Simple model for a two-component gas flow in the presence of macroscopic scatterers

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The nonequilibrium state of a two-component gas mixture in the presence of macroscopic scatterers is investigated. The general solution of a set of the kinetic equations is obtained in the operator form. The model presentation of the collision operator is derived in the spirit of the Bhatnagar-Gross-Krook method. The rigorous solution of a set of the derived model kinetic equations is found.

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I. INTRODUCTION

We aim to extend the simple and effective idea of the Bhatnagar-Gross-Krook (hereafter BGK) approximation [1] to the consideration of nonequilibrium states of a many-component particle system in the presence of macrons. This problem concerns a variety of physical applications such as the kinetics of a phonon-impuriton system of a superfluid mixture of helium isotopes [2–5] and the phonon-electron system in metals [4,6,7], to mention a few.

The first successful attempt to extend the BGK method to the analysis of a gas mixture is due to Morse [8]. Regrettably, this model is restricted to the consideration of a mixture of classical gases and cannot strictly be applied to the above mentioned systems. Furthermore, the model of Morse (in its conventional form) does not make it possible to consider the size effect in the many-component gas mixture. The same holds true for the model introduced by Sirovich [9] and some other extensions of the BGK approach (see [1] and references therein).

The above arguments give a motivation for a refinement of the BGK approximation, namely, we are going to generalize this method in two different respects. First, we will construct an approximation that is applicable to the analysis of not only classical but also quantum gas mixtures. Second, we will adopt our method to the investigation of effects due to particle collisions with external scatterers (macrons).

The results obtained in this paper imply no restrictions to a particular particle statistics. The extension of the Lorentz approximation to the case of a quantum gas mixture presents a nontrivial mathematical problem. Effects due to the gas statistics bring specific features to the exact formulation of the BGK approximation. This allows one to conclude that, as well as the above mentioned physical implications, the mathematical aspects of the problem considered are themselves of interest.

The paper is organized as follows. Section II presents the mathematical treatment of the above described problem in the framework of the exact kinetic approach. In Sec. III, a model for the interparticle collision operator is proposed to investigate the general solution of the kinetic equation. A model kinetic equation is solved in Sec. IV. A brief summary is given in Sec. V.

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II. EXACT CONSIDERATION OF THE KINETIC PROBLEM

Our first step is to solve a set of the kinetic equations in the operator form. For the sake of simplicity we restrict our consideration to the steady state of a two-component gas mixture. In order to formulate exactly the kinetic problem it is necessary to consider the Hilbert spaces $\mathfrak{R}_i (i=1,2)$ of the momentum functions $g(\vec{p}_i)$ with scalar products of the form

$$\langle h(\vec{p}_i), g(\vec{p}_i)\rangle_i = -\int f_0^{(i)\prime}\left(\frac{\varepsilon_i}{T}\right) h^*(\vec{p}_i) g(\vec{p}_i) d\Gamma_i . \quad (1)$$

Here, T is the temperature, $\vec{p}_i$ and ε_i the momentum and energy of a particle of species i, $d\Gamma_i$ the corresponding volume element of the momentum phase space, $f_0^{(i)}$ the equilibrium distribution function of the gas of the ith species, and p_i the moduli of the corresponding vectors. The prime denotes differentiation with respect to the argument, and the Boltzmann constant is set equal to 1.

We introduce further the Hilbert space $\mathfrak{R}$ of vectors $|\phi(\vec{p}_1), \varphi(\vec{p}_2)\rangle^t$ with the components $\phi(\vec{p}_1)$ and $\varphi(\vec{p}_2)$ belonging to the Hilbert spaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively. The scalar product in the space $\mathfrak{R}$ is defined by the equality

$$\langle \phi_1(\vec{p}_1), \varphi_1(\vec{p}_2) | \phi_2(\vec{p}_1), \varphi_2(\vec{p}_2)\rangle = \langle \phi_1(\vec{p}_1), \phi_2(\vec{p}_1)\rangle_1 + \langle \varphi_1(\vec{p}_2), \varphi_2(\vec{p}_2)\rangle_2 . \quad (2)$$

