Twisted Gaussian Schell-model solitons

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We show that a certain class of spatially partially coherent solitons, namely, twisted Gaussian Schell-model solitons, exists in a logarithmically saturable nonlinear medium with a noninstantaneous temporal response. Unlike previously reported Gaussian Schell-model solitons, those discussed here carry a position-dependent *twist* phase, which vanishes in the fully coherent limit. We demonstrate that the presence of the twist phase provides an opportunity for controlling the degree of spatial coherence of such solitons *without* affecting their intensity.

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A theoretical discovery of self-trapping of optical beams [1] has generated a flurry of theoretical and experimental activity in the field of optical spatial solitons [2–16]. Fully spatially coherent solitons were extensively studied in various nonlinear media, (see [2–5] and references therein). Recently, spatially partially coherent solitons have been experimentally realized in a noninstantaneous photorefractive medium [8], and have been theoretically investigated in materials with photorefractive [9–11], Kerr-type [12–14], saturable logarithmic [15] as well as with thresholding [16] nonlinearities.

The case of media with logarithmic nonlinearity stands out because, first of all, fully spatially coherent solitons existing in such media are just the familiar lowest-order Hermite-Gaussian beams, which are generated by some single-mode lasers. This circumstance facilitates the study and application of such solitons. Second, the width of Gaussian solitons in a saturable medium of logarithmic type is independent of the intensity [5], which is a rather unusual and, perhaps, even unique situation for nonlinear media. Furthermore, the possibility of obtaining simple, analytical results makes logarithmically saturable nonlinearity an attractive model for studying generic properties of fully as well as partially spatially coherent solitons in saturable nonlinear media. To our knowledge, however, the *only* partially coherent self-trapped beams found in a nonlinear medium of logarithmic type to date [15] are the so-called Gaussian Schell-model beams (Chap. 5 of [17]).

In this paper, we show that noninstantaneous, logarithmically saturable nonlinear media support another important class of solitons, namely *twisted* Gaussian Schell-model (TGSM) solitons. We apply second-order coherence theory in the space-time representation (Chap. 3 of [17]) together with a self-consistent multimode approach along the lines outlined in [9,12] to obtain a closed-form analytical expression for the mutual intensity of such solitons. The mutual intensity of each TGSM soliton possesses a nontrivial *twist* phase. This appears to be the first demonstration of the existence of partially coherent solitons carrying a position-dependent phase that vanishes in the fully coherent limit. The presence of the twist phase is important because it significantly extends the range of parameters for which stable solitons exist in nonlinear media of logarithmic type. In a sharp contrast to the case of partially coherent solitons in nonsaturable Kerr-like media, where the physical characteristics of a soliton, which is made up of N uncorrelated modes become effectively arbitrary as the number of modes becomes very large [13], the intensity as well as the degree of coherence of *any* TGSM soliton is completely specified by the soliton width, its spatial coherence length and the magnitude of a twist parameter. Moreover, we show that the possibility of varying the strength of phase twisting enables one to generate twisted Gaussian Schell-model solitons of a given width, with prescribed coherence properties in a wide range of soliton parameters. Hence, one can *control* the degree of spatial coherence of these solitons *without* affecting their intensity. We emphasize that gaining control over such a relatively unexplored degree of freedom in soliton physics as spatial coherence of optical solitons may not only augment our knowledge of fundamental properties of such solitons, but it may also open up novel opportunities in several applied areas such as, for example, imaging with solitons of arbitrary degree of spatial coherence. In this connection, it should be noted that advantages of imaging with partially coherent light have long been appreciated in the domain of linear optics [18].

We begin by considering a statistically stationary optical field $U(\boldsymbol{\rho},z,t)$. In order for such a field to represent a spatially partially coherent beam propagating along the z axis, $U(\boldsymbol{\rho},z,t)$ must be of the form: $U(\boldsymbol{\rho},z,t)=v(\boldsymbol{\rho},z,t)e^{ikz}$, where $k=n_0\omega/c$, ω and n_0 are a carrier frequency and a linear refractive index of the medium, respectively. Further, $v(\boldsymbol{\rho},z,t)$ is an envelope field, which varies slowly with respect to z. The envelope field may be represented as a series in the spatial modes:

$$v(\boldsymbol{\rho},z,t)=\sum_s a_s(t)\psi_s(\boldsymbol{\rho},z). \tag{1}$$

here $\psi_s(\boldsymbol{\rho},z)$ stands for a spatial mode function, s is a set of indices labeling the modes, and $a_s(t)=b_s e^{i\theta_s(t)}$, where b_s is a real constant and $\theta_s(t)$ is a random phase. Suppose, further that the response time of the nonlinear medium is much greater than a characteristic correlation time of phase fluctuations across the partially coherent beam. It then follows from the theory presented in Refs. [9,12] that the partially coherent beam will propagate in such a noninstantaneous medium as a stationary soliton if all the spatial modes ψ_s are statistically stationary, mutually uncorrelated linear modes of a self-induced via the nonlinearity waveguide. The resultant waveguide has a nonevolving index profile, and each spatial mode $\psi_s(\boldsymbol{\rho},z)$ obeys the parabolic equation,

$$\left\{2ik\frac{\partial}{\partial z}+\nabla_\perp^2+k^2n_{nl}^2(I)\right\}\psi_s(\boldsymbol{\rho},z)=0, \tag{2}$$

with the nonlinear refractive index $n_{nl}(I)$ depending only on the time-averaged intensity, $I=\langle|v|^2\rangle$. In Eq. (2), $\nabla_\perp$ is a gradient transverse to the direction of propagation of the beam. Since we are interested in spatial coherence properties of optical solitons, we have to calculate the mutual intensity at a pair of points specified by the vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ in the transverse plane of the beam, defined as (Chap. 3 of [17])

$$\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\langle v^*(\boldsymbol{\rho}_1,z,t)v(\boldsymbol{\rho}_2,z,t)\rangle. \tag{3}$$

Here the angular brackets denote the time average. We have also omitted the time dependence of Γ because of the statistical stationarity of the envelope field. In order to perform the time averaging, we recall that the spatial modes are mutually uncorrelated, which implies the cancellation of the cross-mode interference terms. Mathematically, this is expressed as

$$\langle a_s^*(t)a_{s'}(t)\rangle=\lambda_s\delta_{ss'}, \tag{4}$$

where $\lambda_s=\langle|a_s|^2\rangle$ is the average amplitude of each mode. Next, on substituting from Eq. (1) for the envelope field into Eq. (3), and on taking the time average with the help of Eq. (4), we obtain for the mutual intensity the representation

$$\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\sum_s \lambda_s\psi_s^*(\boldsymbol{\rho}_1,z)\psi_s(\boldsymbol{\rho}_2,z). \tag{5}$$

In practice, the coherence properties of statistical fields are quantitatively described in terms of the degree of spatial coherence [19] at a pair of points specified by the vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, defined as

$$\mu(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\frac{\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)}{\sqrt{I(\boldsymbol{\rho}_1)}\sqrt{I(\boldsymbol{\rho}_2)}}, \tag{6}$$

where the average intensity of the beam $I(\boldsymbol{\rho})$ can be readily shown to be expressed as

$$I(\boldsymbol{\rho})=\sum_s \lambda_s|\psi_s(\boldsymbol{\rho},z)|^2. \tag{7}$$

Given an explicit dependence of the material refractive index on the intensity, the problem of finding self-trapped partially coherent beams is reduced to solving the mode equations (2), together with Eq. (7), which plays the role of the self-consistency condition.

We consider the following model for the nonlinear refractive index [5]:

$$n_{nl}^2(I)=(\Delta n)^2\ln(I/I_t). \tag{8}$$

Here Δn specifies the strength of nonlinearity, and I_t is a "threshold" intensity. This expression for the nonlinear refractive index of the logarithmically saturable medium follows from a more realistic model for such a material, $n^2=n_0^2+(\Delta n)^2\ln(1+I/I_t)$, in the limit $I\gg I_t$. An additional justification for the use of the simplified formula (8) is based on a comparison of numerically simulated dynamics of beams in the media with $\ln(I/I_t)$ and with $\ln(1+I/I_t)$ nonlinearities, respectively. Such a comparison indicates that, provided the ratio $I_{\max}/I_t$, $I_{\max}$ being the peak intensity of the beam, is sufficiently large, the former model approximates well the latter [15].

