Asymptotic analysis and renormalized perturbation theory of the non-Hermitian dynamics of an inviscid vortex

Francesco Volponi*
Graduate School of Engineering, The University of Tokyo, Tokyo 113-8656, Japan

Swadesh M. Mahajan
Institute for Fusion Studies, University of Texas at Austin, Austin, Texas 78712-1083

Zensho Yoshida
Graduate School of Frontier Sciences, The University of Tokyo, Tokyo 113-8656, Japan
(Received 25 October 2000; published 26 July 2001)

An analysis of the non-Hermitian fluid systems described by the Rayleigh equation in an unbounded domain has been carried out in the regime of large wave numbers. The evolution of a special class of localized vorticities is also discussed. Asymptotic and perturbative approaches lead to the same final result. In the limit considered, the system is stable. The perturbation analysis reveals interesting pathologies of the non-Hermitian systems. Under specific conditions, the expansion is found to show secular growth. A discussion about the mechanism of insurgence of such singular behavior is presented. It is also shown that the divergent expansion is renormalizable by means of the renormalization group method—the renormalized results are in complete conformity with the asymptotic solutions.

DOI: 10.1103/PhysRevE.64.026312 PACS number(s): 47.15.−x

I. INTRODUCTION

Non-Hermitian operators represent a challenge for functional analysts and mathematical physicists. For such operators, no general theory of spectral resolution in infinite dimensional function spaces is yet available. The main obstacle is that, unlike Hermitian operators, they are not resolvable in terms of orthogonal and complete sets of eigenmodes, making impossible the formulation of an appropriate spectral theorem. Moreover, analogous to nonlinear systems, the coupling induced by non-Hermitian operators renders a decomposition in terms of orthogonal modes (i.e., Fourier modes) quite useless.

The repercussions on the analysis of physical systems, whose dynamics is governed by a non-Hermitian operator, are rather serious. A canonical example is represented by sheared flows, which are described by the ideal Euler equation. When these equations are linearized about an equilibrium shear flow, the generator of the dynamics turns out to be non-Hermitian. The energy of the perturbations, which is decomposed from the energy of the mean field, is not closed and the exchange of the energies among perturbations and the ambient field is extremely complicated. A physical consequence is that the linear equations do not conserve ‘‘energy’’ which is an essential property of the original nonlinear system.

In this paper we will discuss some of the pathologies of non-Hermitian operators by analyzing, perhaps, the simplest example of a shear-flow system, the Rayleigh equation in an unbounded domain. In bounded (channel) regions this equation has been widely analyzed in the framework of the Kelvin-Helmholtz instability [1].

By solving the initial value problem (which for non-Hermitian systems is more appropriate than solving an eigenvalue problem), we will first investigate the time-asymptotic evolution of the system. Then we will rederive this result by means of a perturbative analysis in the regime of small curvature of the ambient flow. The issue of the validity of the perturbation theory for non-Hermitian systems is a central point in this investigation. As pointed out before, non-Hermitian operators induce coupling of orthogonal modes and they mimic the behavior of the nonlinear operators. Since nonlinearities are known to induce divergences in perturbation theory [2,3] we might expect to find the same pathology in non-Hermitian systems. This is indeed the case. We will show that, under specific conditions, the perturbative analysis leads to a singularity. We will also show how to renormalize this divergence by means of the renormalization group method [4,5]. The renormalized expression comes out to be in perfect agreement with the results of the asympototic analysis.

This proof of renormalizability is important in connection with another issue, namely, the suitability of Kelvin's representation to describe strong sheared flows. This method [6], first introduced more than a century ago in the stability analysis of the Couette flow, and widely used in the last decade [7–14] to unveil many important aspects of non-Hermitian, shear-flow systems, is rather limited; it is applicable only if the the shear-flow profile is linear. The renormalized perturbative analysis shows that a departure from the linear profile need not, and perhaps does not lead to singular changes in the time asymptotic behavior. The results of this study puts Kelvin's extremely simple method on a much firmer footing.

