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Time scales in rotating unstable Langevin-type dynamics

J. I. Jiménez-Aquino* and M. Romero-Bastida
Departamento de Física, Universidad Autónoma Metropolitana Iztapalapa, Apartado Postal 55–534, México, Distrito Federal, 09340 Mexico

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In this Rapid Communication we propose a different and general characterization of rotating, unstable Langevin-type dynamics in the presence of an external force in the context of two dynamical representations **x** and **y**, using the passage time distribution. Here **y** is the transformed space of coordinates obtained by means of a time-dependent rotation matrix. The Langevin dynamics in the new **y** space defines an interesting concept of external force and internal noise due to rotation. The theory is applied to the characterization of rotational unstable systems of two (such as the laser system) and three variables, and stimulates its application in other fields, for instance, in plasma physics.

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Stochastic differential equations have become a useful tool in the description of a great variety of physical systems in which the presence of fluctuations plays a fundamental role. During the seventies and eighties the study of transient relaxation of unstable states (or in general, of any initial condition far from the steady state) was proposed as an interesting topic in the study of nonequilibrium phenomena [1]. The decay of unstable steady states has been studied in various specific contexts such as dynamics of phase transitions [2,3], hydrodynamical instabilities [4], spinodal descomposition [5], the switch on of lasers [6], relaxation of chemical instabilities [7], and dynamics of liquid crystals [8]. Among the various methods proposed to study the decay of unstable steady states, we find the method of time scales called the passage time distribution (PTD) and nonlinear relaxation times (NLRT); both theoretically developed in the context of Langevin-type dynamics [9] or Fokker-Planck equation [10]. In the early nineties it was proposed by Vemury and Roy [11] that very weak optical signals can be detected via the transient dynamics of a laser using the laser as a superregenetive receiver. Immediately after the PTD [12] and NLRT methods [13] were used to corroborate the numerical [11] and experimental [14] results. Later Dellunde *et al.* [15] proposed an alternative passage time method to detect efficiently large optical signals in a laser, showing in this case the oscillatory behavior of the system. In the following year the detection of weak optical signals in the same laser system was studied by the same authors, taking into account the phase fluctuations of the injected signals [16]. However, nothing about the oscillatory behavior of the system was discussed, neither why the quasideterministic (QD) approach works well in the time characterization of such a system. Most of the works cited above rely upon a Langevin-type equation, whose associated systematic force is derived from a potential, with the exception of those studied in Refs. [15,16]. Inspired in these last works, a rotating Langevin-type dynamics has recently been proposed in [17,18], in which that laser system is such a particular case. For rotating unstable systems we mean those which, once leaving the initial unstable state, describe practically deterministic rotational trajectories to reach the steady state or some approximation of it. In Ref. [17] the (QD) approach, valid in the limit of long times, is used to study the characterization of those particular systems of two and three variables, whereas in Ref. [18], a study has been made only in the case of two variables for not so large times, where the QD approach is no longer valid. In this Rapid Communication we propose the study, through the PTD, of the decay process of rotating unstable Langevin-type dynamics in the presence of an external force. The theoretical approach, which is quite general with emphasis in a time-dependent orthogonal rotation matrix, is formulated in the context of two Langevin dynamical representations **x** and **y**, **y** being the transformed space of coordinates in which the Langevin dynamics introduces a different concept of rotating external and internal (noise) forces. It is precisely in the **y** scheme where it can be better understood why the QD approach does not describe the rotational evolution of such systems, a fact not explained in Refs. [15,16]. We also show that in the generalized formalism, the systems of two and three variables are just particular examples. We hope that the present paper may serve to stimulate corresponding experiments or theoretical studies in other fields, for instance, the dynamics of particles in a plasma. Our proposal also admits a covariant formulation.