In order to find partially coherent solitons that can exist in logarithmically saturable media, we first assume the intensity profile of a circular soliton to be Gaussian, so that

$$I(\boldsymbol{\rho})=I_0\exp\left(-\frac{\rho^2}{2\sigma_I^2}\right), \tag{9}$$

where I_0 is the axial intensity and the constant σ_I represents the soliton width. Under this assumption, Eq. (2) for the spatial mode ψ_s takes the form

$$\left\{2ik\frac{\partial}{\partial z}+\nabla_\perp^2+k^2(\Delta n)^2[\ln(I_0/I_t)-\rho^2/2\sigma_I^2]\right\}\psi_s=0, \tag{10}$$

which is equivalent to the Schrödinger equation for a two-dimensional isotropic harmonic oscillator. In view of the axial symmetry of the intensity profile assumed in Eq. (9), we express all the eigensolutions to Eq. (10) in polar coordinates $\boldsymbol{\rho}=(\rho,\phi)$ as [20]

$$\psi_{mn}(\boldsymbol{\rho},z)=(\rho/l_\perp)^{|m|}\exp[i(m\phi+\kappa_{mn}z)]L_n^{|m|}(\rho^2/l_\perp^2)\times\exp\left(-\frac{\rho^2}{2l_\perp^2}\right). \tag{11}$$

Here $L_n^{|m|}(x)$ is an associate Laguerre polynomial of order n, $(n=0,1,2,\ldots)$, and with the azimuthal index m, $(m=0,\pm1,\ldots)$; $l_\perp$ is a characteristic width of each mode in the transverse plane,

$$l_\perp=\left[\frac{k^2(\Delta n)^2}{2\sigma_I^2}\right]^{-1/4}, \tag{12}$$

and κ_{mn} is a mode propagation constant given by the expression

$$\kappa_{mn}=k(\Delta n/\sqrt{2})^2\ln(I_0/I_t)-(2n+|m|+1)/kl_\perp^2. \tag{13}$$

Suppose that the modal weights λ_{mn} are distributed according to

$$\lambda_{mn}=A\frac{n!}{(n+|m|)!}\xi^{n+|m|/2}\eta^m, \tag{14}$$

where A is a positive constant, which is to be determined later, ξ and η specify the weight of each mode in the representation (5). We substitute the expressions for λ_{mn} and ψ_{mn} from Eqs. (11) and (14) into Eq. (5). Next, we sum over the modes with the same azimuthal mode index m, but with different radial indices n, using the following formula for a series of the associated Laguerre-Gaussian polynomials [21]:

$$\sum_{n=0}^{\infty} \frac{n!}{(m+n)!} z^n L_n^m(x) L_n^m(y) = \frac{(xyz)^{-m/2}}{1-z} \exp\left[-\frac{z(x+y)}{1-z}\right] I_m\left(\frac{\sqrt{4xyz}}{1-z}\right). \tag{15}$$

Here $I_m(x)$ is a modified Bessel function of order m. We then perform the summation over the azimuthal mode index m utilizing the generating function for the modified Bessel functions [22],

$$\exp\left[\left(t+\frac{1}{t}\right)\frac{z}{2}\right] = \sum_{m=-\infty}^{\infty} t^m I_m(z), \tag{16}$$

as well as the property $I_m(x) = I_{-m}(x)$. The resulting expression for the mutual intensity of a stable [23] partially coherent soliton carrying the twist phase is found to be

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = I_0 \exp\left(-\frac{\rho_1^2+\rho_2^2}{4\sigma_I^2}\right) \exp\left[-\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2}{2\sigma_c^2}\right] \times \exp[iu\rho_1\rho_2 \sin(\phi_1-\phi_2)], \tag{17}$$

where

$$\frac{1}{2\sigma_I^2} = \frac{1+\xi-(\eta+1/\eta)\sqrt{\xi}}{(1-\xi)l_\perp^2}, \tag{18a}$$

$$\frac{1}{\sigma_c^2} = \frac{(\eta+1/\eta)\sqrt{\xi}}{(1-\xi)l_\perp^2}, \tag{18b}$$

and

$$u = \frac{(\eta-1/\eta)\sqrt{\xi}}{(1-\xi)l_\perp^2}. \tag{18c}$$

It follows at once from Eqs. (6) and (17) that the degree of spatial coherence of TGSM solitons is given by

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \exp\left[-\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2}{2\sigma_c^2}\right] \exp[iu\rho_1\rho_2 \sin(\phi_1-\phi_2)]. \tag{19}$$

In these expressions, σ_I is the soliton width, σ_c is the spatial coherence length of the soliton, and u is a twist parameter [24]. We recall that each spatial mode of the TGSM soliton is an isotropic two-dimensional harmonic oscillator. The component of the angular momentum of such an oscillator along the axis of symmetry is known to be conserved [25], which is manifest in polar coordinates, since $e^{im\phi}$ are simultaneous eigenfunctions of the Hamiltonian and the z component of the angular momentum of each mode. The invariant twist phase arises as a consequence of the presence of such additional integral of motion. It should also be noticed that in deriving Eq. (17), we have already imposed the self-consistency condition by requiring that the soliton intensity amplitude I_0 and width σ_I match those of the ansatz (9). Thus the self-consistency requirement specifies the value of the constant A, $A=(1-\xi)I_0$.

