Probability amplitude description of the dynamics of charged particles in a magnetic field in the macrodomain

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The set of Schrödinger-like equations obtained earlier by the author [Phys. Rev. Lett. **26**, 417 (1971), Phys. Rev. A **31**, 3951 (1985)] for the charged particle dynamics in an inhomogeneous magnetic field in the macrodomain, are derived here starting from the quantum-mechanic Schrödinger equation in its path-integral representation. This derivation enables a generalization of the equations to include a curl-free vector potential in the Schrödinger-like equations. In view of the amplitude character of the latter equations, which now descends directly from that of the quantum-mechanic Schrödinger equation, they now predict the existence in the macrodomain of all such phenomena, which are characteristic of a probability amplitude theory, e.g., the interference, and the observation of a curl-free vector à la Aharonov-Bohm. A discrete energy structure, predicted as interference maxima and minima has already been observed by the author with his co-workers [Mod. Phys. Lett. A **8**, 167 (1993)]. A prediction is now made for the observability of a curl-free vector potential in the macrodomain, in the context of the present problem.

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I. INTRODUCTION

A probability amplitude description of particle dynamics is known to be characteristically quantum mechanical, and known to apply in the microdomains of atomic and subatomic dimensions characterized by the quantum of action $\hbar$. An amplitude description is, of course, known to describe the wave phenomena in the macrodomain of the continuous media. But it has not been known to govern the particle dynamics in the macrodomain (excluding the case of many-particle correlated systems that lead to superconductivity and superfluidity in the macrodomain).

The present author had, however, given a "wave-mechanical" model [1] for what was then regarded as the "nonadiabatic" behavior of charged particles in an inhomogeneous magnetic field, which yielded a set of Schrödinger-like equations for the "nonadiabatic behavior." The nonadiabatic leakage of particles from adiabatic traps in this model thus turned out to be in the nature of quantumlike tunneling of the adiabatic potential $(\mu\Omega)$ which governs the dynamics of particles along the field lines (in the adiabatic approximation) through the adiabatic equation of motion (see, for example, Northrop [2])

$$m\frac{dv_\parallel}{dt}=-\nabla_\parallel(\mu\Omega). \tag{1}$$

Here, $v_\parallel=ds/dt$ is the velocity of the particle along the parallel coordinate s of the field line, and μ is an "adiabatic action invariant"

$$\mu=\frac{1}{2}mv_\perp^2/\Omega. \tag{2}$$

This invariance of μ applies, in the limit of the slow variation of the magnetic field as defined by

$$\epsilon=\frac{v_\perp}{\Omega^2}\frac{d\Omega}{ds}\ll 1, \tag{3}$$

where $\Omega=eB/mc$ is the gyrofrequency of the particle and $v_\perp$ is the magnitude of the "perpendicular" velocity of the particle.

Since a nonspecialist reader may not be familiar with these concepts, an elaboration is in order. It has long been known since at least the work of Kruskal [3], who formulated the problem of adiabatic invariance more precisely and elegantly, that the latter concept (of "adiabatic invariance") in classical mechanics is related to a general class of "asymptotic phenomena" and singular perturbation theory in physics, when the Hamiltonian of a system is a function of a small parameter ϵ, which signals a slow variation of the potential or field governing the dynamics. An action $J=\int p\,dq$ can be defined for any bounded degree of freedom of the system, and the problem of adiabatic invariance of the action $J=\int p\,dq$ is one of determining how good the invariance is as the system experiences changes in the field, either explicitly in time, or through motion in space. It has been shown that the action J has an asymptotic expansion in the small parameter ϵ, the first term of which is a good approximation if ϵ is small enough, and is what is widely referred to as the "adiabatic invariant." If ϵ is not "small enough," other terms of the series must be included. When the adiabatic invariance of the gyroaction defined by Eq. (2) is good enough, the three-dimensional motion of the charged particle in a magnetic field reduces to the one-dimensional potential motion along the field line governed by Eq. (1) with the potential $(\mu\Omega)$.

A very crucial aspect of the series for the action J is that besides the terms in various powers of ϵ, there always exists a nonanalytic term of the form $\exp(-c/\epsilon)$ nonexpandible in ϵ that becomes important when ϵ is not small enough. Effects associated with such a term are referred to as "nonadiabatic effects." For example, charged particles that may be trapped in the adiabatic potential by virtue of Eq. (1) may finally leak out, given sufficiently long time. Such a leakage of particles is referred to as a ''nonadiabatic leakage'' or loss. It has been a rather challenging problem mathematically to find a proper way to calculate this loss, because it requires extracting out the contribution due to the nonanalytic term $\sim\exp(-c/\epsilon)$. The author has noted that a formal analogy exists between the phenomena of nonadiabatic leakage of particles from finite adiabatic potential wells and the quantum tunneling of particles from finite classical potential wells. The relationship between quantum mechanics and classical mechanics is also known to be asymptotic in nature as evidenced by the WKB series expansion, where $\hbar$ is formally the small parameter. The nonadiabatic effects, in relation to the adiabatic equation of motion (1) are analogous to what the quantum effects are in relation to the classical equation of motion. And the quantum tunneling probability has the nonanalytic (in $\hbar$) form $\exp(-C/\hbar)$ similar to the form $\exp(-c/\epsilon)$ given above for the term that is responsible for the nonadiabatic leakage of particles.

Based on this formal similarity between the two systems, it was conjectured by the author that a Schrödinger-like description may be possible to describe the nonadiabatic leakage of particles from adiabatic potential wells, $(\mu\Omega)$. Using these ideas, and Feynman path-integral-type considerations, the author was indeed able to derive [1] the following set of Schrödinger-like equations

$$\frac{i\mu}{\lambda}\frac{\partial\Psi(\lambda)}{\partial t}=-\left(\frac{\mu}{\lambda}\right)^2\frac{1}{2m}\frac{\partial^2\Psi(\lambda)}{\partial s^2}+(\mu\Omega)\Psi(\lambda),$$

$$\lambda=1,2,3,\ldots, \tag{4}$$

which are an infinite set of equations for $\lambda=1,2,\ldots$ for the amplitude functions $\Psi(\lambda)$, in which the probability density is given by a generalized expression:

$$\mathcal{P}(s,t)=\sum_{\lambda}\Psi^*(\lambda)\Psi(\lambda). \tag{5}$$

As is obvious from here, the gyroaction μ appears in the role of $\hbar$ and the adiabatic potential $(\mu\Omega)$ appears in the location of potential in the QM-Schrödinger equation. In the limit $\mu\to 0$ (which is formally equivalent to $\epsilon\to 0$) these equations lead to the adiabatic equation of motion (1), while for $\mu\neq 0$, these equations would describe the loss of particles from adiabatic traps analogous to quantum tunneling. The adiabatic equation of motion (1) which follows from Eq. (4) in the limit $\mu\to 0$, bears the same relationship with the latter as the classical equation of motion does with the QM-Schrödinger equation, from which it is similarly obtained in the limit $\hbar\to 0$. These equations were able to describe, quite successfully through the mode $\lambda=1$, the various characteristics of the experimentally determined life terms against nonadiabatic leakage existing at the time (1970), and also predicted the existence of the other multiple lifetimes corresponding to the other modes $\lambda=2,3,\ldots$ for the same energy $\mathcal{E}$ and gyroaction value μ at injection. These multiple lifetimes were subsequently observed [4] through a series of experiments carried out at the Physical Research Laboratory. The success of these predictions by this Schrödinger-like model was strongly suggestive that an amplitude formalism was at work in the macrodomain.