A set of kinetic equations describing a nonequilibrium state of the two-component system can be represented in the operator form as

$$S_{11}|g\rangle + S_{12}|g\rangle + L|g\rangle = |V\rangle \quad (3)$$

where $|g\rangle = |g_1, g_2\rangle^t$, $|V\rangle = |\vec{v}_1\vec{\nabla}_1, \vec{v}_2\vec{\nabla}_2\rangle^t$, $\vec{\nabla}_i = (\partial/\partial\vec{r})(\varepsilon_i - \mu_i)/T + \vec{X}_i/T$, $\vec{X}_i$ are the external forces, $\vec{v}_i = \partial\varepsilon_i/\partial\vec{p}_i$ is the velocity of particles of species i, f_i are the distribution functions, $g_i = [f_0^{(i)\prime}(\varepsilon_i/T)]^{-1}(f_i - f_0^{(i)})$ are small corrections to the equilibrium distribution functions $f_0^{(i)}$, and μ_i are the chemical potentials. The superscript t denotes the transposition, and all gradients and external forces are supposed to be directed along the z axis. We defined the collision operators in Eq. (3) by

$$S_{11}=\begin{pmatrix}\hat{S}_{11} & 0\\ 0 & \hat{S}_{22}\end{pmatrix},\quad S_{12}=\begin{pmatrix}\hat{S}_{12}^{(1)} & \hat{S}_{12}^{(2)}\\ \hat{S}_{21}^{(1)} & \hat{S}_{21}^{(2)}\end{pmatrix},\quad L=\begin{pmatrix}\hat{L}_{1} & 0\\ 0 & \hat{L}_{2}\end{pmatrix},$$

$$\hat{S}_{ii}g_i=\int W_{ii}(\vec{p}_i,\vec{p}_{i_1}|\vec{p}_{i_2},\vec{p}_{i_3})[1\pm f_0^{(i)}(p_i)]^{-1}\times[1\pm f_0^{(i)}(p_{i_2})]f_0^{(i)}(p_{i_1})[1\pm f_0^{(i)}(p_{i_3})]\times[g_i(\vec{p}_{i_2})+g_i(\vec{p}_{i_3})-g_i(\vec{p}_i)-g_i(\vec{p}_{i_1})]d\Gamma_{i_1}d\Gamma_{i_2}d\Gamma_{i_3}, \tag{4}$$

$$\hat{S}_{ij}^{(i)}g_i=\int W_{12}(\vec{p}_i,\vec{p}_j|\vec{p}_{i_1},\vec{p}_{j_1})[1\pm f_0^{(i)}(p_i)]^{-1}\times[1\pm f_0^{(i)}(p_{i_1})]f_0^{(j)}(p_j)[1\pm f_0^{(j)}(p_{j_1})]\times[g_i(\vec{p}_{i_1})-g_i(\vec{p}_i)]d\Gamma_{i_1}d\Gamma_j d\Gamma_{j_1}, \tag{5}$$

$$\hat{S}_{ij}^{(j)}g_j=\int W_{12}(\vec{p}_i,\vec{p}_j|\vec{p}_{i_1},\vec{p}_{j_1})[1\pm f_0^{(i)}(p_i)]^{-1}\times[1\pm f_0^{(i)}(p_{i_1})]f_0^{(j)}(p_j)[1\pm f_0^{(j)}(p_{j_1})]\times[g_j(\vec{p}_{j_1})-g_j(\vec{p}_j)]d\Gamma_{i_1}d\Gamma_j d\Gamma_{j_1}. \tag{6}$$

Hereafter, the upper (lower) signs correspond to bosons (fermions), $\hat{S}_{ii}$ are the linearized collision operators of particles of the same species, $\hat{S}_{ij}^{(i)}$ and $\hat{S}_{ij}^{(j)}$ are the components of the linearized operators of cross collisions that act in the spaces $\mathfrak{R}_i$ and $\mathfrak{R}_j$, respectively, and $\hat{L}_i$ are the operators describing the particle-macron collisions. In the framework of the Lorentz approximation the last operators read [10]

$$\hat{L}_i g_i=N\int W(\vec{p}_i|\vec{p}_{i_1})[g(\vec{p}_{i_1})-g(\vec{p}_i)]\delta(\varepsilon_i(p_i)-\varepsilon_i(p_{i_1}))d\Gamma_{i_1} \tag{7}$$

where N is the macron number density.