In Sec. II, we will present an asymptotic analysis of the unbounded Rayleigh equation followed in Sec. III by its perturbative investigation. Section IV will be devoted to the

*Email address: fvolponi@plasma.q.t.u-tokyo.ac.jp

proof of renormalizability of the divergence found in Sec. III. We summarize our results in Sec. V.

II. ASYMPTOTIC INITIAL VALUE TREATMENT

We start our analysis from the two-dimensional (2D) vorticity equation for inviscid fluids which reads

$$\partial_t \Omega + \{\Phi,\Omega\} = 0, \tag{1}$$

where where $\{,\}$ represents the Poisson bracket. The streamfunction Φ is related to the velocity V and the vorticity Ω through the equations

$$V = \nabla \Phi \times e_z, \tag{2}$$

$$\Omega = -\Delta \Phi, \tag{3}$$

where e_z is the unit vector in the z direction and Δ is the 2D Laplacian. By decomposing the streamfunction as

$$\Phi = \Phi_0 + \phi, \tag{4}$$

where Φ_0 represents the equilibrium and ϕ the perturbative field, we can linearize Eq. (1) to obtain

$$\partial_t(\Delta \phi) + \{\phi, \Delta \Phi_0\} + \{\Phi_0, \Delta \phi\} = 0. \tag{5}$$

For a parallel equilibrium flow of the type $V_0 = (-f(y),0,0)$, where $f(y)$ is an analytic function, we obtain the Rayleigh equation

$$(\partial_t - f(y)\partial_x)\Delta \phi = -f''(y)\partial_x \phi, \tag{6}$$

where $' = \partial_y$. Since the ambient field is homogeneous with respect to x, we can decompose ϕ into Fourier modes proportional to e^{ikx}. Writing $\partial_x = ik$ with a good quantum number k (in what follows, we take $k>0$) Eq. (6) translates as

$$[\partial_t - ikf(y)]\Delta \phi = -ikf''(y)\phi. \tag{7}$$

By inverting the Laplacian operator Δ we obtain

$$\phi = -\Delta^{-1}\omega = \frac{1}{2k}\int_{-\infty}^{+\infty} e^{-k|\overline{y}-y|}\omega(\overline{y},t)d\overline{y}. \tag{8}$$

By using Eq. (8) we can recast Eq. (7) into

$$[\partial_t - ikf(y)]\omega = \frac{if''(y)}{2}\int_{-\infty}^{+\infty} e^{-k|\overline{y}-y|}\omega(\overline{y},t)d\overline{y}. \tag{9}$$

In terms of

$$Q(y,t) = e^{-ikf(y)t}\omega(y,t) \tag{10}$$

the system is described by

$$\partial_t Q(y,t) = \frac{if''(y)}{2}\int_{-\infty}^{+\infty} e^{-k|\overline{y}-y|}Q(\overline{y},t)e^{ikt[f(\overline{y})-f(y)]}d\overline{y}. \tag{11}$$

We will carry out the analysis of Eq. (11) in the regime of short wavelengths (large k) perturbations. In the Appendix we will present a treatment which takes into account special classes of localized perturbations and which is valid for wave numbers $k>1$.

In the first case, since k is large then $e^{-k|y-\overline{y}|}$ is a strong damping factor and it is reasonable to assume that the main contribution to the integral in Eq. (11) comes from the neighborhood of y. By Taylor expanding $f(\overline{y})$ and $Q(\overline{y},t)$ about y we have

$$f(\overline{y}) - f(y) \simeq f'(y)(\overline{y}-y) + \frac{f''(y)}{2}(\overline{y}-y)^2 = az + bz^2, \tag{12}$$

$$Q(\overline{y},t) \simeq Q(y,t) + \partial_y Q(y,t)(\overline{y}-y) = Q(y,t) + \partial_y Q(y,t)z, \tag{13}$$

where $z = \overline{y} - y$, $a = f'(y)$ and $b = f''(y)/2$. Substituting Eqs. (12)–(13) into Eq. (11) we have

$$\partial_t Q = \frac{if''(y)}{2}Q\int_{-\infty}^{+\infty} e^{-k|z|}e^{ikt(az+bz^2)}dz + \frac{if''(y)}{2}\partial_y Q\int_{-\infty}^{+\infty} ze^{-k|z|}e^{ikt(az+bz^2)}dz. \tag{14}$$