The derivation in Ref. [1] was, however, somewhat heuristic, and it is interesting that such a heuristically inductive derivation was able to produce the set of equations that were later confirmed through a deductive derivation based on a known dynamical equation of classical mechanics as a starting point, since the problem ostensibly belongs to the domain of classical mechanics. The equation of classical mechanics in question is the classical Liouville equation for the problem under consideration, and Eqs. (4) and (5) were obtained by constructing what may be regarded as a Hilbert space representation of the former. Needless to say, the classical Liouville equation is just another representation of classical dynamics, since its characteristics are just the Hamilton equations of motion. (It is perhaps desirable to give a brief outline of the derivation of Ref. [5] leading to Eqs. (4) and (5), to motivate the reader who may not be familiar with it. This is given in the Appendix.) With this derivation, Eqs. (4) and (5) can be justifiably raised from the status of a ''model'' of Ref. [1] to that of a theory, with the important claim that it affords an amplitude description of charged particle dynamics in the macrodomain.

It will be noted from this derivation that each of its nonstandard steps are specially designed as follows: (i) carrying out a noncanonical transformation to the initial momenta values, (ii) change of the gyrophase variable ϕ to the action phase Φ, (iii) writing the Liouville density function $f=\psi^2$, to ensure positive definiteness of f, as well as to serve as a step towards constructing a Hilbert space representation, and finally (iv) obtaining the equation for a suitable Fourier transform Ψ of ψ whose evolution is shown to be governed by the set of Eqs. (4) and (5). This derivation amounts to a kind of ''unfolding'' of the classical Liouville density function f to the set (4) for the set of amplitude functions $\Psi(\lambda)$

It may be pointed out that the gyroaction μ, which takes the place of $\hbar$ in Eq. (4) is typically $\mu\approx 10^9\hbar$, so that the corresponding spatial dimensions would be $L\approx 10^9$ Å $=10$ cm. Thus Eq. (4) presents a case of an amplitude formalism complete with the probability prescription (5) for macroscopic dimensions. The main difference from the amplitude formalism of quantum mechanics is that the latter has the fundamental constant $\hbar$, while in the former case, μ (as an initial value and therefore an *exact* constant of motion) can be chosen to have any value in an experiment. But the amplitude character of Eq. (4) prevails.

Based on this fact (the amplitude character) the author had suggested in Ref. [5] the existence of interferencelike effects in the macrodomain for particles with typical deBroglie-like wavelength $\lambda\approx 10$ cm. Such effects have been subsequently observed by Varma and Punithavelu [6,7] in the form of a discrete energy structure, which essentially represents interference maxima and minima, and which have been recently confirmed by Ito and Yoshida [8] (though their explanation is somewhat different from ours).

However, in spite of the remarkable predictive successes of this theory (the prediction and subsequent observation of previously unexpected and unsuspected multiple residence times in adiabatic traps [4] and interferencelike effects [6–8]) it has enjoyed the status of a ''stand-alone'' theory. It does connect to classical mechanics through the derivation of Ref. [5] which derives it from the classical Liouville equation, by constructing its Hilbert space representation. However, it is not connected in a manner that is commonly regarded as manifest, namely, in terms of the equation of motion and the initial value paradigm, or even through the standard application of the Liouville equation. So the origin and significance of its amplitude character have aroused considerable interest over the years. Being an amplitude theory, one would expect it to be connected with quantum mechanics in some way. But such a connection has not yet been properly elucidated.

It, therefore, becomes interesting and necessary to examine whether the wave functions of the Schrödinger-like formalism of Ref. [5] can be related to the wave function of the QM-Schrödinger equation for the same system, namely, charged particles in an inhomogeneous magnetic field.

It is one of the objectives of this paper to establish such a connection. This would enlarge the scope for making, as we shall see, new predictions for this system in the macrodomain, which would not have been otherwise possible. Such a connection was sought to be established right after the first paper of 1971, and an attempt was made in that direction in 1972 [9]. Though this derivation did produce the same set of equations, it did not go far enough and left some issues unanswered. The advantage of a derivation starting from quantum mechanics would, of course, be that the amplitude character of the derived set of equations, inasmuch as it would now flow directly from that of the QM-Schrödinger equation, would now prevail unreservedly. Furthermore, it would afford a closer understanding of the relationship between the QM-Schrödinger equation and these sets of equations, as well as between the present quantum-mechanical derivation and the derivation of Ref. [5].

We also take the opportunity to generalize these sets of equations to include all the three components of vector potential $\mathbf{A}$, but taking only $A_\theta \hat{\mathbf{e}}_\theta$ to have nonzero curl, so that the magnetic field still has only B_r and B_z components and A_r and A_z are curl free in almost the entire region except for a small source region. We assume axisymmetry of the magnetic field. With this generalization, it will be shown that we obtain a set of equations, still one-dimensional, but with a structure similar to that of the QM-Schrödinger equation with a vector potential, which is assumed here to be curl free in the entire region of space except inside a thin torus where the B_θ field is confined.

As will be shown in Sec. III B, the set of equations with the vector potential so obtained predicts the possibility of observing, in the manner of the Aharonov-Bohm effect, the curl-free vector potential in the macrodomain (≈ 10 cm). Should such an effect indeed be observed in the macrodomain, it would constitute a spectacular demonstration of the amplitude character of governing equations in the macrodomain, for it is the amplitude that carries the information of the vector potential in its phase. Moreover, and more importantly, the observation of the curl-free vector potential in the classical macrodomain, as these observations would signify, would appear to manifestly contradict the Lorentz equation of motion. This would entail an enlargement of our understanding of the classical charged particle dynamics in a magnetic field.