The rotating unstable Langevin-type equation for the column vector **x** in the presence of an external force $\mathbf{f}_e$ can be written as

$$\dot{\mathbf{x}} = a\mathbf{x} + W\mathbf{x} + n(r)\mathbf{x} + \mathbf{f}_e + \mathbf{z}(t), \tag{1}$$

where a is real and positive, W is a $n\times n$ real antisymmetric matrix such that $W^T = -W$ and W^T is transposed. The scalar function $n(r)$ accounts for nonlinear contributions due to the fact that $r \equiv x^2 = \mathbf{x}^T\mathbf{x}$, r being the square of the norm of the vector and $\mathbf{z}(t)$ the fluctuating force whose elements $\xi_i(t)$ satisfy the property of Gaussian white noise with zero mean value and a correlation function

$$\langle \xi_i(t)\xi_j(t')\rangle = 2Q_{ij}\delta_{ij}\delta(t-t'), \tag{2}$$

*Email address: ines@xanum.uam.mx

where Q_{ij} is the matrix that represents the noise intensity. The linear systematic force $\mathbf{f}_s = a\mathbf{x} + W\mathbf{x}$ is not in general derived from a potential, because $\nabla \times \mathbf{f}_s = \nabla \times W\mathbf{x} \neq 0$ and therefore the rotating character of the dynamics (1) is due to the properties of the matix W.

A different equivalent Langevin-type dynamics may be obtained if we make the change of variable $\mathbf{y} = \mathrm{e}^{-Wt}\mathbf{x}$, such that

$$\dot{\mathbf{y}} = a\mathbf{y} + n(r)\mathbf{y} + \mathfrak{R}^{-1}(t)\mathbf{f}_e + \mathfrak{R}^{-1}(t)\mathbf{z}(t), \tag{3}$$

where the factor $\mathrm{e}^{Wt} = \mathfrak{R}(t)$ is in general a time-dependent orthogonal rotation matrix [19], satisfying that $\mathfrak{R}^T(t) = \mathfrak{R}^{-1}(t)$ and therefore $\mathrm{e}^{-Wt} = \mathfrak{R}^{-1}(t)$; $n(r)$ remains the same function because r is invariant; i.e., $r \equiv \mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{y}$. Clearly in these dynamics, the external force as well as the internal noise, are rotational.

The PTD characterizes the linear approximation of nonlinear dynamics (1) or (3). Thus, it can be shown in general that the linear solution of Eqs. (1) and (3), assuming $x_i(0) = y_i(0) = 0$, can be written as

$$x_i(t) = \mathrm{e}^{at}\mathfrak{R}_{ij}(t)h_j(t), \quad y_j(t) = \mathrm{e}^{at}h_j(t), \tag{4}$$

with

$$h_j(t) = \int_0^t \mathrm{e}^{-as}\mathfrak{R}_{kj}(s)[f_{e_k} + \xi_k(s)]ds. \tag{5}$$

The dynamical characterization of the system will be given in terms of the square of the norm of vector $\mathbf{x}$ and $\mathbf{y}$, which satisfies

$$r(t) = h^2(t)\mathrm{e}^{2at}, \tag{6}$$

where $h^2(t) \equiv \mathbf{h}^T(t)\mathbf{h}(t)$. In the limit of long times, the process (6) is dominated by the exponential term and the process $h^2(t)$ plays the role of an effective initial condition and therefore the solutions (4) become a quasideterministic process. In this limit of approximation, the linear process associated with Eqs. (1) or (3) can be well characterized by the quasideterministic (QD) approach, but the rotational evolution of the system is not properly described by this approach [16,17].

For not so large time scales, the QD approach is no longer valid and therefore another approach must be proposed. Here, we use the strategy proposed in Ref. [18], where the study has only been made for systems of two variables. The PTD at which the system reaches a reference value R^2 can be calculated from Eq. (6), but it is not an easy task, because the right-hand side also depends on time. However, we must get some profit from the statistical properties of the process $\mathbf{h}(t)$, which are given in general by

$$\langle h_i(t)\rangle = \int_0^t \mathrm{e}^{-as}\mathfrak{R}_{ki}(s)f_{e_k}\,ds, \tag{7}$$

and

$$\langle h_i(t)h_j(t)\rangle = \langle h_i(t)\rangle\langle h_j(t)\rangle + \frac{Q}{a}(1 - \mathrm{e}^{-2at})\delta_{ij}, \tag{8}$$

where we have applied the orthogonality of the rotation matrix $\mathfrak{R}(t)$ and assumed that $Q_{kk} = Q$.

