Multistability in dynamical systems induced by weak periodic perturbations

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It is shown that weak resonant perturbations at subharmonic frequencies can induce multistability in a wide class of nonlinear systems, which display the period-doubling route into chaos or possess isolated subharmonic branches. The number of attractors induced depends on the subharmonic frequency, amplitude, and phase of periodic perturbations, as well as an initial dynamical state of nonlinear systems. Experimental and numerical evidences are given on the basis of a loss-modulated CO_2 laser.

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Nonlinear dissipative systems being in a certain range of parameters, can possess such a fundamental property as multistability, which means a coexistence of several attractors for a fixed set of system parameters. The numerous experimental examples include electric circuits [1–3], lasers [4,5], mechanical systems [3,6], etc. As a rule, such systems are low-dimensional ones, therefore only a small number of attractors can exist simultaneously. Besides, the range of parameters where multistability occurs, can be rather restricted and not always accessible in experimental conditions. For instance, a coexistence of four attractors in a loss-modulated CO_2 laser was experimentally found in the narrow ranges of both the modulation amplitude and the modulation frequency [5]. Potentially, this property can be important for some specific applications, for instance as dynamical memory [7,8]. In order to induce multistability, a delayed feedback [7,8] or coupling nonlinear systems to arrays [9] can be used, which lead to an increase of the system dimensionality and, as a result, to the appearance of multistability.

In this paper, a simple method for obtaining multistability is being demonstrated, which can be achieved in a wide class of nonlinear systems displaying the period-doubling route into chaos and/or possessing isolated subharmonic branches. It is shown here that weak perturbations at the frequencies f_d/n (where f_d is the main driving frequency, $n=3,4,8,16,\dots$) can induce up to n coexisting attractors under an appropriate choice of the bifurcation parameter as well as the amplitude and the phase of the resonant perturbation. Such a perturbation increases the dimensionality of the system by one and therefore can lead to the appearance of multistability. The experimental and numerical evidences are given here on the basis of a loss-modulated CO_2 laser. In the broad sense a loss-modulated CO_2 laser can be classified as a nonautonomous system with a Toda potential [10] and therefore can serve as one of the experimental realizations of periodically forced asymmetric nonlinear oscillators.

A particular case of multistability, bistability induced by the resonant perturbation at the first subharmonic frequency, was numerically and experimentally studied on the basis of a loss-modulated CO_2 laser [11,12]. Similar results were obtained later in a fiber laser pumped with two modulation frequencies [13]. The appearance of bistability was theoretically found also in the quadratic map with additive periodic forcing with a period 2 [14], where this effect was studied in great detail. Besides, the assumption was made there that the additive periodic forcing with a period larger than 2 can induce multistability in the quadratic map. The logistic map is known to be the paradigm in studying the route into chaos via period-doubling bifurcations. From this point of view, experimental and numerical evidence of the assumption made on the basis of this map in the real physical system, are a matter of great importance.

Numerical simulation. Multistability was studied numerically with the help of bifurcation diagrams in the presence of periodic perturbations. The simple rate-equation laser model was used [4]:

$$\frac{du}{dt}=\tau^{-1}(y-k)u, \tag{1}$$

$$\frac{dy}{dt}=(y_0-y)\gamma-uy,$$

where

$$k=k_0+k_d\cos(2\pi f_d t)+k_p\cos(2\pi f_d t/n+\varphi). \tag{2}$$

Here u is proportional to the radiation density, y and y_0 are the gain and the unsaturated gain in the active medium, respectively, τ is a transit time of light in the cavity, γ is the gain decay rate, k is the total cavity losses, k_0 is the constant part of the losses, k_d is the driving amplitude, k_p is the perturbation amplitude, f_d is the driving frequency, φ is the perturbation phase, and $n=3,4,8,16$. The following fixed parameters were used throughout the calculations: $\tau=3.5\times10^{-9}$ s, $\gamma=1.978\times10^5$ s^{-1}, $y_0=0.175$, and $k_0=0.1731$. For all cases of the numerical simulation, the relaxation oscillation frequency of the laser was $f_{ro}=50$ kHz. The parameters k_d and k_p were varied in the numerical simulations. In what follows, the normalized bifurcation parameter μ and perturbation amplitude ϵ were used (defined as $\mu=k_d/k_{1/2}$ and $\epsilon=k_p/k_{1/2}$, respectively, where $k_{1/2}$ is the first period-doubling threshold in the absence of the resonant periodic perturbation).