As will be reported shortly, we have indeed found some preliminary experimental evidence for the effect that a curl-free vector potential has on the electrons *à la* Aharonov-Bohm in the classical macrodomain.

In the next section we derive the required set of equations starting from the Feynman path-integral representation for the quantum-mechanical problem under consideration.

II. A PATH-INTEGRAL REPRESENTATION FOR A CHARGED PARTICLE IN AN INHOMOGENEOUS MAGNETIC FIELD AND THE DERIVATION OF THE SET OF SCHRÖDINGER-LIKE EQUATIONS

Since we want to start with the quantum-mechanical considerations of the charged particle dynamics, it is expedient to employ, as done earlier [9], the path-integral representation. Then if $\psi(r,\theta,z,t+\tau)$ is the probability amplitude for the particle at r,θ,z (cylindrical coordinates) at the time $t+\tau$, it is connected to that at $(r-\Delta r,\theta-\Delta\theta,z-\Delta z,t)$ through the Feynman relation (τ being a small time interval)

$$\psi(r,\theta,z,t+\tau)=\left(\frac{m}{2\pi i\hbar\tau}\right)^{3/2}\int d(\Delta\theta)rd(\Delta r)d(\Delta z) \times\exp\left[\frac{i}{\hbar}\int_t^{t+\tau}L\,dt\right] \times\psi(r-\Delta r,\theta-\Delta\theta,z-\Delta z,t), \tag{6}$$

where L is the Lagrangian for the charged particle in a magnetic field:

$$L=\frac{1}{2}m(\dot{x}^2+r^2\dot{\theta}^2+\dot{z}^2)+\frac{e}{c}(\dot{r}A_r+r\dot{\theta}A_\theta+\dot{z}A_z) \tag{7}$$

and where $\int L\,dt$ in the exponent of Eq. (6) is written in the form

$$\int_t^{t+\tau}dt'L=L\tau=\frac{1}{2}m[(\Delta r)^2+r^2(\Delta\theta)^2+(\Delta z)^2]/\tau +\frac{e}{c}(\Delta r\ \ A_r+r\Delta\theta A_\theta+\Delta z A_z). \tag{8}$$

If we take a Fourier transform of Eq. (6) with respect to the variable θ (taking the functions to be periodic with period 2π) we obtain

$$\psi(r,\nu,z,t+\tau)=\left(\frac{m}{2\pi i\hbar\tau}\right)^{3/2}\int d(\Delta\theta)rd(\Delta r)d(\Delta z) \times\exp\left[\frac{i}{\hbar}L\tau-i\nu(\Delta\theta)\right] \times\psi(r-\Delta r,\nu,z-\Delta z,t), \tag{9}$$

where ν is an integer ($+ve$ or $-ve$), the angular Fourier transform variable.

Next, consider the exponent $[i/\hbar L\tau - i\nu(\Delta\theta)]$ with the expression for $L\tau$ given by Eq. (8). On completing the square in $(\Delta\theta)$, we get

$$\frac{1}{\hbar}L\tau-\nu(\Delta\theta)=\frac{1}{2}\frac{mr^2}{\hbar\tau}\left[\Delta\theta-\tau\left(\hbar\nu-\frac{e}{c}rA_\theta\right)/mr^2\right]^2 -\frac{1}{\hbar}\frac{\tau}{2mr^2}\left(\hbar\nu-\frac{e}{c}rA_\theta\right)^2. \tag{10}$$

Using this in Eq. (9) and carrying out integration with respect to $(\Delta\theta)$, this yields

$$\psi(r,z,\nu,t+\tau)=\left(\frac{m}{2\pi i\hbar\tau}\right)\int d(\Delta z)\,d(\Delta r) \times\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(\Delta r)^2}{\tau}+\frac{1}{2}m\frac{(\Delta z)^2}{\tau}+\frac{e}{c}\Delta rA_r +\frac{e}{c}\Delta zA_z\right]-\frac{\tau}{2mr^2}\left(\hbar\nu-\frac{e}{c}rA_\theta\right)^2\right\} \times\psi(r-\Delta r,z-\Delta z,\ \nu,t). \tag{11}$$

Note that the exponent now has the term $1/2mr^2(\hbar\nu-(e/c)rA_\theta)^2$ which represents an effective potential for the (r,z) motion. Note also that in taking the Fourier transform of Eq. (6) we had assumed the θ dependence of A_θ (if any) to be weak enough to justify its being disregarded (since we have assumed axisymmetry, this will not be required here, however). But, in general, this would indeed be justified if we take $\nu\gg1$.

Since we would eventually be interested in the large quantum number limit (approaching ‘‘classical’’ or macroscopic) we shall take $\nu\gg1$. This would be in the spirit of the Born-Oppenheimer approximation. In fact $M=\hbar\nu(\nu\gg1)$ defines the canonical angular momentum which will not be conserved if A_θ is not strictly independent of θ.

Now, we specialize to the case of the near adiabatic limit defined by the inequality (3) which implies that the particle stays close to the magnetic line around which it gyrates; i.e., the gyroradius r_L of the particle is much less compared to the characteristic length L of the magnetic-field variation. In a curl-free inhomogeneous magnetic field, the field lines would, in general, have a curvature. It is then more appropriate to have a local orthogonal system of coordinates in place of the cylindrical coordinate system.

Following Dykhne and Chaplik [10], we employ, for an axisymmetric magnetic-field configuration, the coordinate system (s,x,θ) where s is the length along the line of force, x a coordinate orthogonal to the particular field line, and θ the angular coordinate orthogonal to both x and s. The line element dl in this coordinate system is given by

$$dl^2=dx^2+h_s^2ds^2+h_\theta^2d\theta^2, \tag{12}$$

where h_s and h_θ are the scale factors, $h_s=(1-x/\rho),h_\theta=r$, and $\rho(s)$ is the radius of curvature of the particular field line. If, for simplicity, we consider the small Larmor radius limit and assume ρ to be large, that is, we select a line of force for the particles to be on, near the axis of the magnetic-field configuration, we then have $h_s\simeq1$. The parametric equation of a line of force is given by

$$r=R(s),\quad z=z(s). \tag{13}$$

In the small Larmor radius limit the coordinate x of the particle will always remain small during the motion; we can thus expand $(e/c)rA_\theta$ in the potential-energy term in the power of x/ρ:

$$rA_\theta=(rA_\theta)_{x=0}+x\frac{\partial}{\partial x}(rA_\theta)|_{x=0}+\ \dots\ . \tag{14}$$