To solve the problem, we propose that,

$$h_i(t) = \langle h_i(t)\rangle + g(t)\eta_i, \tag{9}$$

where $g^2(t) = (1 - \mathrm{e}^{-2at})$ and η_i is a Gaussian random variable with zero mean value and variance $\langle \eta_i\eta_j\rangle = Q/a\,\delta_{ij}$. The process (9) is quite compatible with Eqs. (7) and (8). If we assume that the amplitude of the external force dominates over the intensity of internal noise, we can say that the dominant contribution of the process $h_i(t)$ must be the first term of Eq. (9), and therefore, we can make a series expansion in Eq. (6) up to the first order in powers of η_i, such that

$$t = t_P - \frac{g(t_P)}{a}\sum_i \frac{\langle h_i(t_P)\rangle}{|\langle \mathbf{h}(t_P)\rangle|^2}\eta_i + O(\eta_i^2), \tag{10}$$

where $|\langle \mathbf{h}(t_P)\rangle|^2 = \Sigma_i\langle h_i(t_P)\rangle^2$ and t_P is the zeroth order approximation given by

$$t_P = \frac{1}{2a}\ln\left(\frac{R^2}{|\langle \mathbf{h}(t_P)\rangle|^2}\right). \tag{11}$$

The PTD is then

$$\langle t\rangle = t_P = \frac{1}{2a}\ln\left(\frac{R^2}{|\langle \mathbf{h}(t_P)\rangle|^2}\right), \tag{12}$$

and the variance defined as $\langle(\Delta t)^2\rangle \equiv \langle t^2\rangle - \langle t\rangle^2$ will be

$$\langle(\Delta t)^2\rangle = \frac{Q\,g^2(t_P)}{a^3}\sum_i \frac{\langle h_i(t_P)\rangle^2}{|\langle \mathbf{h}(t_P)\rangle|^4}. \tag{13}$$

Clearly, the PTD is only dominated by the deterministic approximation, whereas the variance contains the effect of both internal noise through its intensity Q and external force through the mean value $\langle h_i(t_P)\rangle$.

In the case of two variables, the matrices W and $\mathfrak{R}(t)$ are given by

$$W = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \quad \mathfrak{R}(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}, \tag{14}$$

and therefore $\nabla \times \mathbf{f}_s = -2\omega\hat{\mathbf{k}}$, which is a vector perpendicular to the rotation plane. We use the same experimental data used in Ref. [15] to show, in Fig. 1, a single stochastic trajectory of the system on the plane (x_1, x_2), which is a circular spiral. On the plane (y_1, y_2) the corresponding stochastic trajectory describes "loops" as shown in Fig. 2. According to the solutions (4), the set of spiral or "loops" trajectories emerge from the origin of coordinates to reach the circle of radius R at random directions because of the rotational character of the noise as given by Eq. (5).

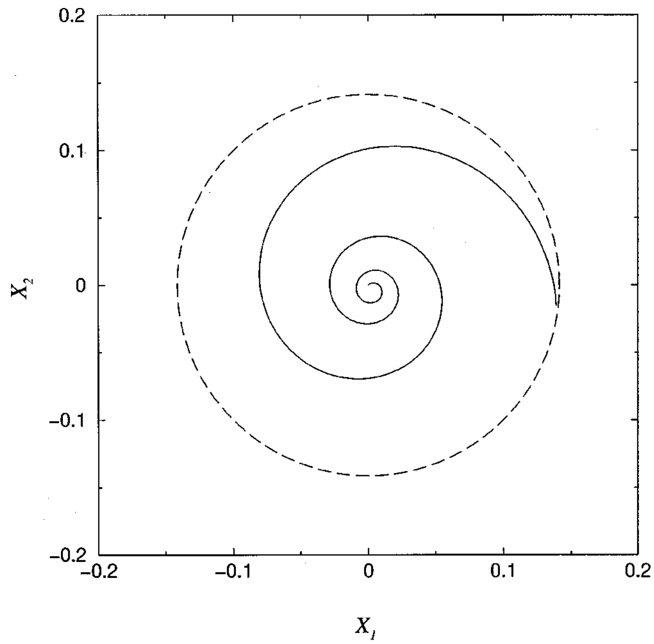

FIG. 1. Linear dynamical evolution of a single stochastic trajectory of Eq. (1) to reach the circle of radius $R^2 = 0.02$ in the case of two variables.