An asymptotic evaluation of the above integrals gives two different limits depending on whether the first derivative $a = f'(y)$ vanishes or not. If $a = f'(y)$ is nonzero then, for a large time, the two integrals [15] yield

$$\partial_t Q = \frac{2ib}{k(1+a^2t^2)}Q - \frac{4abt}{k^2(1+a^2t^2)^2}\partial_y Q. \tag{15}$$

We have gone back to a partial differential equation. However since we are interested in the long-term time behavior, we may neglect the second term on the right side due to its faster decay (t^{-3}). In force of this neglection we can, then, readily integrate Eq. (15) to find

$$Q(y,t) = Q_0(y)e^{i(b/ak)\tan^{-1}(at)}. \tag{16}$$

Therefore, for $t\to\infty$, depending on the sign of a

$$Q(y,t) \to Q_0(y)e^{i(b\pi/2ak)\mathrm{sgn}(a)}, \tag{17}$$

implying a simple phase shift from the initial perturbation. In terms of the vorticity, we recover an oscillatory behavior of the type

$$\omega(y,t) = Q_0(y)e^{i(b\pi/2ak)\mathrm{sgn}(a)}e^{ikf(y)t}. \tag{18}$$

An important deviation from the oscillatory response is found when $f'(y)$ vanishes at some point. In this case, Eq. (11) becomes

$$\partial_t Q = \frac{if''(y)}{2} Q \int_{-\infty}^{+\infty} e^{-k|z|} e^{iktbz^2} dz + \frac{if''(y)}{2} \partial_y Q \int_{-\infty}^{+\infty} z e^{-k|z|} e^{iktbz^2} dz. \tag{19}$$

The second integral vanishes by symmetry, and the asymptotic evaluation of the first leads to the equation

$$\partial_t Q = \sqrt{\frac{b\pi}{kt}} e^{i(3/4)\pi} Q, \tag{20}$$

with the solution

$$Q(y,t) = Q_0(y) e^{2\sqrt{(b\pi/k)} e^{i(3/4)\pi} \sqrt{t}}. \tag{21}$$

We notice that, independent of the sign of b, Q always decays with time. The vorticity ω

$$\omega(y,t) = Q_0(y) e^{2\sqrt{(b\pi/k)} e^{i(3/4)\pi} \sqrt{t}} e^{ikf(y)t} \tag{22}$$

has oscillations imposed on the decay.

The main result of this section is that in an unbounded inviscid fluid with a background parallel shear flow, the vorticity perturbations for large k show asymptotic oscillations if the first derivative of the background flow does not vanish at some point. At the points where the first derivative does vanish, the perturbations suffer an exponential decay of the type $e^{-\sqrt{t}}$.

III. PERTURBATIVE ANALYSIS

In this section, we develop a perturbation analysis of the Rayleigh equation when the flow curvature $f''(y)$ is small. Assuming $f''(y) = \varepsilon^2 g(y)$, where $g(y)$ is an $O(1)$ function and ε is a small positive parameter quantifying the smallness of the curvature with respect to the flow $f(y)$, Eq. (7) may be written as

$$[\partial_t - ikf(y)] \Delta \phi = -\varepsilon^2 ikg(y) \phi. \tag{23}$$

By using Eq. (8) and applying $[\partial_t - ikf(y)]^{-1}$ to both sides of Eq. (23), we arrive at the integral equation

$$\omega(y,t) = \Omega_0(y) e^{ikf(y)t} + \varepsilon^2 \frac{ig(y)}{2} \times \int_{-\infty}^{+\infty} e^{-k|\bar{y}-y|} \left(\int_0^t \omega(\bar{y},\bar{t}) e^{ikf(y)(t-\bar{t})} d\bar{t} \right) d\bar{y}, \tag{24}$$

where $\Omega_0(y)$ is an arbitrary function, to be specified by the initial conditions. We look for an approximate solution of Eq. (24) by means of a Born perturbative series of the type