Further, given σ_I, σ_c, and u, there are two unknown parameters, ξ and η, in three equations (18). Hence, these equations not only provide the values of the modal weights in terms of known parameters, but they also specify a relation between the spatial coherence length of the soliton and the twist parameter for which such TGSM solitons exist. In order to find this relation, we first express the quantities ξ and η in terms of the soliton parameters. It follows from Eqs. (18b) and (18c), that

$$\eta = \sqrt{\frac{1+u\sigma_c^2}{1-u\sigma_c^2}}. \tag{20}$$

The non-negativity of every coefficient λ_{mn}, together with Eq. (20), leads to the condition on the twist parameter, which is familiar from linear optics of partially coherent TGSM beams [26,27]

$$-\frac{1}{\sigma_c^2} \leqslant u \leqslant \frac{1}{\sigma_c^2}. \tag{21}$$

It is evident from this inequality that the twist phenomenon is characteristic of partially coherent solitons, since it vanishes in the fully coherent limit, ($u\to 0$ as $\sigma_c\to\infty$). Next, we infer from Eqs. (18a) and (18b), that

$$\xi = \frac{l_\perp^2/\sigma_{\text{eff}}^2-1}{l_\perp^2/\sigma_{\text{eff}}^2+1}, \tag{22}$$

where

$$\frac{1}{\sigma_{\text{eff}}^2} \equiv \frac{1}{\sigma_c^2} + \frac{1}{2\sigma_I^2}. \tag{23}$$

It then follows, again from the non-negativity of the λ_{mn}'s, that

$$l_\perp \geqslant \sigma_{\text{eff}}. \tag{24}$$

Finally, on substituting for η and ξ from Eqs. (20) and (22) into Eq. (18b), we arrive after some algebra at the expression

$$\frac{\sigma_I}{\sigma_c} = \left[\frac{(\alpha^2-1)/2}{1+\sqrt{1+u^2\sigma_c^4(\alpha^2-1)}}\right]^{1/2}, \tag{25}$$

where $\alpha^2 = 2k^2\sigma_I^2(\Delta n)^2$. Equation (25) defines the range of parameters u and σ_c that can serve as the twist strength and the spatial coherence length of a soliton of the width σ_I. For these soliton parameters, inequality (24) can be shown to be

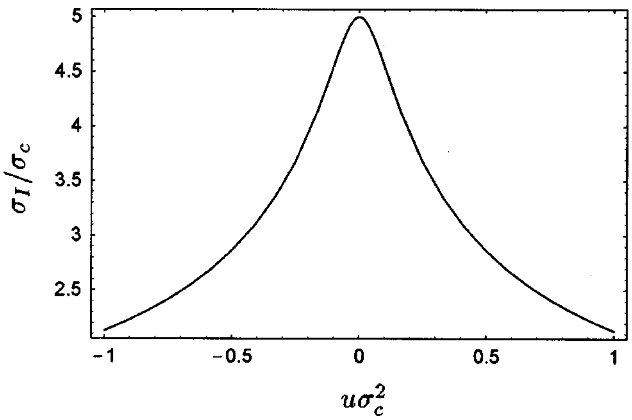

FIG. 1. The ratio σ_I/σ_c of the soliton width σ_I, which is kept fixed, to the variable spatial coherence length σ_c as a function of the parameter $u\sigma_c^2$. The parameter α is taken to be such that $\alpha^2 = 101$. Gaussian Schell-model solitons correspond to the maximum of the curve.

fulfilled. The relation between the soliton parameters σ_I/σ_c and $u\sigma_c^2$ is displayed in Fig. 1. This figure illustrates the dependence of the coherence length of a soliton of a given width on the magnitude of the twist. It is also instructive to plot the soliton width σ_I versus the strength of the nonlinearity Δn of the medium for different values of the twist parameter u (see Fig. 2). It is seen from Fig. 2 that for a fixed value of the coherence length, the dependence of σ_I on u, while strong for relatively weak nonlinearities, tends to become less pronounced as the magnitude of the nonlinear refractive index increases.