To calculate the mean value of each component of Eq. (7), we can assume, without loss of generality, that $f_{e_1} = f_{e_2} = |\mathbf{f}_e|/\sqrt{2}$ with $|\mathbf{f}_e|$ the modulus of vector $\mathbf{f}_e$. Defining $z(t) \equiv |\mathbf{f}_e|/\lambda_2 2\sqrt{2}(1-e^{-\lambda_2 t})$ and $z^*(t) \equiv |\mathbf{f}_e|/\lambda_1 2\sqrt{2}(1-e^{-\lambda_1 t})$, where the asterisk stands for complex conjugate, $\lambda_1 = a + i\omega$ and $\lambda_2 = a - i\omega$, and $\pm i\omega$, are the eigenvalues of matrix W. In this case we get $\langle h_1(t)\rangle = A(t) + iB(t)$, and $\langle h_2(t)\rangle = A(t) - iB(t)$, where $A(t) \equiv z(t) + z^*(t)$ and $B(t) \equiv z(t) - z^*(t)$. The passage time distribution in this case is then

$$t_P = t_0 - \frac{1}{2a}\ln[1+\phi(t_P)], \tag{15}$$

where

$$t_0 = \frac{1}{2a}\ln\left[\frac{(a^2+\omega^2)}{|\mathbf{f}_e|^2}\right] \tag{16}$$

and $\phi(t) = [e^{-2at} - 2e^{-at}\cos\omega t]$. In the limit of long times, t_P goes to t_0, which is the deterministic limit of the QD approach in the case of a large amplitude of external force and is not appropriate to characterize the rotation of the system [12,16]. The second term of Eq. (15) has an oscillatory behavior due to the function $\phi(t)$, and therefore, for not very long times the time scale t_P must be the appropriate one to characterize the rotating evolution of the system.

For the variance we can get, for the large amplitude of an external force such that $\beta^2 = a|\mathbf{f}_e|^2/Q(a^2+\omega^2) \gg 1$, the following approximation

$$\langle(\Delta t)^2\rangle = \frac{g^2(t_P)}{a^2\beta^2[1+\phi(t_P)]}\left[1+\frac{\phi'(t_P)}{2a(1+\phi(t_P))}\right]^{-2}. \tag{17}$$

The time scale t_P as well as the variance can be calculated through the iterative procedure

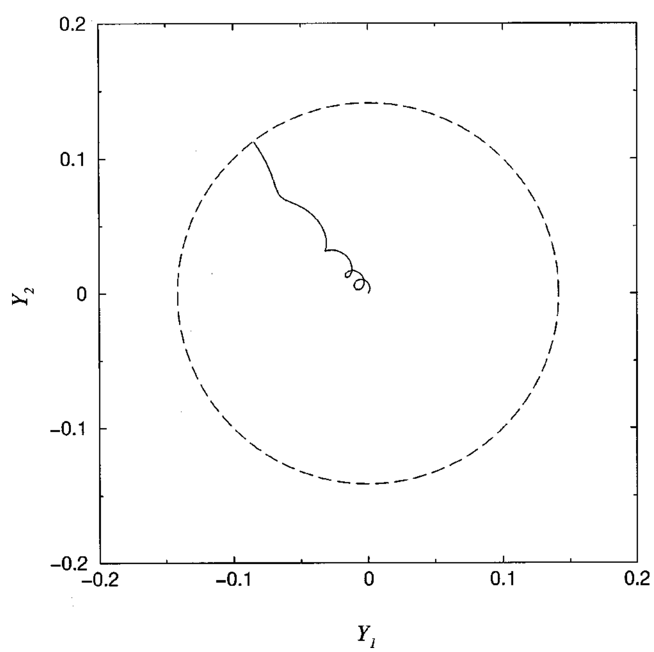

FIG. 2. Linear dynamical evolution of a single stochastic trajectory of Eq. (3) to reach the same circle as in Fig. 1.

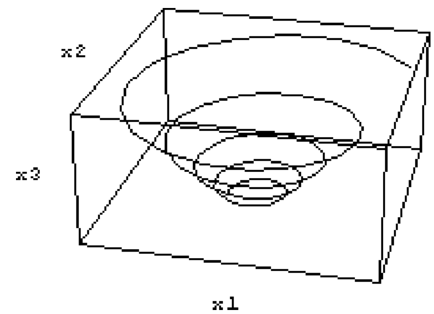

FIG. 3. Same as Fig. 1, but in three dimensions.

