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Noise-enhanced stability of periodically driven metastable states

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We study the effect of noise-enhanced stability of periodically driven metastable states in a system described by piecewise linear potential. We find that the growing of the average escape time with the intensity of the noise is depending on the initial condition of the system. We analytically obtain the condition for the noise enhanced stability effect and verify it by numerical simulations.

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Escape from a metastable state is a phenomenon observed in several scientific areas. Among them there are the theory of diffusion in solids, chemical kinetics, and transport in complex systems [1].

The mean first passage time (MFPT) of a Brownian particle moving in potential fields usually decreases with noise intensity according to the Kramers formula [2] or some universal scaling function of the system parameters [3,4]. The dependence on the noise intensity of the MFPT for metastable and unstable systems was revealed to have resonance character first noted by Hirsch *et al.* [5] and then observed in different physical systems [6–11]. The most important conclusion of these studies is that the noise can modify the stability of the system in a counterintuitive way. The system remains in the metastable state for a longer time than in the deterministic case and the escape time has a maximum at some noise intensity. Noise-enhanced stability (NES) was originally found numerically by Dayan *et al.* [6], and observed experimentally in a tunnel diode by Mantegna and Spagnolo [8]. More recently, it was found that the noise-induced slowing down [10] and the noise-induced stabilization [11] are related to NES phenomenon [8].

Some questions arise from previous studies. (i) What is the reason of the increase of the average escape time with the noise intensity? (ii) What about the condition for which the NES effect takes place? To answer both questions we investigate the escape time from a periodically driven metastable state for a piecewise linear potential. We study the nature of this phenomenon analytically. We find that for fixed potential the decay time of unstable initial state can be dramatically increased by the presence of a small noise depending on the initial condition of the system. We obtain the condition for the NES effect, as an explicit relation between the driving frequency and the parameters of the potential.

We consider the model of overdamped Brownian motion described by the equation

$$\frac{dx}{dt} = -\frac{\partial U(x)}{\partial x} + F(t) + \xi(t), \tag{1}$$

where $\xi(t)$ is the white Gaussian noise with zero mean, $\langle \xi(t)\xi(t+\tau)\rangle = 2q\delta(\tau)$, $F(t)$ is the dichotomous driving force, and $U(x)$ is a potential profile defined as

$$U(x) = \begin{cases} \infty, & x=0 \\ hx, & 0<x\leqslant \ell \\ E-k(x-\ell), & \ell \leqslant x < b, \end{cases} \tag{2}$$

with $E = h\ell$. Specifically we assume $k>0$ and $|F(t)|<k$.

First we consider the system governed by Eq. (1) with $F(t)=0$ and potential (2) with arbitrary h. If $h>0$ ($E>0$) the states at $x<\ell$ become metastable. The exact expression of the MFPT from initial position x_0 to boundary b is known for the case $F(t)=0$,

$$\tau(x_0,b,q) = \frac{1}{q}\int_{x_0}^{b} e^{u(z)} \int_{-\infty}^{z} e^{-u(y)} dy\, dz, \tag{3}$$

where $u(x)=U(x)/q$ is a dimentionless potential profile. Evidently in physical systems we cannot observe the microscopic initial conditions. However the MFPT of Eq. (3) it is sufficient to obtain the MFPT with arbitrary macroscopic initial distribution by simple integration. Therefore we further study $\tau(x_0,b,q)$ because it contains the full information about the system. If $0<x_0<\ell$, the decay time $\tau=\tau_1$ for the potential profile of Eq. (2) is

$$\tau_1(x_0,b,q) = \frac{b-\ell}{k} - \frac{\ell - x_0}{h} + \frac{q(h+k)}{h^2 k} e^{E/q} - \frac{q}{h^2} e^{hx_0/q} - \frac{h+k}{k^2 h}(1-e^{-A/q}) - \frac{q}{kh} e^{(E-A)/q}. \tag{4}$$

If $\ell < x_0 < b$, the decay time $\tau=\tau_2$ is

$$\tau_2(x_0,b,q) = \frac{1}{k}\left[\frac{q(h+k)}{hk}(e^{-A/q} - e^{-\Delta E/q}) + b - x_0 + \frac{q}{h}(e^{(E-\Delta E)/q} - e^{(E-A)/q})\right]. \tag{5}$$

Here $A=k(b-\ell)$, and $\Delta E = k(x_0-\ell)$.