First, we shall consider the effect of the resonant perturbation with a frequency $f_d/4$ [15]. In the absence of the perturbation the system displays the usual route into chaos via

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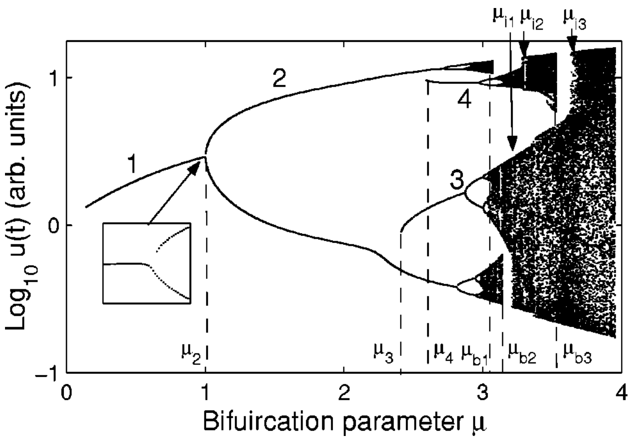

FIG. 1. The numerical bifurcation diagram for the laser vs μ in the presence of the resonant perturbation at the frequency $f_d/4$ showing a coexistence of four attractors, which are denoted by the numbers 1, 2, 3, and 4 ($f_d=100$ kHz, $\epsilon=3.53\times10^{-3}$, $k_{1/2}=2.478\times10^{-3}$, $\varphi=0$).

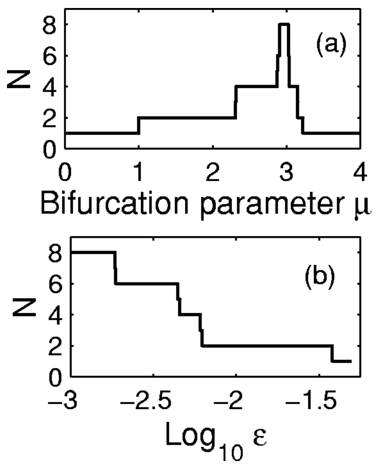

FIG. 2. The number of attractors N induced by the perturbation at $f_d/8$ vs (a) μ ($\epsilon\approx7.06\times10^{-4}$, $f_d=100$ kHz, $k_{1/2}\approx2.478\times10^{-3}$, $\varphi=0$) and (b) ϵ ($\mu\approx3$).

period-doubling bifurcations. No other attractors were found up to $\mu\approx3.065$ (for this case, $k_{1/2}\approx2.478\times10^{-3}$), where period-3 branch appears by a saddle node bifurcation. In the presence of the perturbation at $f_d/4$, as the bifurcation parameter increases, new attractors appear in succession in the systems, which are shown in the bifurcation diagram in Fig. 1. This diagram was obtained by a superimposition of several ones generated with randomly distributed initial conditions. In order to simplify the diagram, only one of four subbands of each new attractor is shown. This means that every subband corresponds to one attractor. For the given perturbation amplitude ($\epsilon\approx3.53\times10^{-3}$) the second attractor appears at $\mu_2\approx1.006$, directly above the first original period-doubling threshold via a saddle node bifurcation. This new isolated branch destroys the first period-doubling bifurcation (Fig. 1) which is typical for imperfect bifurcations [16]. A similar situation takes place above the second period-doubling threshold where, for instance, the third attractor appears at $\mu_3\approx2.407$ and the fourth at $\mu_4\approx2.59$, destroying the second period-doubling bifurcation. This means that the periodic perturbation at $f_d/4$ acts as an imperfection in the problem for the first and second period-doubling bifurcation points [17]. It is interesting to note, at the same time this perturbation also destroys such a global bifurcation as the band-merging crisis, inducing three boundary crises (onsets of which are located at $\mu_{b1}\approx3.073$, $\mu_{b2}\approx3.175$, and $\mu_{b3}\approx3.527$, Fig. 1) and three interior crises ($\mu_{i1}\approx3.2$, $\mu_{i2}\approx3.283$, and $\mu_{i3}\approx3.634$, shown in Fig. 1 by arrows) instead of two last band-merging crises. Thus, the presence of weak resonant perturbations induces other types of global bifurcations such as dangerous and explosive bifurcations in place of safe bifurcations (in accordance with a classification given in Ref. [18]). All new attractors appear, bifurcate, and disappear at different values of the bifurcation parameter. Therefore, as μ increases, the number of coexisting attractors passes in series from 1 up to 4 and then decreases again in turn to one. In the region of $2.59\lesssim\Delta\mu^{(4)}\lesssim3.073$, four attractors exist simultaneously, each of which can be at different periodicity.