In order to inverse the collision operator in Eq. (3) we proceed with an analysis of the kernels of the collision operators introduced above. For the sake of brevity we associate with the n-dimensional orthonormal basis $\{|\phi_i\rangle\}_{i=1}^n$ the projector P of the form

$$P[\{|\phi\rangle\}_{i=1}^n]=\sum_{i=1}^n|\phi_i\rangle\langle\phi_i|. \tag{8}$$

Let us consider a set of vectors defined by

$$|e_l^k\rangle=\frac{(\langle p_1|p_1\rangle_1\langle p_2|p_2\rangle_2)^{-(k-1)/2}}{[2(2-l-2l^2)\pi]^{-1/2}\||p_1,p_2\rangle\|}\left|\begin{matrix}(Y_1^1(\theta_1,\phi_1)+lY_1^{-l}(\theta_1,\phi_1))\langle p_2|p_2\rangle_2^{k-1}p_1\\(-1)^{k-1}(Y_1^l(\theta_2,\phi_2)+lY_1^{-l}(\theta_2,\phi_2))\langle p_1|p_1\rangle_1^{k-1}p_2\end{matrix}\right\rangle,$$
$$|e_3^k\rangle=\frac{[\langle\varepsilon_1(p_1)|\varepsilon_1(p_1)\rangle_1\langle\varepsilon_2(p_2)|\varepsilon_2(p_2)\rangle_2]^{-(k-1)/2}}{\||\varepsilon_1(p_1),\varepsilon_2(p_2)\rangle\|}\left|\begin{matrix}\varepsilon_1(p_1)\langle\varepsilon_2(p_2)|\varepsilon_2(p_2)\rangle_2^{k-1}\\(-1)^{k-1}\varepsilon_2(p_2)\langle\varepsilon_1(p_1)|\varepsilon_1(p_1)\rangle_1^{k-1}\end{matrix}\right\rangle, \tag{9}$$
$$|e_4^k\rangle=\frac{(\langle 1|1\rangle_1\langle 1|1\rangle_2)^{-(k-1)/2}}{\||1,1\rangle\|}\left|\begin{matrix}\langle 1|1\rangle_2^{k-1}\\(-1)^{k-1}\langle 1|1\rangle_1^{k-1}\end{matrix}\right\rangle\quad(l=0,1,2;k=1,-1)$$

where $Y_l^m(\theta_i,\phi_i)$ $(i=1,2)$ are the spherical harmonics [11] expressed in terms of spherical coordinates (p_i,θ_i,ϕ_i) in the momentum spaces. The double brackets $\|\cdots\|$ denote the norm of the vector. Let us also note that $\langle e_l^1|e_l^2\rangle=0$.

The set of vectors introduced above makes it possible to find the kernels of the collision operators in an explicit form. The conservation laws in the cross collisions can be expressed as

$$S_{12}|e_l^1\rangle=|0\rangle, \tag{10}$$

with $l=0,-1,1$ corresponding to the conservation of z, x, and y components of the particle momenta, and $l=3,4$ corresponding to the conservation of energy and particle number density, respectively, $|0\rangle\equiv|0,0\rangle^t$.

It should be stressed that the collision operators S_{11} and S_{12} have kernels of different dimensions. The cross-collision operator S_{12} has five collision invariants due to the above conservation laws in collisions of particles of different species. The operator S_{11} has ten independent collision invariants: five for mutual collisions of the particles of each of the two species. Thus the corresponding conservation laws can be written as

$$S_{11}|e_l^{1,2}\rangle=|0\rangle. \tag{11}$$

The set of vectors (9) must be properly normalized to constitute a basis in the null space of the operator S_{11}. This basis is found to be

$$|\xi^k_{0,1,2}\rangle \equiv |e^k_{0,1,-1}\rangle, \quad |\xi^1_{3,4}\rangle = \frac{1}{\sqrt{2(1\pm\langle e^1_3|e^1_4\rangle)}}(|e^1_3\rangle \pm |e^1_4\rangle),$$

$$|\xi^2_3\rangle = \frac{(E-P[|\xi^1_3\rangle,|\xi^1_4\rangle])|e^2_3\rangle}{\sqrt{\langle e^2_3|(E-P[|\xi^1_3\rangle,|\xi^1_4\rangle])|e^2_3\rangle}},$$

$$|\xi^2_4\rangle = \frac{(E-P[|\xi^1_3\rangle,|\xi^1_4\rangle,|\xi^2_3\rangle])|e^2_4\rangle}{\sqrt{\langle e^2_4|(E-P[|\xi^1_3\rangle,|\xi^1_4\rangle,|\xi^2_3\rangle])|e^2_4\rangle}}, \tag{12}$$

so that $\langle \xi^k_l|\xi^n_m\rangle = \delta^m_l \delta^n_k$, where δ^m_l is the Kronecker symbol and E is the unity operator.

