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On-ramp simulations and solitary waves of a car-following model

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An on-ramp simulation of a car-following model reveals qualitatively similar results to previous simulations of continuum models carried out by Helbing *et al.* [Phys. Rev. Lett. **82**, 4360 (1999)] and by Lee, Lee, and Kim[Phys. Rev. E **59**, 5101 (1999)]. Here, we discuss the solitary solution type in greater detail. It can be approximated by a Kortweg–de Vries equation derived from the analogous continuum version. Hence, this establishes a further link between these two traffic simulation types and supports the idea that models of either kind lead to similar results when they contain a relaxation term.

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In this paper we use the discrete optimal velocity (OV) model

$$\dot{v}_n = a[V(b_n) - v_n] \tag{1}$$

suggested by Bando *et al.* [1] with a monotonically increasing OV function

$$V(b_n) = \tanh(b_n - 2) + \tanh(2). \tag{2}$$

Each car's acceleration is proportional to the difference between the desired speed V given as a function of its *headway* b_n (the distance to the car in front) and its current speed v_n. This relaxation term is characteristic of most of the recently suggested traffic models, both car-following and continuum [2]. It seems to be an essential feature of traffic modeling in order to reproduce phenomena such as stop-and-go traffic and its upstream traveling shock fronts.

So far, there have been two attempts to model oncoming vehicles near a ramp. First, Helbing *et al.* [3] and Lee *et al.* [4] have simulated an on-ramp in their continuum models by introducing a source term to the right-hand side of the equation for the conservation of cars

$$\rho_t + (\rho v)_x = q_{in}(x,t), \tag{3}$$

where ρ is the density and v is the speed of cars. Second, Helbing used a macroscopic model to represent the on-ramp and its vicinity, whereas the remaining stretch of road was simulated by the corresponding microscopic model [5,6]. Here, the question remains crucial as to how to incorporate the interaction of both systems near their interfaces.

In contrast to this two-phase approach, we aim for a direct on-ramp simulation of the discrete model Eq. (1). Cars are inserted in an open system ("infinitely" long road represented by 2500 vehicles in the numerical simulations) at $x = 0$ with constant flux q_{ramp} after $t = 0$ (dimensionless running time of programs between 1000 and 2500), their speed matching the speed of the surrounding cars at the ramp. For $t < 0$ the traffic state consists of equally distributed cars of density $\rho_0 = 1/b_0$ along the road with constant speed $v_0 = V(b_0)$.

Inserting cars now becomes a discrete process both in space and time in contrast to continuum simulations. To avoid crashes, vehicles only enter the road if a safety distance d, both to the car in front and behind, is given. It turns out that this is fulfilled in the overwhelming majority of the cases.

The traffic states that can be found after the on-ramp flux sets in [Fig. 1(a)] turn out to be qualitatively the same as in Helbing's continuum model [3]. The simulations are carried out with a *sensitivity* $a = 1.5$, for which the model Eq. (1) is linearly unstable in a density regime $\rho \in [0.39, 0.69]$. This in turn corresponds to a headway range $b \in [1.45, 2.55]$ [Fig. 1(b)]. Depending on the original density ρ_0 and the on-ramp flow q_{ramp}, we find triggered stop-and-go (TSG) traffic upstream (Fig. 2), oscillatory congested traffic upstream, homogeneous congested traffic upstream, and homogeneous "congested" traffic downstream. Helbing correctly refers to the latter as free traffic, since the upstream flow is not affected by the on-ramp at all, and the slightly higher downstream density still corresponds to the free flow regime.