Moreover, we have the total magnetic field on a field line

$$B=-\frac{1}{r}\frac{\partial}{\partial x}(rA_\theta). \tag{15}$$

Hence,

$$\left(\hbar\nu-\frac{e}{c}rA_\theta\right)^2=\left[\left(\hbar\nu-\frac{e}{c}(rA_\theta)_{x=0}\right)+\frac{eB}{c}rx\right]^2 =\left(\hbar\nu-\frac{e}{c}(rA_\theta)_o\right)^2+\left(\frac{eB}{c}\right)^2x^2r^2+2xr\left(\frac{eB}{c}\right) \times\left(\hbar\nu-\frac{e}{c}(rA_\theta)_{x=0}\right)+\cdots\ . \tag{16}$$

Note that $(rA_\theta)_{x=0}$ refers to the value on the particular field line from which x is measured. $(rA_\theta)_{x=0}$ basically represents the flux coordinate of the field line. For the axisymmetric case, ν is a constant of motion and $\hbar\nu(\nu\gg1)$ is identified as the canonical angular momentum $M\equiv\hbar\nu$, which is further identified with $e/c(rA_\theta)_{x=0}\equiv M$ (when the departures from axisymmetry as small ν will be an adiabatic invariant). Then Eq. (16) gives the potential-energy term in Eq. (11) as

$$\frac{1}{2mr^2}\left(\hbar\nu-\frac{e}{c}rA_\theta\right)^2=\frac{1}{2}m\Omega^2x^2. \tag{17}$$

Equation (11) then reads in the local coordinate system, as

$$\psi(x,s,\nu,t+\tau)=\left(\frac{m}{2\pi i\hbar\tau}\right)\int d(\Delta s)\,d(\Delta x) \times\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(\Delta x)^2}{\tau}+\frac{1}{2}m\frac{(\Delta s)^2}{\tau}+\frac{e}{c}\Delta xA_x +\frac{e}{c}\Delta sA_s-\frac{\tau}{2}m\Omega^2(s)x^2\right]\right\} \times\psi(x-\Delta x,s-\Delta s,\nu,t). \tag{18}$$

Now define a function

$$\tilde{\psi}(x,s,t;\nu)=\psi(x,s,t,\nu)\exp\left\{-\frac{i}{\hbar}\frac{e}{c}\int^{x}A_x dx\right\}. \quad (19)$$

In terms of $\tilde{\psi}$ we have Eq. (18) as

$$\tilde{\psi}(x,s,\nu,t+\tau)=\left(\frac{m}{2\pi i\hbar\tau}\right)\int d(\Delta s)\,d(\Delta x)\times\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(\Delta x)^2}{\tau}-\frac{1}{2}m\Omega^2(s)x^2\tau+\frac{1}{2}m\frac{(\Delta s)^2}{\tau}+\frac{e}{c}\Delta sA_s+\right]\right\}\times\tilde{\psi}(x-\Delta x,s-\Delta s,\nu,t). \quad (20)$$

Next we consider an eigenfunction expansion [11] of a part of the kernel in Eq. (18), i.e.,

$$\left(\frac{m}{2\pi i\hbar\tau}\right)^{1/2}\exp\left\{\frac{i}{\hbar\tau}\left[\frac{1}{2}m(\Delta x)^2-\frac{1}{2}m\Omega^2\tau^2x^2\right]\right\}=\sum_{n'}\chi_{n'}(x)e^{-iE_{n'}\tau/\hbar}\chi^{*}_{n'}(x-\Delta x). \quad (21)$$

This part of the kernel represents a harmonic oscillator with frequency Ω, where the χ_n are the harmonic-oscillator wave functions and

$$E_n=\left(n+\frac{1}{2}\right)\hbar\Omega(s) \quad (22)$$

are the Landau energy levels. If we now use the expansion (21) in Eq. (20), multiply both sides by χ^{*}_{n}, and integrate over x, using the orthonormality of the eigenfunction χ_n, we obtain

$$\Psi(s,n,\nu,t+\tau)=\left(\frac{m}{2\pi i\hbar\tau}\right)^{1/2}\int d(\Delta s)\times\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(\Delta s)^2}{\tau}+\frac{e}{c}\Delta sA_s-n\hbar\Omega(s)\tau\right]\right\}\times\Psi(s-\Delta s,n,\nu,t+\tau). \quad (23)$$

Equation (23) follows on carrying out the integration over (Δx) on the right-hand side of Eq. (20). Consider now the exponent in Eq. (23), and note that it can be written as

$$\frac{1}{2}m\frac{(\Delta s)^2}{\hbar\tau}+\frac{e}{\hbar c}\Delta sA_s-n\Omega\tau=n\left[\frac{1}{2}\frac{m}{\hbar n}\left(\frac{\Delta s}{\tau}\right)^2+\frac{e}{\hbar nc}\left(\frac{\Delta s}{\tau}\right)A_s-\Omega\right]\tau. \quad (24)$$

Now note also that when A_s and Ω are independent of s (homogeneous field) n is a strict constant of motion; call it n_o. When Ω and A_s are slowly varying functions of s, n_o is an adiabatic invariant, which is identified with $\mu=n_o\hbar$, $n_o\gg1$ the gyroaction. However, transitions will, in general, occur from n_o to $n=n_o+\lambda$, where $n_o\gg\lambda>1$, when Ω and A_s vary with s. Therefore, λ represents a change in the quantum number from n_o induced by the motion in a varying magnetic field and vector potential A_s. (We may call it ''nonadiabaticity.'' But this should be properly considered as ''quantum nonadiabaticity'' as we are still in the quantum domain, albeit with large quantum numbers.) We therefore note that

$$n_o\left[\frac{1}{2}\frac{m}{\hbar n_o}\left(\frac{\Delta s}{\tau}\right)^2+\hbar n_o\frac{e}{c}\left(\frac{\Delta s}{\tau}\right)A_s-\Omega\right]\tau=n_oL_A\tau/\mu\to nL_A\tau/\mu=(n_o+\lambda)L_A\tau/\mu, \quad (25)$$

where

$$L_A=\frac{1}{2}m\left(\frac{\Delta s}{\tau}\right)^2-\frac{e}{c}\left(\frac{\Delta s}{\tau}\right)A_s-\mu\Omega \quad (26)$$

is the effective Lagrangian in the presence of a vector potential A_s. It is actually the adiabatic Lagrangian but for the vector potential term. In view of, Eqs. (24) and (25) we therefore, write the exponent (24) in Eq. (23) as

$$n_oL_A\tau/\mu+\lambda L_A\tau/\mu=\frac{1}{2}\frac{m}{\hbar}\left[\frac{(\Delta s)^2}{\tau}+\frac{e}{c}\Delta sA_s-n_o\hbar\Omega\tau\right]+\lambda L_A\tau/\mu. \quad (27)$$