The vectors $|\xi^1_l\rangle^4_{l=0}$ form the basis of the kernel of operator S_{12}. This allows one to introduce the projectors

$$P_{12} = \sum_{i=0}^{4} |\xi^1_i\rangle\langle\xi^1_i|, \quad P_{11} = P_{12} + \sum_{i=0}^{4} |\xi^2_i\rangle\langle\xi^2_i| \tag{13}$$

into the kernels of the operators S_{12} and S_{11}, respectively.

The structure of the kernel of operator L merits a few comments. In the framework of the Lorentz approximation [10], the particle-macron collisions lead to a change in the direction of particle motion so that their energy remains unchanged. The energy conservation law in the particle-macron collisions is reflected by the δ factor in Eq. (7). Hence, the collision operators $\hat{L}_i$ act only on the polar and azimuthal components of an arbitrary function of the momentum. Thus, any vector of $\mathfrak{R}$ formed by the angle independent functions belongs to the kernel of L. The basis of the kernel of L can be obtained by applying the Gram-Schmidt procedure [11] to the set of vectors constituted by monomials of the form $\{|p^n_1,0\rangle^t,|0,p^n_2\rangle^t\}^\infty_{n=0}$. For our purposes, however, it is enough to introduce the projector P_L into the kernel of L associated with the above basis by means of formula (8). Relations of further importance are

$$P_1 \equiv P_{12}P_L = \sum_{l=3}^{4} |\xi^1_l\rangle\langle\xi^1_l|,$$

$$P_2 \equiv (P_{11}-P_{12})P_L = \sum_{l=3}^{4} |\xi^2_l\rangle\langle\xi^2_l|. \tag{14}$$

The projectors derived above can be conveniently used to solve the kinetic equation (3) in operator form. Projecting the latter onto the orthogonal subspaces associated with the projectors $E-P_1$, $P_{12}(E-P_L)$, P_2, and P_1 one finds, respectively,

$$\left(\sum_{i=1}^{2} (E-P_{1i})S_{1i}(E-P_{1i}) + (E-P_L)L(E-P_L)\right)|g\rangle = (E-P_1)|V\rangle, \tag{15}$$

$$P_{12}(E-P_L)L(E-P_L)|g\rangle = P_{12}(E-P_L)|V\rangle, \tag{16}$$

$$P_2S_{12}(E-P_{12})|g\rangle = P_2|V\rangle, \quad P_1|V\rangle = 0. \tag{17}$$

Because $P_L|V\rangle = 0$, the second equality in Eq. (17) is fulfilled trivially and can be omitted. According to the first equality in Eq. (17) the vector $S_{12}|g\rangle$ must belong to the kernel of projector P_2.

The operator on the left side of Eq. (15) can be inverted uniquely. This follows from the fact that the latter presents a sum of the negative operators of the form $(E-P_{ij})S_{ij}(E-P_{ij})$. Thus, the general solution of the kinetic equation (3) can be written in the form

$$|g\rangle = |g_0\rangle + |g'\rangle \tag{18}$$

with an arbitrary vector $|g_0\rangle$ belonging to the subspace associated with the projector P_1 and

$$|g'\rangle = \left(\sum_{i=1}^{2} (E-P_{1i})S_{1i}(E-P_{1i}) + (E-P_L) \times L(E-P_L)\right)^{-1} (E-P_1)|V\rangle. \tag{19}$$

The exact solution (18),(19) of the kinetic problem can be used for analysis of the steady nonequilibrium states of the two-component gas mixture. Two further comments are important. First, the solution (18) has no restrictions on the particle energy-momentum relation. That is, it can be applied to quasiparticle gas mixtures also. Second, the result (19) takes into account different dimensionalities of the kernels of the operators S_{12} and S_{11}. It gives the correct limits when certain types of collision are neglected.