$$\omega = \sum_{n=0}^{+\infty} \varepsilon^{2n} \omega_n . \tag{25}$$

Plugging Eq. (25) into Eq. (24), we obtain the following recursion relations:

$$\omega_0 = \Omega_0(y) e^{ikf(y)t}, \tag{26}$$

$$\omega_1 = \frac{ig(y)}{2} \int_{-\infty}^{+\infty} e^{-k|\bar{y}-y|} \left(\int_0^t \omega_0(\bar{y},\bar{t}) e^{ikf(y)(t-\bar{t})} d\bar{t} \right) d\bar{y}, \tag{27}$$

$$\vdots$$

$$\omega_m = \frac{ig(y)}{2} \int_{-\infty}^{+\infty} e^{-k|\bar{y}-y|} \left(\int_0^t \omega_{m-1}(\bar{y},\bar{t}) e^{ikf(y)(t-\bar{t})} d\bar{t} \right) d\bar{y}. \tag{28}$$

Carrying out the integration in $\bar{t}$ the first order correction ω_1 reads

$$\omega_1(y,t) = \frac{e^{ikf(y)t} g(y)}{2k} \times \int_{-\infty}^{+\infty} e^{-k|\bar{y}-y|} \Omega_0(\bar{y}) \frac{e^{ik[f(\bar{y})-f(y)]t} - 1}{f(\bar{y}) - f(y)} d\bar{y}. \tag{29}$$

To determine the temporal behavior, we must evaluate the integral on the right side of Eq. (29)

$$S(y,t) = \int_{-\infty}^{+\infty} e^{-k|\bar{y}-y|} \Omega_0(\bar{y}) \frac{e^{ik[f(\bar{y})-f(y)]t} - 1}{f(\bar{y}) - f(y)} d\bar{y}. \tag{30}$$

Following the argument used in the previous section, for large k, we can approximate

$$\Omega_0(\bar{y}) \simeq \Omega_0(y) + (\bar{y} - y) \partial_y \Omega_0(y), \tag{31}$$

and $f(\bar{y})$ by Eqs. (12). Using (12) and Eq. (31), S becomes

$$S = \Omega_0(y) \int_{-\infty}^{+\infty} e^{-k|z|} \frac{e^{ik(az+bz^2)t} - 1}{az + bz^2} dz + \partial_y \Omega_0(y) \int_{-\infty}^{+\infty} z e^{-k|z|} \frac{e^{ik(az+bz^2)t} - 1}{az + bz^2} dz. \tag{32}$$

By differentiating with time, we obtain

$$\partial_t S = ik \left[\Omega_0(y) \int_{-\infty}^{+\infty} e^{-k|z|} e^{ik(az+bz^2)t} dz + \partial_y \Omega_0(y) \int_{-\infty}^{+\infty} z e^{-k|z|} e^{ik(az+bz^2)t} dz \right]. \tag{33}$$

The two integrals in the above equation are the same already calculated in Eq. (14). Analogously to what done before we neglect the second integral on the right hand side due to its faster decay, obtaining for S

$$\partial_t S = \frac{2i\Omega_0(y)}{1+a^2t^2}, \tag{34}$$

which is easily integrated as

$$S = \frac{2i}{a}\Omega_0(y)\tan^{-1}(at). \tag{35}$$

The first order correction ω_1 is given by

$$\omega_1(y,t) = e^{ikf(y)t}\frac{2ig(y)}{ak}\Omega_0(y)\tan^{-1}(at). \tag{36}$$

The result of the perturbative analysis is

$$\omega(y,t) = e^{ikf(y)t}\Omega_0(y)\left[1+\varepsilon^2\frac{ig(y)}{ak}\tan^{-1}(at)\right], \tag{37}$$

in "full" agreement with the result of Eq. (18).