This leads Eq. (23) to the form

$$\Psi(s,t+\tau,n_o+\lambda;\nu)=\left(\frac{m}{2\pi i\hbar\tau}\right)^{1/2}\int d(\Delta s)\times\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(\Delta s)^2}{\tau}+\frac{e}{c}\Delta sA_s-n_o\hbar\Omega(s)\tau\right]_{(1)}+\frac{i\lambda}{\mu}\left[\frac{1}{2}m\frac{(\Delta S)^2}{\tau}+\frac{e}{c}\Delta SA_s-\mu\Omega\tau\right]_{(2)}\right\}\times\Psi(s-\Delta s,t,n_o+\lambda;\nu). \quad (28)$$

We note that the two terms in the exponent in Eq. (28) lead to widely different scales of variation of the wave function. The first one [in the square bracket marked with the subscript (1)] leads to the variation on the microscopic scale characterized by the denominator $\hbar$, while the second one [marked with the subscript (2)] leads to a variation on the macroscale characterized by the denominator $\mu=n_o\hbar$. We also accordingly denote the interval Δs in the second bracket by ΔS to emphasize the slower variation of the macroscale. Now, a factor of the kernel with the subscript (1) term in the exponent in Eq. (28), namely,

$$\mathcal{K}_s=\left(\frac{m}{2\pi i\hbar\tau}\right)^{1/2}\times\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(\Delta s)^2}{\tau}+\frac{e}{c}\Delta s A_s-n_o\hbar\Omega(s)\right]\right\}, \quad (29)$$

which represents a motion along the coordinate s in the potential $n_o\hbar\Omega(s)$, and curl-free vector potential A_s, can be expanded in terms of the energy eigenfunctions [11] of the Hamiltonian corresponding to the Lagrangian in the exponent of the kernel $\mathcal{K}_s$ in Eq. (29):

$$\mathcal{K}_s=\sum_{\kappa'}\varphi_{\kappa'}(s)e^{-iE_{\kappa'}\tau/\hbar}\varphi^*_{\kappa'}(s-\Delta s), \quad (30)$$

where

$$E_\kappa=\frac{1}{2m}\left(\hbar\kappa-\frac{e}{c}A_s\right)^2+n_o\hbar\Omega=\frac{1}{2m}(\hbar K)^2+n_o\hbar\Omega \quad (31a)$$

with

$$K=\left(\kappa-\frac{e}{\hbar c}A_s\right)=\left(\frac{2m}{\hbar^2}\right)^{1/2}[E_K-n_o\hbar\Omega]^{1/2} \quad (31b)$$

and

$$\varphi_\kappa(s)=\left(\frac{m}{2\pi K\hbar^2}\right)^{1/2}\frac{1}{2i}\left[\exp\left\{i\int\kappa_+ds\right\}-\exp\left\{i\int\kappa_-ds\right\}\right]$$
$$=\left(\frac{m}{2\pi K\hbar^2}\right)^{1/2}\exp\left[\frac{ie}{\hbar c}\int^s A_s ds\right]\sin\left[\int^s ds K(s)\right], \quad (32a)$$

where

$$\kappa_\pm=\frac{e}{\hbar c}A_s\pm K. \quad (32b)$$

We shall now transform, away from Eq. (28) the rapidly varying part $\mathcal{K}_s$ of the kernel in this equation by transforming it into the Fourier space with respect to that part of s, which accounts for the rapid variation. Thus using Eq. (30) in Eq. (28) multiplying both sides by φ^*_κ, and integrating over s, and later over Δs (on the right-hand side) one gets

$$\Psi(K,\lambda,t+\tau;n_o,\nu)=e^{-iE_\kappa\tau/\hbar}\exp\left[\frac{i\lambda}{\mu}\left\{\frac{1}{2}m\frac{(\Delta S)^2}{\tau}+\frac{e}{c}\Delta S A_S-\mu\Omega\tau\right\}\right]\Psi(K,\lambda,t;n_o,\nu), \quad (33)$$

where the weak dependence of Ω and A_s on S has been disregarded in taking the Fourier transform. Writing $K=K_o+k$, where K_o is a large constant wave number $K_o\gg 1/L$ (L, characteristic length of Ω and A_s) and $k\ll K_o$ ($k\sim 1/L$). Then

$$E_K=E_{K_o}+\frac{\hbar k}{m}(\hbar K_o)$$
$$=E_{K_o}+(\hbar k)v_o, \quad (34)$$

noting that $(\hbar K_o)=mv_o$, with v_o being the velocity corresponding to the wave number K_o. Using Eq. (34) in Eq. (33) we get (dropping the subscript o on v_o)

$$\Psi(k,\lambda,t+\tau;K_o,n_o,\nu)=e^{-iE_{K_o}\tau/\hbar}\exp\left[\frac{i\lambda}{\mu}\left\{\frac{1}{2}m\frac{(\Delta S)^2}{\tau}+\frac{e}{c}\Delta S A_S-\mu\Omega\tau\right\}-ikv\tau\right]\times\Psi(k,\lambda,t,K_o,n_o,\nu). \quad (35)$$

Equation (35) is still left with a rapid time dependence characteristic of the microdomain of $\hbar$ [showing through the factor $\exp(-iE_{K_o}\tau/\hbar)$ on the right-hand side]. To "remove" (transform away) this (rapid) time dependence, multiply both sides by $\exp[iE_{k_o}t/\hbar]$, and integrate over a time interval Δt, $T\gg\Delta t\gg\hbar/E_{\kappa_o}$ (T being the characteristic macroscopic time $T\simeq L/v$) which finally yields

$$\Psi(k,\lambda,t+\tau;E_{K_o},K_o,n_o,\nu)=\exp\left[i\frac{\lambda}{\mu}\left\{\frac{1}{2}m\frac{(\Delta S)^2}{\tau}+\frac{e}{c}\Delta S A_s-\mu\Omega\tau\right\}-ikv\tau\right]\times\Psi(k,\lambda,t;E_{K_o},K_o,n_o,\nu). \quad (36)$$

There are now two ways to proceed from Eq. (36). One, is to take the inverse transform with respect to k, which gives [the momentum parameters (ν,n_o,K_o,E_{K_o}) will be suppressed hereafter]

$$\Psi(S,\lambda,t+\tau;\nu,n_o,K_o,E_{K_o})=\exp\left[\frac{i\lambda}{\mu}L_A\tau\right]\Psi(S-v\tau,\lambda,t;\nu,n_o,K_o,E_{K_o}), \quad (37)$$

where

$$L_A=\frac{1}{2}m\left(\frac{\Delta S}{\tau}\right)^2+\frac{e}{c}\frac{\Delta S}{\tau}A_S-\mu\Omega \quad (38)$$

is the reduced Lagrangian. We no longer call it "adiabatic" Lagrangian because of the presence of the curl-free vector potential A_S, which would have no effect on the adiabatic (classical) equation of motion. But in Eq. (37) for the amplitude Ψ, it will have a nontrivial effect.