At first sight it is very surprising that the car-following model reveals the same results as continuum models. However, as shown by Berg *et al.* [7], there is an analogous continuum counterpart of the OV model Eq. (1), which is in good agreement with its discrete version for moderate gradients of the density. Moreover, it also resembles the former models of Helbing and Kerner *et al.*

We can now prove that the TSG traffic state can be approximated analytically using this continuum analogue

$$v_t + v v_x = a[\overline{V}(\rho) - v] + a\overline{V}'(\rho)\left[\frac{\rho_x}{2\rho} + \frac{\rho_{xx}}{6\rho^2} - \frac{\rho_x^2}{2\rho^3}\right], \tag{4}$$

$$\overline{V}(\rho) = V(1/\rho). \tag{5}$$

This model consists of an equation for the conservation of cars [Eq. (3)] and a governing equation for their acceleration [Eq. (4)]. In the following we neglect the nonlinear term in

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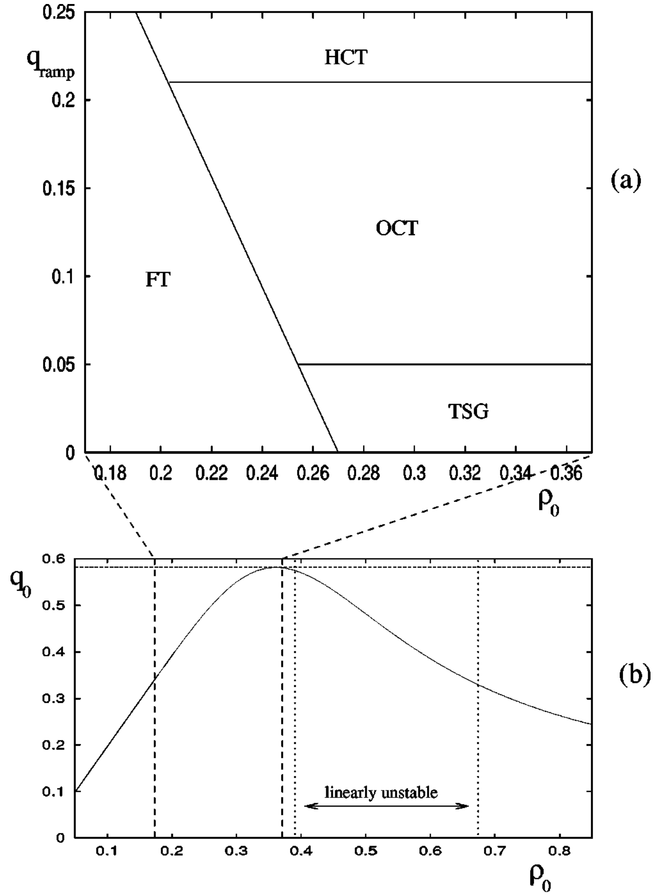

FIG. 1. (a) Phase diagram of traffic states as a function of the original homogeneous flow of density ρ_0 and the on-ramp flux q_{ramp}. It qualitatively resembles the phase diagram found in Helbing's continuum model [3]. (b) The fundamental diagram for $a = 1.5$, showing the original flow q_0 of density ρ_0 before the on-ramp perturbation sets in, contains an unstable region.

ρ_x^2, since the same wave type also occurs in the above-mentioned continuum models, which do not include this term. Being that we are in the linear stable regime [Fig. 1(b)], we can try to interpret the solitary wave type (Fig. 2) as an upstream traveling wave away from the on-ramp. Hence, there will be no source term on the right-hand side of Eq. (3), $q_{in}=0$.

Since the waves of Fig. 2 resemble *solitary waves*, which are solutions of the Kortweg–de Vries (KdV) equation

$$\rho_t+(\alpha+\beta\rho)\rho_x+\rho_{xxx}=0, \tag{6}$$

we are motivated to try and extract a KdV equation from Eqs. (3) and (4) by balancing nonlinear and dispersive effects, as well as neglecting the dissipation in the first place.