The term $|g_0\rangle$ in Eq. (18) presents the solution of the homogeneous equation (3) with $|V\rangle = 0$. The term $|g'\rangle$ expresses a particular solution of the nonhomogeneous equation (3). It should be emphasized that this latter term must also fulfill equality (16). In order to illustrate that, let us consider the hydrodynamic limit $L\equiv 0$ ($P_L \equiv E$). For brevity we also set $\vec{X}_i = \vec{0}$. In this case, condition (16) reduces to $P_{12}|V\rangle = 0$. In explicit form this reads

$$\frac{\partial}{\partial \vec{r}} \sum_{l=1}^{2} \int \vec{p}_l \frac{\partial \varepsilon_l}{\partial \vec{p}_l} f^{(l)}_0 d\Gamma_l = 0. \tag{20}$$

Equality (20) yields nothing but the condition of mechanical equilibrium in a *closed* particle system: the total pressure must be constant all over its volume. The presence of macrons makes a particle system unclosed. Condition (20) breaks down due to particle-macron collisions. In order to obtain the condition describing the steady state of a system in this case one has to analyze the solution (19) with $L \neq 0$.

III. A MODEL FOR THE COLLISION OPERATOR

Our next step is to derive a model representation for the collision operators to analyze the solution (18) in physically explicit terms. In order to be physically reliable, our model has to reflect the following essential features of true collision operators. (1) The model collision operators must satisfy conditions (10) and (11) expressing the conservation laws in the collisions. (2) Linearized collision operators must be selfadjoint. (3) The H theorem [10] must be satisfied.

According to the first two conditions the collision operators must satisfy the relations

$$S_{1i}|g\rangle=(E-P_{1i})S_{1i}(E-P_{1i})|g\rangle \quad (i=1,2). \tag{21}$$

Within linear theory, the average effect of collisions reduces to a change in the distribution function by a small amount proportional to the correction $|g\rangle$. The coefficients of proportionality, the so-called collision frequencies [1,10], qualitatively describe the effect of collisions. Extending the idea of this commonly used trick we approximate S_{11} and S_{12} by multiplication operators by taking into account conditions (21). This results in the model

$$S_{1i}=(E-P_{1i})\hat{\nu}_{1i}(E-P_{1i}) \quad (i=1,2) \tag{22}$$

where $\hat{\nu}_{11}=\text{diag}\{-\nu_{11},-\nu_{22}\}$ and $\hat{\nu}_{12}=\text{diag}\{-\nu_{12},-\nu_{21}\}$ are diagonal matrices with real elements.

Let us emphasize that the collision frequencies ν_{ij} must be positive. This follows from the H theorem written in the form

$$\langle g|S_{11}+S_{12}|g\rangle\leqslant 0. \tag{23}$$

Condition (23) can be derived by taking the time derivative of the system entropy

$$S=-\sum_{l=1}^{2}\int[f_l\ln f_l\mp(1\pm f_l)\ln(1\pm f_l)]d\Gamma_l \tag{24}$$

and linearizing the result with respect to $|g\rangle$.

According to condition (23) the model operators must be negative and, consequently, the above mentioned parameters must satisfy $\nu_{ij}>0$. Let us also note that the equals sign in (23) holds if (and only if) the vector $|g\rangle$ belongs to the kernel of the operator $S_{11}+S_{12}$.

Another restriction related to the values of ν_{ij} arises from their physical nature, namely, the frequencies ν_{12} and ν_{21} are not independent. Indeed, according to Eqs. (5) and (6) the following equality holds:

$$\int \overline{w}_{12}f_0^{(1)'}d\Gamma_1=\int \overline{w}_{21}f_0^{(2)'}d\Gamma_2 \tag{25}$$

where

$$\overline{w}_{ij}(\vec{p}_i)=\int W_{12}(\vec{p}_i,\vec{p}_{j_1}|\vec{p}_{i_2},\vec{p}_{j_3})[1\pm f_0^{(i)}(p_i)]^{-1}\times[1\pm f_0^{(i)}(p_{i_2})]f_0^{(j)}(p_{j_1})\times[1\pm f_0^{(j)}(p_{j_3})]d\Gamma_{i_2}d\Gamma_{j_1}d\Gamma_{j_3}. \tag{26}$$

Applying the relation (25) for the true collision integrals to its model presentation (22) one finds

$$\int \nu_{12}f_0^{(1)'}d\Gamma_1=\int \nu_{21}f_0^{(2)'}d\Gamma_2. \tag{27}$$

The relation (27) is particularly simple for the momentum independent frequencies ν_{12} and ν_{21}. It reads

$$\nu_{12}\int f_0^{(1)'}d\Gamma_1=\nu_{21}\int f_0^{(2)'}d\Gamma_2. \tag{28}$$

In the case of Maxwell distribution functions $f_0^{(i)}$, equality (28) reduces to the relation used in [8].