As can be seen, Eq. (37) has been obtained from Eq. (16) by systematically transforming away all the rapid dependencies on θ, the coordinates x and s, and the time t, characteristic of the microscale of $\hbar$. The corresponding ''central'' quantum wave numbers (ν, n_o, K_o, E_{K_o}) appear as parameters in the argument of the wave functions in Eq. (37). These wave functions then represent the transition amplitudes from the large quantum numbers (n_o, K_o) to $(n_o + \lambda, K_o + k)$ as a consequence of magnetic-field inhomogeneity, with the quantum numbers characterizing the large quantum number state (ν, n_o, K_o, E_{K_o}), which may be regarded as initial values. It must be emphasized that these transition amplitudes Ψ are probability amplitudes nevertheless, and exhibit all the properties that are characteristic of a quantum-mechanical probability amplitude, but now in the macroscopic domain characterized by the wave numbers $k \ll K_o$, $\lambda \ll n_o$, and the action $\mu = n_o \hbar \gg \hbar$. Note that since we considered an axisymmetric magnetic field, where $M = \hbar \nu (\nu \gg 1)$ is an exact constant of motion, no division such as $\nu = \nu_o + \delta (\delta \ll \nu_o)$ was required to be done. It would, however, be necessary if one were to consider a nonaxisymmetric field, in which case δ would correspond to the change from ν_o due to nonaxisymmetry. Such a case is considered later.

Equation (37) is of the Feynman path-integral form. One way to proceed is to integrate the right-hand side over $(\Delta S) = v\tau$, which yields

$$\Psi(S, \lambda, t+\tau) = \left(\frac{m\lambda}{2\pi i \mu \tau}\right)^{1/2} \int d(\Delta S) \times \exp\left[i \frac{\lambda}{\mu} L_A \tau\right] \Psi(S - \Delta S, \lambda, t). \quad (39)$$

Using the standard procedure (Feynman and Hibbs [11]) this gives

$$\frac{i\mu}{\lambda} \frac{\partial \Psi(\lambda)}{\partial t} = -\frac{1}{2m}\left(\frac{\mu}{\lambda i} \frac{\partial}{\partial S} - \frac{e}{c} A_S\right)^2 \Psi(\lambda) + \mu \Omega \Psi(\lambda), \quad \lambda = 1,2,3, \ldots, \quad (40)$$

where the probability density $\mathcal{P}(S,t)$ is then given by

$$\mathcal{P}(S,t) = \sum_{\lambda} \Psi^*(\lambda) \Psi(\lambda). \quad (41)$$

This set of equations [(38) and (39)] is then a generalization of the set obtained earlier [5] to include a curl-free vector potential. The other way to proceed from Eq. (36) is to follow the procedure given in Ref. [5] where apart from the vector potential term A_S, the same equation was obtained [Eq. (31) of Ref. [5]]. This method does not appeal to the Feynman procedure as carried out in Eq. (39).

III. DISCUSSION

A. The nature of the Schrödinger-like formalism

The first thing to note about the Schrödinger-like formalism, represented by Eqs. (40) and (41) is that the wave functions Ψ governed by these equations must necessarily be amplitudes in the sense of wave mechanics, as they flow directly from the wave amplitudes of the QM-Schrödinger formalism. The second thing to note is that these equations are identical in form [except the generalization to include the vector potential in Eq. (40)] to those obtained earlier [5] as a Hilbert space representation of the classical Liouville equation, or to those obtained even earlier [1] through a heuristic derivation. In fact, it is interesting to note that Eq. (36) is the same as Eq. (31) of Ref. [5] (apart from the vector potential term in Eq. (36), which is absent from Eq. (31) of Ref. [5]). Also the parameter argument $\alpha' s$ of the amplitude functions of Ref. [5] which were taken to be the initial values of the momenta, $\{\alpha\} \equiv (M, p_o, \mu_o, \mathcal{E})$ [being, respectively, the canonical angular momentum M, the linear momentum p_o, gyroaction μ_o, and energy $\mathcal{E}$, are identical with the parameter argument of the functions of the present derivation which are {refer to Eq. (36)} ν, K_o, n_o, E_{K_o} and essentially the set $(M = \hbar \nu, p_o = \hbar K_o, \mu = \hbar n_o, \mathcal{E} = E_{K_o})$ of Ref. [5]. Hence there is a one to one correspondence between the amplitude functions of Ref. [5] and those of this paper.

Therefore the present derivation from quantum mechanics vindicates the earlier derivations and assignment to the functions $\Psi(n)$ of Ref. [5] and the meaning of wave amplitude *à la* wave mechanics, and therefore justifies the prediction made about their describing interferencelike phenomena, for which the evidence has already been reported [6–8]. The significant point to be appreciated is that these Eqs. (40) and (41) refer now to the macroscopic dimension of 10–50 cm, characterized by the magnitude of $\mu \approx 10^9 \hbar$ (typically) rather than to the microdomain of ~ 1 Å which is characteristic of $\hbar$. This presents a severe dilemma *vis a vis* the standard classical-mechanical (equation of motion–initial value) paradigm, which does not support such an interference phenomena. (It may be mentioned, however, that there does exist, to be sure, wave and interference phenomena in classical physics, for example, waves on a string, or in any other continuous medium. But the dilemma mentioned above refes only to the equation of motion-initial value paradigm for single-particle dynamics, which does not support interference phenomena.) The author has been faced with this dilemma, and he has suggested [12] that topological considerations in classical mechanics may be at play. Topological properties are global properties of a system, and cannot be captured by the standard equation of motion-intial value paradigm, which represents only a local evolution. He has in fact shown [12] that the Einstein-Bohr-Sommerfeld kind of quantization conditions can be obtained for a classical-mechanical system as a consequence of their topological properties, where the role of $\hbar$ is enacted by an appropriate action (Poincaré invariant) belonging to the classical-mechanical system. It would thus seem that the Hilbert space representation of the classical Liouville equation captures the global topological properties of the system configuration space, and the wave amplitude character of the equations so obtained is a reflection of that fact.