If we consider traveling-wave solutions away from the on-ramp, changing to the reference system of the wave by introducing the coordinate

$$z=x-ct \tag{7}$$

with the wave speed c, the system of Eqs. (3) and (4) turns into

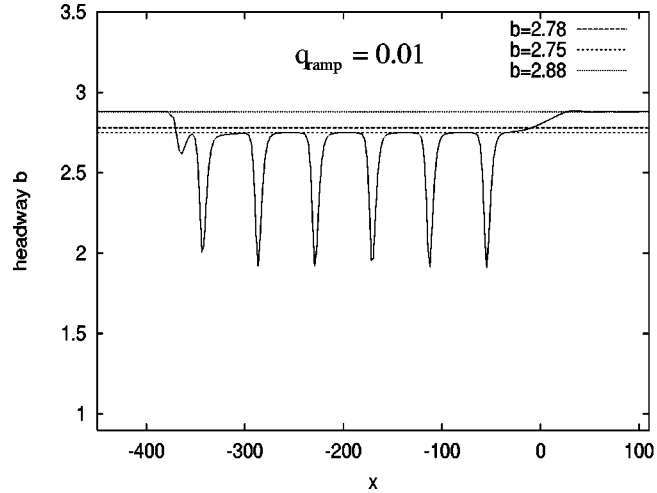

FIG. 2. The TSG traffic state can be considered as a train of solitary waves, which decay only slowly when traveling upstream due to small dissipation.

$$-c\rho_z+q_z=0, \tag{8}$$

$$-cv_z+vv_z=a[\overline{V}(\rho)-v]+a\overline{V}'(\rho)\left[\frac{\rho_z}{2\rho}+\frac{\rho_{zz}}{6\rho^2}\right], \tag{9}$$

with the flow q being the product of the density and the speed

$$q=\rho v\Rightarrow v_z=\frac{c\rho_z}{\rho}-\frac{q\rho_z}{\rho^2}. \tag{10}$$

This way the dynamic equation (9) can be written as

$$-\frac{\rho_z}{\rho^3}q^2+\left(\frac{2c\rho_z}{\rho^2}+\frac{a}{\rho}\right)q=a\overline{V}+\left(\frac{a\overline{V}'}{2\rho}+\frac{c^2}{\rho}\right)\rho_z+\frac{a\overline{V}'}{6\rho^2}\rho_{zz}. \tag{11}$$

We then approximate the flow q to lowest order as

$$q=\rho\overline{V}+a_1\rho_z+a_2\rho_{zz}. \tag{12}$$

The leading term $\rho\overline{V}$ represents the fundamental diagram [Fig. 1(b)]. It describes homogeneous, stationary, and stable flow q of density ρ and speed $\overline{V}(\rho)$. The parameters a_1 and a_2 can be found by substitution of Eq. (12) into Eq. (11) as

$$a_1=\frac{\overline{V}'}{2}+\frac{1}{a}(\overline{V}^2-2c\overline{V}+c^2), \tag{13}$$

$$a_2=\frac{\overline{V}'}{6\rho}. \tag{14}$$

This leads to

$$q=\rho\overline{V}+\left[\frac{\overline{V}'}{2}+\frac{1}{a}(\overline{V}^2-2c\overline{V}+c^2)\right]\rho_z+\frac{\overline{V}'}{6\rho}\rho_{zz}. \tag{15}$$

If the second term in ρ_z can be neglected in comparision to the third term in ρ_{zz}, we can rewrite the equation of conservation of cars Eq. (8) as

$$-c\rho_z + (\rho\overline{V})_z + \frac{\overline{V}'}{6\rho}\rho_{zzz} = 0. \tag{16}$$

If we consider perturbations of the density

$$\rho(x,t) = \rho^* + \hat{\rho}(x,t) \tag{17}$$

near the maximum flow $q_{max} = q(\rho_{max} = 1/2.78) \approx 0.58$, $\rho\overline{V}$ can be approximated by the first terms of a Taylor series

$$\rho\overline{V} = \rho^*\overline{V}(\rho^*) + (\rho\overline{V})_\rho\big|_{\rho=\rho^*}\hat{\rho} + \frac{1}{2}(\rho\overline{V})_{\rho\rho}\Big|_{\rho=\rho^*}\hat{\rho}^2 + \cdots. \tag{18}$$