These expressions show that at large noise intensity the decay time $\tau(q)$ decreases with noise as $1/q$ for arbitrary h. When the noise intensity is small, $q \ll |E|$ the influence of the potential barrier becomes significant. For $E<0$ and $h<0$, the barrier is absent, and the NES effect, also known as noise delayed decay, appears when $|h|<k$ and x_0 is near ℓ [7]. Indeed, for $q\ll|E|,A$, $|E-A|$ the decay time $\tau_2(\ell,b,q)$ grows with noise temperature q. When $x_0=0$, the decay time always decreases with q.

In the case with potential barrier ($E>0$, $h>0$) the escape time depends on the initial position of the particle with respect to the potential barrier. When the particle is within the potential well ($x_0<\ell$) the decay time of metastable state increases infinitely when $q\to 0$, because if the noise is absent, the particle can never surmount the potential barrier. For $q\ll E$ decay time (4) coincides with the Kramers' time, which in this case reads

$$\tau_1(x_0,b,q)\simeq\tau_k=\frac{q(h+k)}{h^2k}e^{E/q}\underset{q\to 0}{\longrightarrow}\infty. \tag{6}$$

When $\ell<x_0<b$ the initial state of the particle is unstable. In the absence of noise, the escape time from this unstable state is a finite value: $\tau_2(x_0,b,0)=(b-x_0)/k$, which does not depend on the potential well. When we add the noise the influence of potential well becomes important. It follows from the exact expression of Eq. (5) that the MFPT rises to infinity if $\Delta E<E$,

$$\tau_2(x_0,b,q)\simeq\frac{q}{kh}e^{(E-\Delta E)/q}\underset{q\to 0}{\longrightarrow}\infty, \tag{7}$$

while for the case $\xi(t)=0$ we have the decay time obtained from the deterministic Eq. (1), i.e., the MFPT has a singularity at $q=0$, when $\Delta E<E$. From a physical point of view this singularity can be explained as follows: When the particle is initially located in the region $\ell<x_0<b$, a small quantity of noise added in the system can eventually push the particle into potential well. Then, the particle will be trapped there for a long time because the well is very deep. This type of trajectories of the Brownian particles therefore leads to a big "tail" in the first passage time distribution (FPTD) $w(t)$. If the potential well is very deep, namely, $E>\Delta E$, the trapping time is so long that the integral for MFPT,

$$\tau=\int_0^\infty tw(t)dt, \tag{8}$$

diverges when $q\to 0$. The FPTD obeys the backward Fokker-Planck equation and can be obtained for piecewise linear potential using the Laplace transform method [12]. The Laplace transform of the FPTD for our potential, when $h=k$ and $\ell<x_0<b$, reads

$$\hat{w}(s)=\int_0^\infty w(t)e^{-st}dt=\frac{B(x_0,s)}{B(b,s)}, \tag{9}$$

where

$$B(x,s)=c\mu e^{\lambda x-2\mu\ell}-c\lambda e^{-\mu x+2\lambda\ell}+\gamma^2e^{-\beta}(e^{-\mu x}+e^{\lambda x})$$