A pathology of the perturbative analysis is found in the case when $f'(y)$ vanishes for some $y=y_s$. In fact for $a=0$, S reads

$$S = \Omega_0(y)\int_{-\infty}^{+\infty} e^{-k|z|}\frac{e^{ikbz^2t}-1}{bz^2}dz + \partial_y\Omega_0(y)\int_{-\infty}^{+\infty} z e^{-k|z|}\frac{e^{ikbz^2t}-1}{bz^2}dz. \tag{38}$$

Since the second integral on the right hand side vanishes for symmetry considerations, we are lead to

$$\partial_t S = ik\Omega_0(y)\int_{-\infty}^{+\infty} e^{-k|z|}e^{ikbz^2t}dz. \tag{39}$$

Evaluating the integral by familiar means, we find

$$\partial_t S = k\Omega_0(y)\sqrt{\frac{\pi}{kbt}}e^{i(3/4)\pi}, \tag{40}$$

which, when integrated, yields

$$S = \Omega_0(y)\sqrt{\frac{k\pi}{b}}e^{i(3/4)\pi}2\sqrt{t}. \tag{41}$$

The resulting $\omega(y,t)$

$$\omega(y,t) = e^{ikf(y)t}\Omega_0(y)\left(1+\varepsilon\sqrt{\frac{g(y)\pi}{2k}}e^{i(3/4)\pi}2\sqrt{t}\right) \tag{42}$$

shows secular growth in time. The perturbative analysis diverges, and breaks down at time scales $t\sim\varepsilon^{-2}$. In Sec. IV, we will show that this secularity is renormalizable.

We end this section with a remark about the insurgence of this secular behavior. Let us consider, for simplicity, the real part of S

$$\mathrm{Re}(S) = \Omega_0(y)\int_{-\infty}^{+\infty} e^{-k|\bar{y}-y|}\frac{\cos(k[f(\bar{y})-f(y)]t)-1}{f(\bar{y})-f(y)}d\bar{y}. \tag{43}$$

If $f(\bar{y})-f(y)$ is an odd function of $(\bar{y}-y)$, $\mathrm{Re}(S)=0$ due to symmetry. For

$$f(\bar{y})-f(y) = F_e(\bar{y}-y), \tag{44}$$

with $F_e(\bar{y}-y)$ an even function of $\bar{y}-y$, $\mathrm{Re}(S)$ reads

$$\mathrm{Re}(S) = \Omega_0(y)\int_{-\infty}^{+\infty}\frac{e^{-k|y-\bar{y}|}}{F_e(\bar{y}-y)}[\cos(ktF_e(\bar{y}-y))-1]d\bar{y}. \tag{45}$$

We further assume (a) $F_e(\bar{y}-y)$ is a function of definite sign, (b) $\lim_{\bar{y}\to y}F_e(\bar{y}-y)=0$, and (c) $e^{-k|y-\bar{y}|}/F_e(\bar{y}-y)$ decays at infinity faster than $1/(\bar{y}-y)$.

Due to (b) the envelope of the integrand, $e^{-k|\bar{y}-y|}/F_e(\bar{y}-y)$ is singular. We want to classify the dependence of $\mathrm{Re}(S)$ on t according to the type of singularity that $e^{-k|y-\bar{y}|}/F_e(\bar{y}-y)$ has at $\bar{y}=y$. To this end let us rewrite Eq. (45) as

$$\mathrm{Re}(S) = \Omega_0(y)\int_{-\infty}^{+\infty}\Xi(t,z,k)dz, \tag{46}$$

where

$$\Xi(t,z,k) = \frac{e^{-k|z|}}{F_e(z)}[\cos(ktF_e(z))-1]. \tag{47}$$

If $1/F_e(z)$ diverges at $z=0$ as $1/z^n$ with $n\geqslant 1$, then the envelope $e^{-k|z|}/F_e(z)$ is not integrable. Therefore for increasing t , since the effective frequency in $\cos[ktF_e(z)]$ increases and $\cos[ktF_e(z)]-1$ has a definite sign, the integral of $\Xi(t,z,k)$ in Eq. (46) tends to some fraction of the infinite area under $e^{-k|z|}/F_e(z)$. This means that $\mathrm{Re}(S)$ is an increasing function of t; in other words $\mathrm{Re}(S)$ has secular growth. On the other end if $e^{-k|z|}/F_e(z)$ is integrable, secularity will not occur since, no matter what the increase in t, the area under $e^{-k|z|}/F_e(z)$ will always be finite.