Inserting this into Eq. (16) yields (dropping the "hat")

$$-c\rho_z + [(\rho\overline{V})_\rho + (\rho\overline{V})_{\rho\rho}\rho]\rho_z + \frac{\overline{V}'}{6\rho}\rho_{zzz} = 0, \tag{19}$$

which is the KdV equation that we were looking for. This equation is only a good approximation under the assumption $|a_1\rho_z| \ll |a_2\rho_{zz}|$. One way to show this is to derive the solution of Eq. (19) and then observe how accurately this condition is fulfilled.

We first transform the KdV equation into a standard form found in most books about this topic [8]. In order to do so, we introduce a new variable

$$u(z) = \frac{1}{6}[(\rho\overline{V})_\rho + (\rho\overline{V})_{\rho\rho}\rho]. \tag{20}$$

Now Eq. (19) turns into (c and $V'/6\rho$ negative)

$$-(-c)u_z - 6uu_z + \left(-\frac{\overline{V}'}{6\rho}\right)u_{zzz} = 0. \tag{21}$$

A coordinate transformation

$$\overline{z} = \sqrt{-\frac{6\rho}{\overline{V}'}}z \tag{22}$$

yields

$$-(-c)u_{\overline{z}} - 6uu_{\overline{z}} + u_{\overline{zzz}} = 0, \tag{23}$$

which is the standard form whose solution is

$$u(\overline{z}) = \frac{c}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{-c}\,\overline{z}\right). \tag{24}$$

This in turns delivers

$$\rho(x,t) = \frac{3c}{(\rho\overline{V})_{\rho\rho}}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{6c\rho}{\overline{V}_\rho}}(x-ct)\right] - \frac{(\rho\overline{V})_\rho}{(\rho\overline{V})_{\rho\rho}}, \tag{25}$$

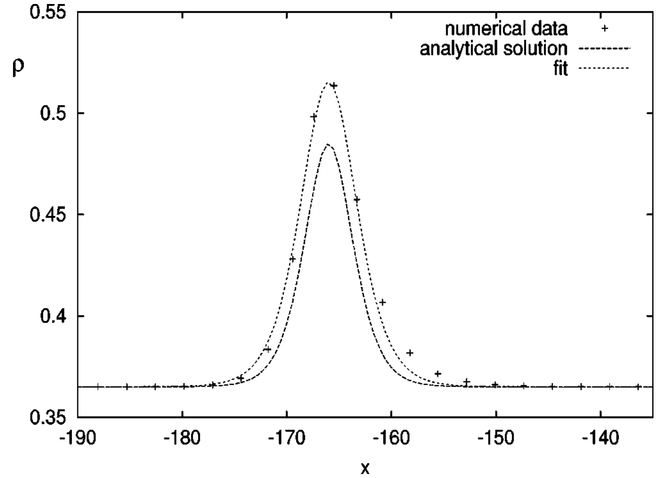

FIG. 3. The solitary wave solution of the car-following simulation (TSG) is only partially matched by the analytical solution (25) due to higher-order and dissipative effects. The latter reaches about 82% of the actual amplitude represented by the sech^2 fit.

which has to be added to ρ^*. Therefore, we end up with two parameters to be fitted, the wave speed c and ρ^*. They are not independent of each other, since they should ideally fulfill the *flow conservation criterion*: the downstream flow must equal the sum of the on-ramp flow and the average flow of the upstream solitonlike profile. This is how the soliton is being selected.

However, ρ^* determines all remaining parameters in Eq. (25) by taking their values at $\rho = \rho^*$. An analysis of the individual terms reveals that the solution does not vary much with ρ^* being close to the maximum of the flow curve.