A more complicated picture is in the presence of the perturbation at the frequency $f_d/8$, where for $\epsilon\approx7.062\times10^{-4}$ a coexistence of eight attractors in the range of the bifurcation parameter $2.9504\lesssim\Delta\mu^{(8)}\lesssim3.0264$ was found. Figure 2(a) shows how the number of attractors N depends on the bifurcation parameter μ. As μ increases, N passes from 1 to 8 and then decreases to 1. Generally, all attractors appear and disappear but not simultaneously. Though, for instance, after the second and third bifurcation points, three pairs of attractors appear for only slightly different values of μ. As perturbation amplitude ϵ increases, the critical points associated with new attractors are shifted to different values of μ, which results in a successive disappearance of attractors. For an illustration, the dependence of N on ϵ is shown in Fig. 2(b).

Based on the numerical results one can generalize the effect of a periodic perturbation at the frequency f_d/n (where $n=2^k$, $k=1,2,3,4,\ldots$) on the period-doubled systems. This perturbation destroys first k period-doubling bifurcations and correspondingly k last band-merging crises giving rise to the appearance of up to n coexisting attractors. This effect is a nonthreshold one. Even extremely weak resonant perturbations induce multistable states, though with a very small difference between them. For small enough perturbation amplitude ϵ the number of coexisting attractors passes in series from 1 up to n and decreases through boundary crises again in turn to one as the bifurcation parameter increases. The lowest periodicity associated with each attractor is nT ($T=1/f_d$) because of the effect of the perturbation at the frequency f_d/n. Each attractor undergoes the route into chaos via period-doubling bifurcations. With increasing n, the range of the bifurcation parameter $\Delta\mu^{(n)}$, where the maximal number n of coexisting attractors occurs, rapidly decreases and is bounded by the range of μ between the kth period-doubling bifurcation point and the critical point for 2^k subbands merging crisis. As n increases, the perturbation amplitude ϵ needs to be decreased in order to observe the maximal number of coexisting attractors. All coexisting attractors appear, bifurcate, and disappear at different values of the bifurcation parameter. An increase of ϵ for given μ results in

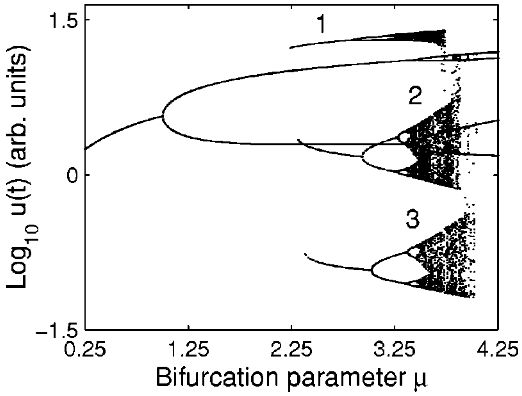

FIG. 3. The numerical bifurcation diagram for the laser vs μ in the presence of the resonant perturbation at $f_d/3$ showing the splitting of the period-3 attractor into three ones which are denoted by the numbers 1, 2, and 3 ($\epsilon = 1.838\times 10^{-3}$, $f_d = 120$ kHz, $k_{1/2} = 4.78\times 10^{-3}$, $\varphi = 0$).

a decrease of the maximal number of coexisting attractors as illustrated in Fig. 2(b) for the case of the perturbation at $f_d/8$ [19]. As in the case of bistability induced by the perturbation at $f_d/2$ [11–13], the phase of the resonant perturbation here plays an important role in obtaining multistability, in the sense that a change in the phase is similar to a change in the perturbation amplitude. This can lead to a change of the number of new coexisting attractors and the overlapping of different dynamical states associated with them.

A common feature of periodically driven nonlinear systems along with a period-doubling scenario to chaos is the existence of nT subharmonic isolated branches (so-called isolas) which result from saddle node bifurcations of a period n. Therefore, we shall consider the effect of the perturbation at the frequency f_d/n on them by the example of period-3 branch. This situation is different from the previous case, where resonant perturbations act on the period-doubling bifurcations. Here the effect of resonant perturbations on saddle node bifurcations inherent in the system is considered. The laser parameters were chosen so that two attractors of period-1 and period-3 can coexist in the absence of the periodic perturbation. The period-3 branch appears by a saddle node bifurcation at $\mu \approx 2.308$ (in this case $k_{1/2} = 4.76 \times 10^{-3}$). Figure 3 shows a bifurcation diagram of the laser in the presence of periodic perturbation at the frequency $f_d/3$. As in previous cases, each subband on the figure corresponds to one attractor. One can see that the periodic perturbation splits primary period-3 branches into three branches of the same period 3, which appear via saddle node bifurcations at different values of μ ($\mu_1^{(3)} \approx 2.236$, $\mu_2^{(3)} \approx 2.310$, and $\mu_3^{(3)} \approx 2.388$).