From the point of view of the present derivation (from quantum mechanics) it is interesting to examine the meaning of the $\Psi(\lambda)$ and the index λ. We recall that λ was taken to be the change in the Landau-level quantum number from n_o

to $n_o+\lambda$, induced by the inhomogeneity in the magnetic field as the motion takes place along the field line. Thus $\Psi(\lambda)$ has the interpretation of the transition probability amplitude for finding the particle in the state $n_o+\lambda$ (n_o being the level number in the absence of inhomogeneity). This transition to the states $(n_o\pm\lambda)$ induced by the inhomogeneity may be termed the "quantum nonadiabaticity."

B. Observation of the curl-free vector potential *à la* Aharonov-Bohm in the macrodomain

The set of Eqs. (40) and (41) generalized as they are to include a curl-free vector potential, afford the possibility of making yet another spectacular prediction, namely, the observability of the curl-free vector potential in the macrodomain, in the manner of the Aharonov-Bohm effect in the microdomain of $\hbar$. Since Eq. (40) is one-dimensional (along the field line coordinate) the situation here is somewhat different, however, compared to the standard Aharonov-Bohm effect. From Eq. (40), for $\lambda=1$, we get the condition for the one-dimensional interference maxima

$$\int_o^L\left(mv_s+\frac{e}{c}A_s\right)ds=2\pi n\mu, \tag{42}$$

where v_s is the velocity of the particle along the field line. If we consider the passage of electrons from an electron source S to a Faraday cup detector D, through a Rowland ring [a torus of a high-magnetic permeability material wound around by current carrying wires, so that the magnetic B_θ field is completely confined in it, and there is only a curl-free vector potential (A_r, A_z) in the space outside] then Eq. (42) yields, on carrying out integration between the source and detector

$$\int_o^L mv_s\,ds+\beta\frac{e}{c}\Phi=2\pi n\mu, \tag{43}$$

where Φ is the flux $\Phi=\int B_\theta\,d\sigma$ enclosed in the Rowland ring, and β is a geometrical factor that is determined by the distance L between the gun and the detector and the mean radius of the Rowland ring.

It may be mentioned that condition (42) for the interference maxima follows by considering at the plate, the interference of two waves: (i) the one originating at the source (the electron gun) with the amplitude given by $\exp[(i/\mu)\int_o^L ds(mv+eA/c)]$ and (ii) the other one a scattered wave, scattered off the grid sitting just next to the plate, with the amplitude given by $\exp[(i/\mu)\int_{L-\delta}^L ds(mv+eA/c)]$, where δ is the plate-grid separation, $\delta\ll L$. It is interesting to note that in this situation, there are no two paths that encircle the flux topologically. On the other hand, we have two open paths, one from the source to the plate, and the other from the grid to the plate detector (corresponding to the scattered wave from the grid). The two paths span different extents of the one-dimensional space with the vector potential A, thereby leading to the condition (42) for the maxima arising out of the two waves mentioned above. Equation (43) therefore, involves only a fraction β of the total flux Φ in the ring, with β being unity when there is complete linkage as in the case of the standard A-B effect.

From Eq. (43) it follows that the value of Φ would clearly affect the position of the interference maxima exhibiting a periodicity with $(e/c)(\beta\Phi/\mu)$. One can vary Φ by varying the current in the Rowland ring and look for a periodic change in the interference maxima.

In experiments carried out at the Physical Research Laboratory, [13] we have indeed found some evidence for the existence of these effects. These are at present under scrutiny, and should be reported shortly.

IV. CONCLUDING REMARKS

The problem of charged particle dynamics in a magnetic field that we have studied over the last 30 years, both theoretically and experimentally, has been a rather interesting venture. The Schrödinger-like formalism for its description, which we discovered 30 years back and checked experimentally for its predictions over the years has, however, presented an enigma. Does it represent a description of the classical-mechanical attributes of the system? Looking at the macrodimensions of its domain, it ought to be so. But looking at its amplitude character, which, in view of the present derivation, descends directly from that of quantum mechanics, must imply in it all the physical attributes of an amplitude theory *à la* wave mechanics; for instance, the consequences of the existence of phase (such as the interference phenomena, observability of a curl-free vector potential). Its predictions on the existence of the multiplicity of lifetimes and their subsequent experimental confirmation, followed directly from the amplitude character of the equations. Likewise the observations of the curl-free vector potential, which is completely outside the domain of classical mechanics, is possible only through the agency of the phase of the amplitude. Of course, the results on the observation of the vector potential are still being scrutinized and need to be confirmed by other workers later. But, as is clear from Eq. (40), the vector potential appears in it in a significant way, and must have its consequences.

What then is the nature of this theory? The present derivation certainly establishes its connection with quantum mechanics. Its amplitude character, is, to be sure, a signature of quantum mechanics, as per this derivation, though it carries no other signature of the latter (for instance, no $\hbar$). One may say that it carries some remnants of the quantum structure into the macrodomain, a kind of "quantum wings" of the macrodomain of classical mechanics. On the other hand, its amplitude character had also been obtained [5] from the classical Liouville equation without appeal to quantum mechanics. Thus, notwithstanding its derivation from quantum mechanics, its amplitude character must be regarded as independent of the latter. It may be concluded that the amplitude character of the formalism of Ref [5] must represent some fundamental property of classical particle dynamics itself (in relation to at least the particular dynamical system) which has not been revealed so far. It is possible that it is related to the topological structure of classical dynamics along the lines elucidated by the author earlier, which can be captured by an amplitude or Hilbert space representation. Clearly, further investigations of these equations are desirable and should help understand the issues better.

Meanwhile, another incidental but important comment may, however, be made about Eqs. (40) and (41). Clearly these equations have been obtained using the large quantum number *ansatz*. As per the correspondence principle, they ought to belong to the macrodomain of classical dynamics. But they are still amplitude equations. In this sense these equations may be considered to represent a relationship between classical and quantum mechanics, though the classical limit is not visible in terms of the Hamilton-Jacobi representation as the standard quasiclassical (WKB) approximation yields, but rather in terms of the action-angle formalism. We shall dwell on this issue in detail later.

APPENDIX: AN OUTLINE OF THE DERIVATION OF EQS. (4) AND (5) FROM R. K. VARMA, PHYS. REV. A 31, 3952 (1985)

As noted in Ref. [1], the adiabatic equation of motion (1) for the dynamics along the field line, can be obtained from the Lagrangian $L=\frac{1}{2}mv_\parallel^2-\mu\Omega$, through the stationarity of the action

$$S=\int_{t_1}^{t_2}dt\left(\frac{1}{2}m\dot{X}^2-\mu\Omega\right). \tag{A1}$$

One may define an action phase Φ through

$$\Phi=S/\mu=\frac{1}{\mu}\int dt\frac{1}{2}mv^2-\int\Omega\,dt\equiv\frac{1}{\mu}\int\frac{1}{2}mv^2\,dt+\phi, \tag{A2}$$

where $\phi=-\int^t\Omega\,dt$ is the gyrophase of the particle in the magnetic field; where μ is taken here to be the *initial value* of the gyroaction that is an *exact* constant of motion, rather than just an adiabatic invariant.