One way to determine the two parameters is to use the numerical results, which both deliver to the wave speed

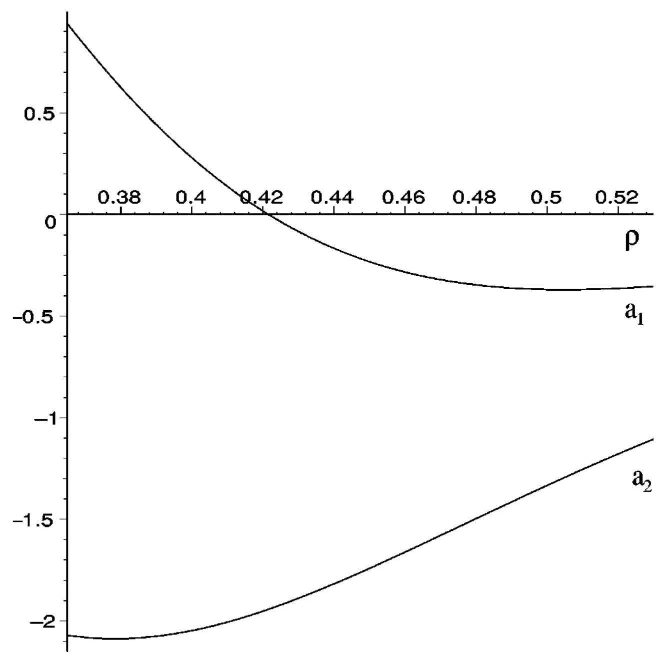

FIG. 4. The coefficients of the expansion Eq. (12) in the density range of the soliton solution. The solitons form in the regime of negligible dissipation.

$$c = -0.64 \pm 0.03 \tag{26}$$

and the density

$$q^* = 0.30 \pm 0.01. \tag{27}$$

These lead to a soliton solution, as shown in Fig. 3, which illustrates that the analytical solution does not quite reach the right amplitude or width. This might be based on the fact that due to its high amplitude, higher-order terms come into play that have been neglected in this analysis and, futhermore, dissipation might broaden the distribution. Nevertheless, the analytical solution can be regarded as a good first-order approach, even though the flow criterion cannot be exactly fulfilled. This is different for the fit, which shows in addition that the numerical data really has the shape of a sech^2 soliton.

If we now consider the values of a_1 and a_2 as shown in Fig. 4, it becomes clear that the soliton solution appears in the regime where the dissipation term becomes negligible, and $|a_1(\rho \approx 0.365)\rho_z| \ll |a_2(\rho \approx 0.365)\rho_{zz}|$ applies. Therefore, as a first-order approach, we can neglect this term and the KdV equation is justified. Nevertheless, dissipation would also contribute an asymmetric correction term that explains the discrepancy between the left-hand and the right-hand side of the maximum in Fig. 3 [9]. It can be treated as a small perturbation $a_2\rho_{zz}$ to the KdV equation (19), and the solution is then found by introducing multiple time scales [9]. This comprehensive method goes beyond the scope of this publication.

Another effect, which might explain a further discrepancy, is that we assumed $a_2 = V'/6\rho$ to be constant. In fact, it varies with ρ between -2.2 and -1.2 across the density range of the soliton solution. If the term depends on ρ, however, it will not allow for an analytical solution any longer. Similar arguments hold for the other parameters in Eq. (25).

As mentioned above, the results of this paper coincide with earlier publications, which consider various OV models. It, therefore, supports the idea that OV models that contain an unstable density region in the fundamental diagram lead to similar predictions of on-ramp states [10]. It does not imply that all states of Fig. 1(a) can be actually found in real traffic [11,12]. It has to be regarded as a sheer mathematical theory, whose predictions might not fully cover real traffic events, since it lacks stochasticity, varying vehicle parameters, and time delay. Hence, it is doubtful if the TSG state can really be interpreted as *synchronized flow* just because of its high flux and slowly varying speed [13].

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