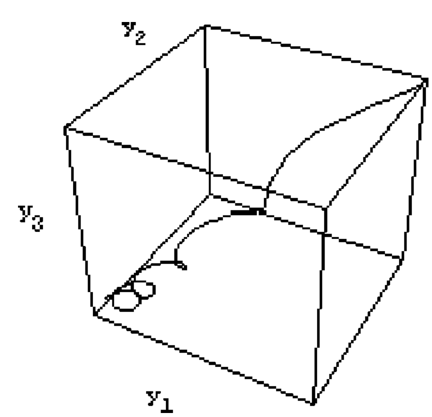

FIG. 4. Same as Fig. 2, but in three dimensions.

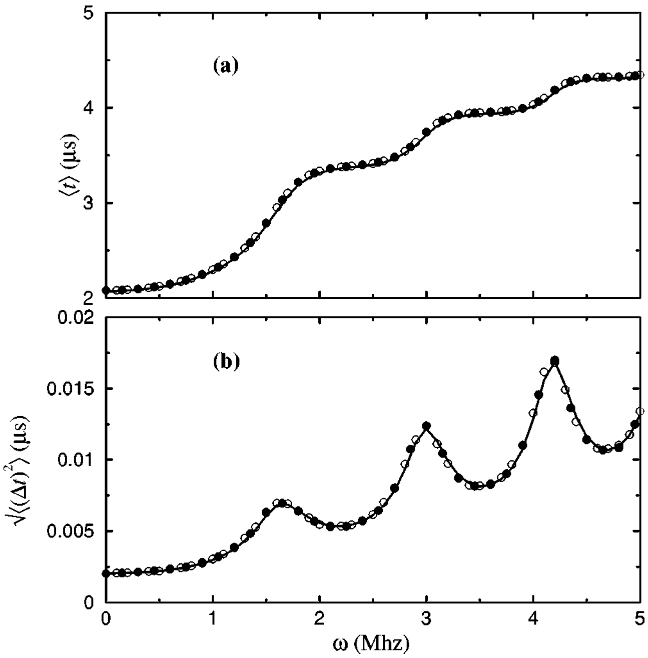

FIG. 5. (a) Mean first time and (b) variance (jitter) as a function of the rotation parameter ω. The solid line corresponds to (a), the iteration of Eq. (18), and to (b), the analytical result of Eq. (17); open circles (filled circles) are the simulation results for the case of two (three) variables.

$$t_P^{(0)}=t_0, \quad t_P^{(n+1)}=t_0-\frac{1}{2a}\ln[1+\phi(t_P^{(n)})]. \tag{18}$$

In the case of three variables, it can be shown that all the 3×3 antisymmetric matrix W' can be reduced to a 3×3 antisymmetric matrix W, very similar to that given in the case of two variables [17]. That is, given the matrix W', it can be reduced to the matrix W and its corresponding associated rotation matrix $\Re(t)$ as follows:

$$W=\begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Re(t)=\begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{19}$$

where now $\omega^2=\omega_1^2+\omega_2^2+\omega_3^2$, where ω_i are the elements of matrix W'. Similarly $\nabla\times\mathbf{f}_s=-2\omega\,\hat{\mathbf{k}}$. Under these circumstances the dynamical evolution of only one stochastic trajectory of the linear solution (4) to reach the sphere (not shown) of radius R, in the space of variables (x_1,x_2,x_3), is also a circular spiral but now growing along the x_3 axis as seen in Fig. 3. Also, the set of the stochastic trajectories leave from the origin of coordinates at random directions due to rotational effects of the noise. Seen along the x_3 axis, the spiral trajectories are essentially the same as those described by the system of two variables. In the **y** representation a single stochastic trajectory of the system to reach the value R is also quite similar to that of Fig. 2, but in the three dimensional space (y_1,y_2,y_3), as shown in Fig. 4. Due to this fact, we can assume that $f_{e_3}=0$ and $\xi_3(t)=0$ in the linear dynamics of Eq. (1) and therefore $\langle h_3(t)\rangle=0$. According with this, the passage time distribution and the variance will be the same as those given by Eqs. (15) and (17), respectively. In Fig. 5 we show the excellent agreement between the theoretical results (15) and (17) and numerical simulations for the systems of two and three variables.

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