Experimental results. The experimental setup was similar to one that was described in Ref. [12]. A CO_2 laser with an acousto-optic modulator inserted in the laser cavity was used [12]. Two sine-wave electric signals from oscillators were applied to the modulator providing the time-dependent cavity losses. The driving signal $V_d(t) = V_d \cos(2\pi f_d t)$ had the frequency f_d and the amplitude $V_d(t)$. In what follows, V_d can be considered as a control or bifurcation parameter. The perturbation signal $V_p(t) = V_p \cos(2\pi f_p t + \varphi)$ had the frequency f_d/n ($n = 3,4, \ldots$), the amplitude V_p and the phase φ. The laser responses were detected with a CdHgTe detector and a digital oscilloscope coupled to PC. The technique of stroboscopic data recording with a sampling period of T ($T = 1/f_d$) was used. For finding new attractors, the technique of a short pulse of loss perturbations was used [12].

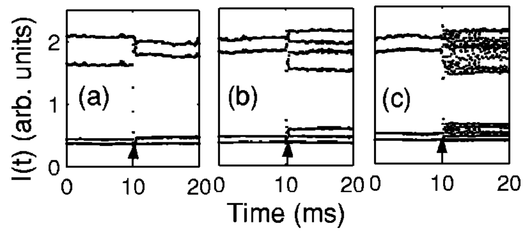

FIG. 4. Experimental stroboscopic laser responses showing switchings between four coexisting attractors induced by the resonant perturbation at $f_d/4$ ($f_d = 100$ kHz, $\varphi \approx 0$). The action of the pulsed perturbation is shown by the arrow.

The experimental results have confirmed the main conclusions made above. First of all, the appearance of bistability induced by weak periodic signal at the frequency f_d/n ($n = 2^k$, $k = 1,2,3,4$) in the parameter range directly above the first unperturbed period doubling threshold was found. This means that bistability in period-doubled systems can be induced by any perturbations that are resonant to the main driving frequency, even with very low frequency.

The appearance of four attractors induced by the perturbation at $f_d/4$ is shown in Fig. 4 where experimental switchings between new attractors are demonstrated. They appear in the system instead of a primary 4T limit cycle in the range of the bifurcation parameter above the second period-doubling threshold. One can see that two attractors have the same 4T periodicity but different amplitudes, one attractor is a 8T limit cycle and one is in a chaotic state. In the absence of the periodic perturbation only one attractor was found for the given laser parameters.

The splitting of the period-3 branch, induced by periodic signal at the frequency $f_d/3$ in the bistability domain between period-1 and period-3 branches, is shown in Fig. 5, where experimental switchings between new attractors are shown.

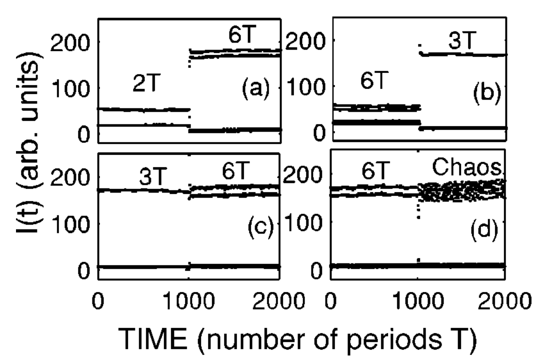

FIG. 5. Experimental stroboscopic laser responses showing switchings between new coexisting attractors induced by the resonant perturbation at $f_d/3$ in the bistability domain of period-1 and period-3 branches ($f_d = 141$ kHz, $\varphi \approx 0$). For all cases the pulse loss perturbation influences at $t = 1000$.

In the absence of the periodic perturbation, two coexisting 2T and 6T limit cycles belonging to period-1 and period-3 branches, respectively, were found [Fig. 5(a)]. In the presence of the perturbation, the 6T limit cycle splits into 3 attractors, which are in different dynamical states (3T, 6T, and chaos). Thus this means the appearance of three attractors in the system in place of the original period-3 attractor.

In summary, it was shown that weak resonant periodic perturbations can noticeably increase the complexity of driven nonlinear systems by inducing a large number of coexisting attractors. Since the period-doubling cascade and the existence of isolas are universal features of nonlinear dynamical systems, one can expect that the results presented here can be applicable to a large variety of such systems in different fields. The main advantage of the considered method is its simplicity and the possibility to induce bistability and multistability in a wide range of modulation frequencies and the bifurcation parameter.

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