We conclude this section with further simplification of the model (22). Because the model operators (22) present the operators of multiplication, the collision operator $S_{11}+S_{12}+L$ maps the subspace of $\mathfrak{R}$ associated with $Y_1^0(\theta_i)$ harmonics into itself. It allows one to look for a particular solution $|g'\rangle$ of the kinetic equation (3) in the subspace corresponding to the same harmonic as that to which the vector $|V\rangle$ belongs. The corresponding nonhomogeneous equation for $|g'\rangle$ reads

$$\left[\left(E-\sum_{i=1}^{2}|e_0^i\rangle\langle e_0^i|\right)\hat{\nu}_{11}\left(E-\sum_{i=1}^{2}|e_0^i\rangle\langle e_0^i|\right)+(E-|e_0^1\rangle\times\langle e_0^1|)\hat{\nu}_{12}(E-|e_0^1\rangle\langle e_0^1|)+\hat{\nu}_L\right]|g'\rangle=|V\rangle \tag{29}$$

where $\hat{\nu}_L=\text{diag}\{-\nu_{1L},-\nu_{2L}\}$.

The explicit forms for the particle-macron collision frequencies ν_{1L},ν_{2L} are defined by the cross sections of particle scattering by macrons. For a classical gas of particles with the mass m_i this reads [10]

$$\nu_{iL}=\frac{p_i}{\sqrt{2m_iT}}\upsilon_{iL},$$

$$\upsilon_{iL}=2\pi N\sqrt{\frac{2T}{m_i}}\int[1-P_1(\cos\alpha)]\sigma_i(p_i,\alpha)\sin\alpha d\alpha \tag{30}$$

where $\sigma_i(p_i,\alpha)\sin\alpha d\alpha$ are the differential scattering cross sections of the quasiparticles of the species i and $P_1(\cos\alpha)$ is the Legendre polynomial of the first order.

The model presentation of the collision operator (22) makes it possible to obtain observable physical results avoiding exact consideration of the nonessential details of the interparticle interaction. It allows one to investigate explicitly the effect of particle-macron collisions on the formation of a steady nonequilibrium state of a gas mixture.

IV. THE RIGOROUS SOLUTION OF THE MODEL KINETIC EQUATION IN THE STATIONARY CASE

The model kinetic equation (29) can be trivially reduced to a set of linear equations in the moments of the distribution functions. For the sake of simplicity we facilitate the solution of Eq. (29) by further considering that the frequencies ν_{1i} are momentum independent. Then the set of vectors $|e_0^1\rangle,|e_0^2\rangle$ presents an irreducible group with respect to multiplication by the operators $\hat{\nu}_{1i}$. Thus we can write

$$\hat{\nu}_{1i}|e_0^l\rangle=\sum_{m=1}^{2}\kappa_i^{(m,l)}|e_0^m\rangle,\quad \kappa_i^{(m,l)}=\langle e_0^m|\hat{\nu}_{1i}|e_0^l\rangle. \tag{31}$$

Using the latter equalities we represent Eq. (29) in the form

$$|g'\rangle=\left(\sum_{i=1}^{2}\hat{\nu}_{1i}+\hat{\nu}_L\right)^{-1}\left(|V\rangle+\sum_{i=1}^{2}\sum_{l=1}^{2}\kappa_{li}|e_0^i\rangle\langle e_0^l|g'\rangle\right) \tag{32}$$

where $\kappa_{ij}=\Sigma_{l=1}^2\kappa_l^{(i,j)}(1-\delta_i^2\delta_j^2\delta_l^2)$.

Multiplying equality (32) by the bra vectors $\langle e_0^l|(l=1,2)$ one derives the set of coupled equations in the moments $\langle e_0^l|g'\rangle$. After some elementary algebra one finds

$$\langle e_0^l|g'\rangle=\Delta^{-1}\left[V_l\left(1-\sum_{m=1}^{2}\kappa_{im}\eta_{im}\right)+V_i\sum_{m=1}^{2}\kappa_{im}\eta_{lm}\right]\quad (l,i=1,2;\ i\neq l) \tag{33}$$

where $\Delta=1-\Sigma_{i,l=1}^2\kappa_{il}\eta_{il}+\kappa\eta,\quad x=x_{12}^2-x_{11}x_{22}(x=\kappa,\eta)$,

$$\eta_{ml}=\langle e_0^m|\left(\sum_{i=1}^{2}\hat{\nu}_{1i}+\hat{\nu}_L\right)^{-1}|e_0^l\rangle,$$

$$V_l=\langle e_0^l|\left(\sum_{i=1}^{2}\hat{\nu}_{1i}+\hat{\nu}_L\right)^{-1}|V\rangle. \tag{34}$$