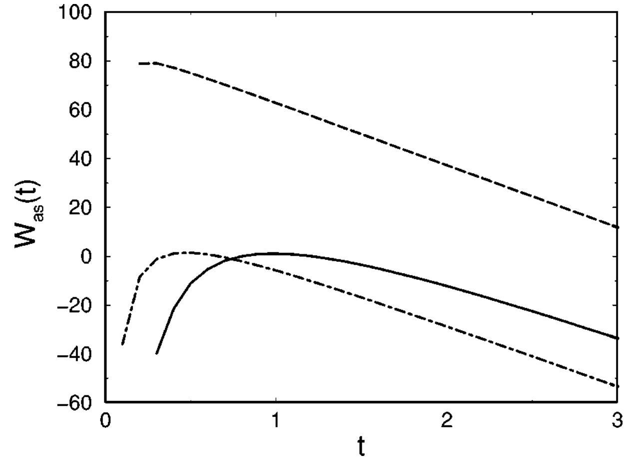

FIG. 1. Semilogarithmic plot of the asymptotic behavior of the FPTD $w_{as}(t)$ vs time t for three values of the initial position $x_o=2.5$ ($\Delta E>E$, dotted-dashed line), $x_0=2$ ($\Delta E=E$, solid line), and $x_0=1.2$ ($\Delta E<E$, dashed line). The parameters are $b=3$, $k=h=1$, $l=1$, $q=0.01$.

and $c=k/2q$, $\gamma=\sqrt{s/q}$, $p=\sqrt{c^2+\gamma^2}$, $\lambda=p-c$, $\mu=p+c$, $\beta=k\ell/q$. Using the limit theorems of Laplace transform, we can obtain from Eq. (9) the asymptotic expression for $t\to\infty$ and $q\ll E$,

$$w_{as}(t)=G(t)\left[1+\frac{1}{2}\left(\frac{t_1+\frac{1}{2}\tau_k}{t_0}-\frac{t_1^2}{t_0t}\right)e^{[-4\theta(x_0)\theta(b)/\tau_kt]}\right],$$

$$G(t)=\frac{t_0}{\sqrt{\pi\tau_kt^3}}\exp\left(-\frac{(t-t_0)^2}{\tau_kt}\right),$$

where $\theta(x)=(x-2\ell)/k$, $t_0=\theta(b)-\theta(x_0)$, $t_1=\theta(b)+\theta(x_0)$, and $\tau_k=4q/k^2$. The time dependence of $w_{as}(t)$ is shown in Fig. 1 for three different values of the initial position $x_0=2.5$ ($\Delta E>E$), $x_0=2$ ($\Delta E=E$), and $x_0=1.2$ ($\Delta E<E$). One can see that the tail of the FPTD rises when ΔE decreases. If $\Delta E>E$, the trapping time in the well is not very long and the integral of Eq. (8) always converges. Nevertheless, the average decay time increases with small noise, reach the maximum and, then, decreases. The plots of $\tau_2(q)$ for various relations between ΔE and E are shown in Fig. 2. The main conclusion from the above analysis is that the strong effect of NES can appear for the fixed potential profile with barrier, if the initial probability distribution is located within the interval (l,b), i.e., in an unstable state beyond the potential well. The physical system can be brought in this nonequilibrium state by a sudden change of control parameters. Examples of such situations include spinodal decomposition in the dynamics of phase transitions and the process of laser switch-on [4,13]. Such relaxation processes in the systems which are far from equilibrium attract now a great deal of attention [10,11,14].

The main aim of this Rapid Communication, however, is to study the escape from the metastable state with an initial distribution located within the potential well in the presence of periodical driving. Further we will apply the above results,

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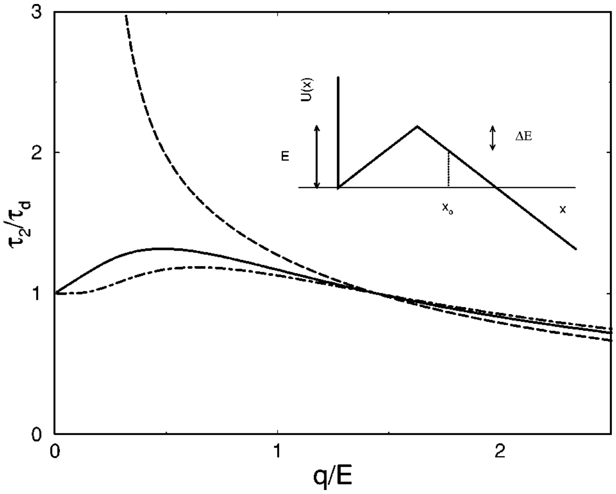