The analysis of Eq. (25) indicates that in the fully coherent limit ($\sigma_c \to \infty$), the soliton width is equal to $\sigma_I = 1/\sqrt{2k\Delta n}$, in agreement with earlier results [5]. Furthermore, the coherence length of a partially coherent soliton with no twist, ($u=0$), is *uniquely* determined by its width, which is again in accord with previously reported results [15]. The case of a nonzero twist is essentially different, though, because, given the soliton width σ_I, there is a range

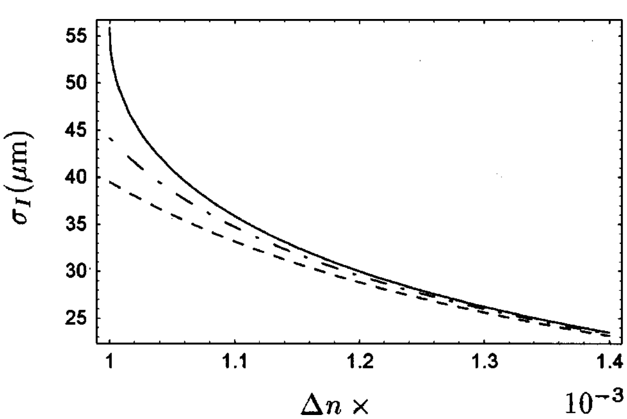

FIG. 2. The soliton width σ_I versus the strength of the nonlinearity Δn for different values of the twist parameter u. Solid, dash-dotted, and dashed lines correspond to the cases $u\sigma_c^2=1$, $u\sigma_c^2 = 0.8$, and $u=0$, respectively. The soliton coherence length is $\sigma_c = 79$ μm.

of attainable soliton coherence lengths. Thus the presence of the twist provides an opportunity for generating solitons with prescribed width *as well as* prescribed spatial coherence length. To produce such TGSM solitons experimentally, one can follow the method, which is somewhat similar to the scheme proposed theoretically in [26] and implemented experimentally in [28] in the context of linear optics. First of all, one can pass a fully coherent, elliptic Gaussian beam of given widths σ_{Ix} and σ_{Iy}, which propagates in a linear medium, through a rotating diffuser. Provided the absorption of light is negligible, the diffuser acts as a random phase screen producing an anisotropic, GSM beam. The mutual intensity of such a beam at a pair of points, specified by the vectors $\boldsymbol{\rho}_1 = (x_1, y_1)$ and $\boldsymbol{\rho}_2 = (x_2, y_2)$ in the transverse plane of the beam, is given by the expression

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = B \exp\left(-\frac{x_1^2 + x_2^2}{4\sigma_{Ix}^2}\right) \exp\left(-\frac{y_1^2 + y_2^2}{4\sigma_{Iy}^2}\right) \times \exp\left[-\frac{(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2}{2\sigma_{c0}^2}\right]. \tag{26}$$

Here B is a positive constant, and σ_{c0} is the coherence length of the anisotropic GSM beam. If the resulting anisotropic GSM beam is then passed through a system of cylindric lenses similar to the one utilized in [28], a TGSM beam emerges at the entrance to the logarithmically saturable nonlinear medium. Finally, we can adjust u and σ_c so that the self-consistency condition (25) is fulfilled. Since such a TGSM beam is an incoherent superposition with the proper modal weights of the modes of the nonlinear waveguide, which is induced by the beam upon its entrance into the logarithmically saturable nonlinear medium, the partially coherent TGSM beam will propagate in such a medium without spreading. Thus given the knowledge of the strength of logarithmic nonlinearity, the coherence control of such solitons can be handled, in principle, by varying certain parameters of a linear optical system at the stage of preparation of a TGSM beam.

To conclude, we have obtained a closed-form analytical expression for the mutual intensity of a class of partially coherent solitons with a position-dependent twist phase, which can exist in logarithmically saturable media. Our analytical results may help to shed light on the properties and behavior of partially coherent solitons in other saturable media. Moreover, since the degree of spatial coherence represents a relatively unexplored dimension in physics of spatial solitons, we believe that the demonstrated possibility of controlling the soliton width and its spatial coherence length simultaneously (and independently) may stimulate a search for effective means of *coherence control* of solitons in saturable nonlinear media. In this connection, it should be mentioned that the possibility of manipulating the degree of spatial coherence of a soliton without affecting the soliton intensity profile is not unique to saturable nonlinearities. The similar possibility has been recently shown to exist in the context of the partially coherent solitons with $\text{sech}^2(x)$ intensity profile that are supported by noninstantaneous Kerr media [14]. We expect solitons with controllable spatial coherence properties to be useful in futuristic all-optical networks for guiding other coherent or partially coherent beams as well as for a distortion-free image transmission through nonlinear media. Another potential application of partially coherent solitons is related to the control of the modulational instability of spatial solitons, which, as was recently demonstrated in Ref. [29], occurs whenever the value of the nonlinearity of a medium exceeds a threshold imposed by the degree of spatial coherence of the soliton.

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