One starts in Ref. [5] from the classical Liouville equation for an ensemble of charged particles in an inhomogeneous, static magnetic field. The ensemble chosen is taken to be what has been termed as a "coherent system of trajectories," by Synge [15] and a "family" by Dirac [16]. This corresponds to a δ function in the initial momenta values, and a distribution in the position coordinates (or vice versa). The Liouville equation is thus transformed to the initial momenta values α_i (from the "current momenta") which are by definition, the exact constants of motion. The corresponding terms $\dot{\alpha}_i\partial f/\partial\alpha_i$ in the Louville equation thus disappear from it, and α_i appear as parameters in the Liouville density function. These are taken to be $P_\parallel^{(o)}$, with the initial momentum parallel to the magnetic field $\mathcal{P}_\theta^{(o)}$, the canonical angular momentum for the assumed axisymmetric field, and $\mu^{(o)}$, the initial gyroaction value and the energy $\mathcal{E}$, all assumed to have δ-function distributions. The Liouville equation is then left with the terms corresponding to the conjugate position coordinates, namely, $X_\parallel$, the "parallel" coordinate ϕ, the gyrophase, and, of course, the time t, and is given as

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial X}-\Omega\frac{\partial f}{\partial\phi}=0, \tag{A3}$$

where the subscript parallel on X is dropped, and $v=\dot{X}$ is now considered as a function of the X and appropriate initial momenta values, rather than being independent of X as in the Liouville equation before this transformation. This is obviously a noncanonical transformation. The Liouville density function f is thus a function of (X,ϕ,t) and the initial momenta values $\{\alpha_i\}\equiv(p_\parallel^{(o)},\mu^{(o)},\mathcal{P}_\theta,\mathcal{E})$ appear as parameters $f=f(X,\phi,t;\{\alpha_i\})$. Two important steps are next carried out: First, the gyrophase angle ϕ is transformed to the action phase Φ defined by Eq. (A2), and second, the Liouville density function f is written as $f=\psi^2$, with ψ as a real quantity. This ensures the positive definiteness of f, as well as paves the way, at the same time, to construct a Hilbert space representation of the Liouville equation. It should be obvious that ψ must also satisfy the transformed Liouville equation (A3) with $\psi=\psi(X,\Phi,t;\alpha_i)$. The importance of the transformation to the action phase Φ lies in the fact that when $\Phi=(1/\mu)\int L\,dt$ appears as an exponent in the form $\exp[(i/\mu)\int L\,dt]$ it is reminiscent of the Feynman kernel, where now μ appears in place of $\hbar$. This is indeed the origin of μ appearing in the role of $\hbar$ in Eqs. (4), and the adiabatic potential $(\mu\Omega)$ appearing in the location of potential in the Schrödinger equation.

A Fourier series expansion of ψ with respect to Φ,

$$\psi=\sum_n\hat{\psi}(n)e^{-in\Phi}, \tag{A4}$$

is employed in the finite-time representation of the equation for ψ, yielding an equation of the form

$$\hat{\psi}(X,l,t+\tau)=e^{i(l/\mu)\int_t^{t+\tau}dt'\,L}\hat{\psi}(X-v\tau,l,t). \tag{A5}$$

This is clearly of the Feynman path-integral form although not exactly. It is, therefore, dealt with in a different way as given in Ref. [5], leading finally to the Schrödinger-like equation (4) of the text. Furthermore, if one uses the expression (A4) in the relation $f=\psi^2$, an integration of this over the unobservable action phase Φ, yields the convolution $\Sigma_l\Psi^*(l)\Psi(l)$ on the right-hand side of Eqs. (5) or (42) for the probability density $\mathcal{P}(X,t)=\int d\Phi f(X,\Phi,t)$, where $\Psi(X,l,t)$, which is governed by Eq. (40), is related to $\hat{\psi}(X,l,t)$ in a manner given in Ref. [5].

It may be noted that the set of equations (4) bear the same relationship to the adiabatic equation of motion (1), as the quantum-mechanical Schrödinger equation does to the classical equation of motion. The meaning of the mode number λ appearing in Eq. (4) is now clear from Eq. (A4). It is interesting to note that the mode number l here is the same as λ of Eq. (40), which has been identified in the derivation of this paper as the change in the Landau quantum number from n_o to $n_o+\lambda$, as a consequence of the magnetic-field inhomogeneity. It is quite fascinating to see that this change λ in the Landau quantum number should appear as a Fourier index related to the series (A4) in the derivation of Ref. [5].

If one wishes to reconstruct the Liouville density function f as a solution of the Louville equation from the solutions $\Psi(n)$ of the Schrödinger-like equations, one can do so by backtracking the various steps used to obtain $\Psi(n)$ from f. If, however, we regard Eq. (A4) not just as a Fourier series expansion but also as an asymptotic expansion in the spirit of the work of Kruskal [3], (see also Rosenbluth and Varma [14]) then $\Psi(n)$ have magnitudes $\sim\epsilon^{1/2|n|-1}\Psi(1)$, with ϵ as the adiabaticity parameter defined by Eq. (3). So the most dominant term corresponds to $n=1$, and others fall off as $\epsilon^{|n|}$ for $n\geqslant 2$.

In the context of the derivation of the same equations (40) in the present paper, the λ represents the change in the Landau-level quantum number from n_o to $n_o+\lambda,(n_o\gg 1)$ as a consequence of the magnetic-field inhomogeneity (or any other nonadiabatic perturbation). Thus $\Psi(\lambda),\lambda=1$, would be the most dominant term in the summation (5) or (41) as the transition to $n_o\pm 1$ would be the most dominant. Transitions to $n_o\pm 2,n_o\pm 3,\ldots$ corresponding to $\lambda=2,3,\ldots$ would be progressively smaller and smaller in magnitude. It is interesting to note how the Fourier index n in Eq. (A4) in the derivation of Ref. [5], has come to be identified as the change in the Landau-level number in the derivation of the present paper.

It may also be mentioned that the derivation of Ref. [5] is *exact* (except for neglecting the higher-order terms in an expansion of the form (16) in Ref. [5]) representing, as it does, the transformation of Liouville equation to Eqs. (4). Therefore, while the system of Eq. (4) and (5) do appear analogous to the Schrödinger theory, they are not obtained just through an analogy, but as a consequence of an exact derivation, particularly including the probability connection (5).

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