FIG. 2. Normalized decay time $\tau_2(q)/\tau_d$ (with τ_d the deterministic time) vs the dimensionless noise intensity q/E for the same parameters as Fig. 1. Inset: the potential $U(x)$ of Eq. (2).

obtained in the static case, for analysis of periodical force effect. Let us consider the same potential profile $U(x)$ of Eq. (2) but with $h=0$. The driving force is $F(t)=a\nu(t)$, where $\nu(t)$ is the dichotomous signal switching between ± 1 with period T and a is the amplitude. We choose $\nu(t)=-1$ (i.e., the barrier is absent) for the first half of the period. The exact expression for the MFPT when the potential varies with time is unknown. Three recent papers [14] develop a theory of escape rates for periodically driven systems. Smelyanskiy *et al.* consider the case of a high potential barrier and small driving amplitude. Lehmann *et al.* consider the regime of strong and moderately fast driving. Maier and Stein analyze the crossover regime. In all these cases the deterministic escape time is infinity. In the present work we consider the different regime of strong and moderately slow modulation when the deterministic escape time is finite.

Therefore, we first start our investigation from the deterministic case $q=0$, and second we define the condition for the NES effect when noise intensity is small. If $x_0=0$, Eq. (1) has a periodic solution in the deterministic regime for $T<2\ell/a$. In this case $x(t)<\ell$ for any t and the particle always remains in the metastable state. If the period is

$$T>2\ell/a, \tag{10}$$

the particle surmounts the potential barrier at time $t=\ell/a$. To obtain the NES effect we should consider only the cases when the states are unstable without noise [8]. Consequently the first condition for NES is given by Eq. (10). It follows from the above analysis that the decay time in the presence of noise strongly depends on potential barrier, namely, the barrier is responsible for the strong increasing of decay time with noise. The exact expressions for fixed potential show that increasing of decay time in the case without barrier is much smaller, and it appears only if the particle is near the point $x_0=\ell$ [see Eqs. (4) and (5)]. Therefore, it is important to consider two cases: (i) $\tau(0,b,0)<T/2$ and (ii) $\tau(0,b,0)>T/2$. In the first case the modulation frequency is so low that during the entire process of decay the potential profile $\Phi(x,t)$ has no barrier. In this case the average escape time is not increased by the noise [see Eq. (4) at $x_0=0$ and $h=-a$]. In the case of $\tau(0,b,0)>T/2$, average escape time can be represented as follows:

$$\tau(0,b,0)=\frac{T}{2}+\tau[x(T/2),b,0], \tag{11}$$

where $x(T/2)$ is the position of the particle at time $t=T/2$. The conditions (10) and $\tau(0,b,0)>T/2$ together mean that

$$\ell<x(T/2)<b. \tag{12}$$

Now we add a small quantity of noise into the system. The MFPT $\tau(0,b,q)$ can be written as

$$\tau(0,b,q)=\tau(0,x_i,q)+\tau(x_i,b,q), \tag{13}$$

where x_i is an arbitrary point between 0 and b. For $x_i=x(T/2)$ and very small noise, the first term on the right-hand side of Eq. (13) is approximately equal to the deterministic time: $\tau[0,x(T/2),q]\simeq\tau[0,x(T/2),0]=T/2$, because the MFPT varies smoothly with noise, when the barrier is absent [see Eq. (4)]. In this case, $\tau(0,b,q)\simeq T/2+\tau[x(T/2),b,q]$, and $\tau[x(T/2),b,q]\gg\tau[x(T/2),b,0]$ because of the potential barrier, which makes the average escape time very large just for $q\rightarrow 0$ [see Eq. (7)]. As a result the decay time $\tau(0,b,q)$ will increase with q and the NES appears. [The decay time $\tau(0,b,q)$ will not grow infinitely at $q\rightarrow 0$ because the barrier exists for only a half of the period.] Thus, we may conclude that inequality (12) must be the condition for NES. This inequality can be rewritten as follows:

$$2\frac{\ell}{a}<T<2\frac{ab+k\ell}{a(a+k)}. \tag{14}$$

Inequality (14) and the condition $a<k$ give the area on the (T,a) plane where the NES effect takes place. In Fig. 3 we show this area for $k=1$, $\ell=2$, and $b=7$, and the results of numerical simulations (shaded area). We perform 3000 different realizations of the decay process $x(t)$ to determine the average escape time for each couple of values of the amplitude a and the period T of the driving force. We consider more than 100 points on the (T,a) plane. We find that within the area defined by inequality (14) the NES effect is very strong: the average escape time increases more than 10% above the deterministic escape time. Outside this area and below the lower boundary the deterministic decay time becomes infinite and NES disappears. In the presence of noise we obtain Kramers-like behavior. This case was studied in detail by Lehmann *et al.* [14]. Inside the area the magnitude of the NES effect decreases from the lower to the upper boundary. Above the upper boundary the NES effect decreases sharply. When the period T and the amplitude a of the driving force are chosen near the upper boundary of Eq. (14), the potential barrier is very small or absent during the process of decay. It explains why the effect is very small when we are near this boundary. We also carried out simulations for $a>k$ and found the NES effect. This parameter

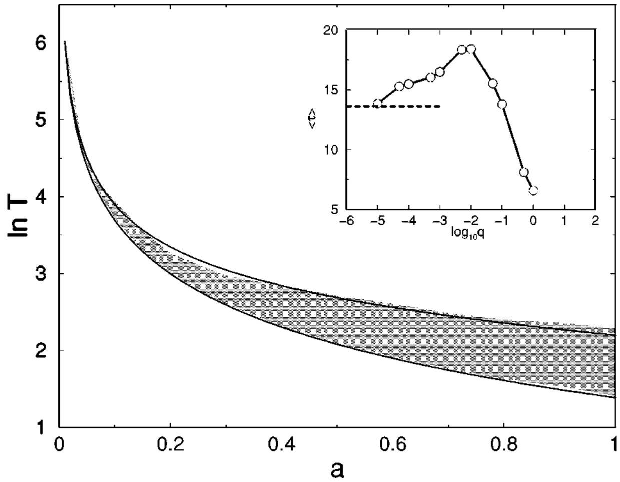

FIG. 3. Shaded area is the region of the plane $(\ln T, a)$, where the NES effect is very strong: the average escape time is greater than 10% above the deterministic escape time. The parameters are $b=7$, $k=1$, $\ell=2$. Inset: the average escape time vs the noise intensity for $a=0.3$ and $T=13.5$. The dashed line indicates the deterministic escape time.

region, however, gives less information about the mechanisms of the NES effect from the viewpoint of the interplay between the regular, random, and periodical forces. In this region in fact the deterministic motion of the particle is characterized by oscillations and the driving force prevails the regular one described by the potential $U(x)$.

Therefore, we conclude that the numerical simulations are in good agreement with the theory. The main mechanism of NES is defined correctly: it is the role of the potential barrier which appears after the particle has crossed the point l of maximal potential. Consequently inequality (12) is the most general condition for the NES effect because it can be applied to a system described by an arbitrary potential with metastable state and where ℓ is the x-coordinate of the maximum of the barrier appearing at $t=T/2$. The mechanism of NES explains why the FPT distributions obtained in simulations and in experiments are multipeaked, periodic, and with an exponential time decaying envelop [6,8]. The peaks appear only for small noise intensity, where the NES effect occurs. The first peak corresponds to the deterministic escape time. The second peak arises because the small noise provides the above-considered inverse probability current, which moves some particles into the potential well. The returned particles can escape only in one period. Therefore, the second peak is one period apart from the first one. After each period we have the same physical situation, and as a consequence, fewer particles go back into the potential well. Therefore, the probability peaks have period T and they decrease with time. The probabilities of escape are independent and equal for successive oscillations of the potential. So if the escape probability per oscillation is p, the probability to escape at the nth cycle is $(1-p)^{n-1}p \simeq pe^{\alpha}e^{-(\alpha t/T)}$, where $-\alpha=\ln(1-p)$ and $n \simeq t/T$. Therefore, the magnitude of the FPT peaks are exponentially decreasing with time.

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