} \sigma^{2/3}. \quad (\text{E4})$$

## APPENDIX F: FRG TO TWO LOOPS

The EMRG method allows to obtain the FRG equation to two loops both at zero and finite temperature.

### 1. Method with $\mathcal{V}$

The exact RG equation to order  $U^3$  given in Appendix C4 when expanded to two loops yields the following finite-temperature RG equation for  $\tilde{R}_l(u)$  at large  $l$  for the periodic case (the derivation is detailed in [16]):

$$\begin{aligned} \partial_l \tilde{R}_l = & \epsilon \tilde{R}_l + \hat{T}_l \tilde{R}_l'' + K_{l\mu} \left( \frac{1}{2} \tilde{R}_\mu''(u)^2 - \tilde{R}_\mu''(0) \tilde{R}_\mu''(u) \right) \\ & + K^A [\tilde{R}'' - \tilde{R}''(0)] \tilde{R}''' + K^C [\tilde{R}'' - \tilde{R}''(0)]^2 \tilde{R}'''' \\ & + K^S \partial_{12} \tilde{S}_l(u, u, 0) + \hat{T}_l K_{l\mu}^E \{ \tilde{R}_\mu^{iv} [\tilde{R}_\mu''(u) - \tilde{R}_\mu''(0)] \\ & - \tilde{R}_\mu'''(0) \tilde{R}_\mu'' \} + \hat{T}_l K_{l\mu}^F \tilde{R}_\mu'''^2. \end{aligned} \quad (\text{F1})$$

In this formula, all terms of order  $R^2$  and higher are retarded, and integrals  $\int_0^l d\mu$  are understood. For the  $R^3$  terms we have omitted the retardation integrals (which involve an additional integral  $\int_0^\mu d\nu$ ) because near the fixed point they can be replaced by a single number. The feedback of the three-replica term is through its partial derivatives. This three-replica term satisfies its own RG equation given in [16]. The precise values of all coefficients are detailed in [16] where the large  $l$  limit is also studied, using the fact that  $\hat{T}_l \rightarrow 0$ , property (93) and the third cumulant RG equation that admits the fixed-point value (80).

On the other hand, one can also directly work at  $T=0$ . Then one obtains [16] the following fixed-point equation for the general case  $\zeta \geq 0$ :

$$0 = \partial_l \bar{R} = (\epsilon - 4\zeta) \bar{R} + \zeta u \bar{R}' + \left[ \frac{1}{2} (\bar{R}'')^2 - \bar{R}''(0) \bar{R}'' \right] \quad (\text{F2})$$

$$+ \left[ \frac{1}{2} + \bar{K}^C \right] \{ (\bar{R}''')^2 [\bar{R}'' - \bar{R}''(0)] - \bar{R}'''(0^+)^2 \bar{R}'' \} \quad (\text{F3})$$

$$+ \bar{K}^C \bar{R}^{\text{iv}} [\bar{R}'' - \bar{R}''(0)]^2 \quad (\text{F4})$$

$$+ \zeta \left( \int_0^l dl' l' \bar{K}_{l'} \right) \left\{ u \bar{R}''' [\bar{R}'' - \bar{R}''(0)] \right. \quad (\text{F5})$$

$$\left. - 4 \left[ \frac{1}{2} (\bar{R}''')^2 - \bar{R}'''(0) \bar{R}'' \right] \right\}, \quad (\text{F6})$$

we have absorbed in  $\bar{R}$  the exact one-loop coefficient  $K$  obtained in Eq. (68). The third line comes from retardation of the flow expanding to first order in  $\zeta$ . This complicated-looking equation can be simplified drastically. First, it depends on only one nonuniversal coefficient  $\bar{K}^C$ . In terms of the cutoff function  $c(s) = \int_a \hat{c}(a) e^{-as}$  (or its expression as a sum of exponentials) one has:

$$K = \frac{1}{2(2\pi)^{d/2}} \left[ 1 + \epsilon \left( \frac{1}{2} - \int_a \hat{c}(a) \ln a \right. \right. \\ \left. \left. + \frac{3}{2} \int_{ab} \hat{c}(a) \hat{c}(b) \ln(a+b) \right) \right],$$

$$\bar{K}^C = - \int_a \hat{c}(a) \ln a + \int_{ab} \hat{c}(a) \hat{c}(b) \ln(a+b) \\ = \int_0^\infty ds \frac{c(s)[1-c(s)]}{s},$$

to the desired order in  $\epsilon$ . The other coefficient is not independent since:

$$\int_l l \bar{K}_l = \frac{2}{S_4} \int_l l e^{-2l} \int_q c'(q^2/2) c'(q^2/2 e^{-2l}) + O(\epsilon) \quad (\text{F7})$$

$$= \bar{K}^C + O(\epsilon). \quad (\text{F8})$$

Then, one can use the one loop equation  $(\epsilon - 2\zeta)R^{*''} + \zeta u R^{*'''} = -\{R^{*'''}[R^{*''} - R^{*''}(0)] + (R^{*''})^2\} + O(\epsilon^3)$  to eliminate the term containing the fourth derivative and one gets after some manipulations absorbing an extra  $(1 - 2\epsilon \bar{K}^C)$  coefficient in  $R$  the universal two loop fixed point RG equation:

$$0 = (\epsilon - 4\zeta)R^* + \zeta u R^{*'} + \left[ \frac{1}{2} (R^{*''})^2 - R^{*''}(0) R^{*''} \right] \quad (\text{F9})$$

$$+ \frac{1}{2} [(R^{*''})^2 \{R^{*''} - R^{*''}(0)\} - R^{*''}(0^+)^2 R^{*''}], \quad (\text{F10})$$

which coincide with the one obtained in [15] by a different method. For short range disorder, study of this fixed-point equation yields [15] no supercusp and the roughness exponent  $\zeta = 0.208\,298\,04\epsilon + 0.006\,858\epsilon^2$ .

## 2. Method with $\hat{\mathcal{V}}$

The method using the Wick ordered functional can also be used. Using the RG equation to third order in  $\hat{U}$  and two-loops (C17) one finds the equation for the corresponding two replica part of  $\hat{U}$ :

$$\partial_l \bar{R}_l = (\epsilon - 4\zeta) \bar{R}_l + \zeta u \bar{R}'_l + K \left( \frac{1}{2} \bar{R}''_l(u)^2 - \bar{R}''_l(0) \bar{R}''_l(u) \right) \\ + K^A [\bar{R}'' - \bar{R}''(0)] \bar{R}''^2 + \hat{T}_l K_l^F \bar{R}''^2, \quad (\text{F11})$$

with:

$$K = - \frac{4}{T^2} I_l^D \Lambda_l^\epsilon, \quad (\text{F12})$$

$$K_{l\mu}^A = \frac{4}{T^4} (I_{l\mu}^A + I_{l\mu}^{A'}), \quad (\text{F13})$$

$$K_{l\mu}^F = - \frac{6}{T^2} I_{l\mu}^F \Lambda_l^\epsilon, \quad (\text{F14})$$

where the integrals have been defined in Sec. III 3. Calculation shows that the constant  $K^A = 1/2$  (to lowest order in  $\epsilon$ ) independent of the cutoff function  $c(s)$ . The coefficient of this term is in agreement with [19,15]. The analysis of the boundary layer is more intricate in this formulation [16].

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- with
- $$A_{np} = \prod_{k=n}^{p-1} (n+2k)/(1+2k)$$
- and
- $$\tilde{a}_{2n} = \sum_{p \geq n} [1/(p-n)!] \tilde{g}_{2p} A_{np} (K/2)^{p-n}$$
- with
- $$K = \Lambda_l^{2-D} G_l^{x=0} = S_D \int_{s>0} (2s)^{-\epsilon/2} c(s).$$
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