The moments (33) are to be substituted in expression (32). The result reads

$$|g'\rangle=\left(\sum_{i=1}^{2}\hat{\nu}_{1i}+\hat{\nu}_L\right)^{-1}\left[|V\rangle+\Delta^{-1}\sum_{l=1}^{2}|e_0^l\rangle\times\left(\sum_{m=1}^{2}\kappa_{lm}V_m+\kappa(V_l\eta_{ii}-V_i\eta_{il})\right)\right](i=1,2;\ i\neq l). \tag{35}$$

The result (35) presents the general solution of the kinetic problem (3) in the framework of approximation (22). It expresses the corrections to the equilibrium distribution functions in terms of the matrix elements (31),(34). Because the operators $\hat{\nu}_{1i}$, $\hat{\nu}_L$ are presented in the form of diagonal matrices, the explicit expressions for the above matrix elements can be written down trivially.

Let us emphasize that the results (35) and (33) imply no restrictions to the kind of particle statistics. They can strictly be applied to gases of fermions and bosons and to their binary mixtures. In addition the results obtained allow one to extend the framework of the conventional Lorentz approximation [10] to consideration of diffusion of mixtures of quantum gases.

In order to give insight into physical applications of the presented mathematical formalism we will now consider a simple illustrative example. Let us investigate the phonon contribution to the Knudsen effect in a classical (Boltzmann) gas using the results (33) and (35). For the sake of simplicity we neglect the phonon-macron collisions ($\nu_{2L}=0$) and consider that $\nu_{1L}\gg\nu_{11},\nu_{12}$ to ensure the Knudsen regime. We also use the simple evaluation (30) for ν_{1L} and put $\mu_2=0$, $\varepsilon_2=cp_2$ for a phonon gas. The classical Boltzmann (phonon) gas is regarded as species 1 (2) in the general formulas.

The condition

$$\langle p_1|g_1\rangle_1=0$$

ensuring the absence of particle mass flow can be expressed explicitly in the form of the relation between temperature and gas number density gradients. Using Eq. (33) one finds after a significant simplification

$$\nabla(n_B\sqrt{T})+\frac{4}{3}\frac{E_{ph}}{\sqrt{T}}\frac{\nabla T}{T}=0 \tag{36}$$

where n_B is the particle number density, $E_{ph}=(4\pi^5T/15)(T/2\pi\hbar c)^3$ is the energy of the phonon gas per unit volume and c is the sound velocity.

The first summand in Eq. (36) presents the gradient of the conventional Knudsen parameter [10]. The second term describes the effect of the phonon drag on the formation of the Knudsen steady state in a classical gas. According to Eq. (36), the phonon drag leads to an effective increase in gas number density gradient equilibrating the phonon contribution to the gas thermal diffusion.

V. CONCLUSIONS

The steady nonequilibrium state of a two-component gas mixture in the presence of macroscopic scatterers was investigated. An analysis of the solvability of the kinetic equation was performed. The solution of the kinetic equation was presented as the sum of the particular solution of the nonhomogeneous equation and the general solution of the corresponding homogeneous one. A set of projectors (13) to the kernel of the collision operators was introduced. It allowed us to obtain the particular solution of the operator kinetic equation. As a result, the general solution (18),(19) of the kinetic problem (3) was obtained in the operator form.

A model presentation (22) was derived for the interparticle collision operators. This presentation is based on the form of the exact solution (18),(19) obtained and the essential features of the true collision operator reflected in the model. The proposed model of the collision operator allows it to introduce the collision frequencies self-consistently. The exact relations between the cross-collision frequencies (25), (27) were obtained and used in the model.

The rigorous solution (35) of the model kinetic equation (29) was obtained. The moments (33) of the distribution functions were calculated. The results obtained are valid for any kind of particle statistics and arbitrary relations between the collision frequencies. They describe an effect of diffusion of a quantum gas mixture in the presence of macroscopic scatterers.

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