Summarizing, the origin of the secularity is in the singularity of the envelope $e^{-k|z|}/F_e(z)$ at zero. In order to obtain a nondivergent behavior, there is a need to introduce an appropriate cutoff at small z.

IV. RENORMALIZATION

In this section by means of the renormalization group method [4,5] we will uniformize the divergence met above. In order to renormalize the expansion given in Eq. (42) we introduce in the system an arbitrary parameter τ (a cutoff), and a renormalization constant

$$R(y,\tau)=1+\sum_{n=1}^{+\infty}\varepsilon^n a_n(y,\tau) \tag{48}$$

such that

$$\Omega_0(y)=R(y,\tau)\Omega(y,\tau). \tag{49}$$

The coefficients $a_n(y,\tau)$ are chosen order by order in ε to eliminate the secular terms.

By adding and subtracting $\sqrt{\tau}$ to $\sqrt{t}$ and using Eq. (48) in Eq. (42) we obtain, to order ε,

$$\omega(y,t)=\Omega(y,\tau)e^{ik_x f(y)t}\left[1+\varepsilon\sqrt{\frac{g(y)\pi}{2k}}e^{i(3/4)\pi}2(\sqrt{t}-\sqrt{\tau})\right]$$
$$+\Omega(y,\tau)e^{ik_x f(y)t}\varepsilon\left(\sqrt{\frac{g(y)\pi}{2k}}e^{i(3/4)\pi}2\sqrt{\tau}+a_1\right)$$
$$+O(\varepsilon^2). \tag{50}$$

It is clear that by choosing

$$a_1=-\sqrt{\frac{g(y)\pi}{2k}}e^{i(3/4)\pi}2\sqrt{\tau} \tag{51}$$

the secular term vanishes.

Since τ does not appear in the original problem, $\omega(y,t)$ should not depend on it. Therefore, we impose the renormalization group equation

$$\left(\frac{d}{d\tau}\omega\right)_{\tau=t}=0, \tag{52}$$

which gives for $\Omega(y,\tau)$ the equation

$$\frac{d}{d\tau}\Omega=\varepsilon\Omega\sqrt{\frac{g(y)\pi}{2k}}e^{i(3/4)\pi}\frac{1}{\sqrt{\tau}} \tag{53}$$

with the solution

$$\Omega(y,t)=\tilde{\Omega}_0(y)e^{\varepsilon\sqrt{[g(y)\pi/2k]}e^{i(3/4)\pi}2\sqrt{t}}, \tag{54}$$

where $\tilde{\Omega}_0(y)$ is the initial value of Eq. (53).

The final renormalized expansion is therefore

$$\omega(y,t)=\tilde{\Omega}_0(y)e^{2\varepsilon\sqrt{[g(y)\pi/2k]}e^{i(3/4)\pi}\sqrt{t}}e^{ikf(y)t}, \tag{55}$$

which is in agreement with Eq. (22). This shows that the first-order correction divergence is renormalizable.

V. SUMMARY

The evolution of a fluid system described by the unbounded Rayleigh equation has been investigated. Even in this simple context, pathologies due to the non-Hermitian nature of the problem arise. We first carried out an asymptotic nonperturbative analysis of the system in the regime of large *k*. The analysis shows the stability of the system. The type of stable behavior depends on whether the first derivative of the flow vanishes at a particular point. At points y_0 where $f'(y_0)\neq 0$, the vorticity simply oscillates in time (the same result holds for localized perturbations). For y_d with $f'(y_d)=0$, it experiences an exponential decay of the type $e^{-\sqrt{t}}$.

We recall that the analysis of the Rayleigh equation in a channel also predicts the stability for large *k*, while Kelvin-Helmholtz unstable modes are found for large wavelength perturbations. However a close comparison between the present system and the one described by the channel Rayleigh equation is difficult since the absence of boundaries drastically alters the physical and mathematical nature of the problem.

The perturbative analysis of the system shows a possible pathology of non-Hermitian systems, namely, a divergence of the first-order correction in the perturbation expansion. This problem is reminiscent of the analogous pathologies met in nonlinear systems; the common origin is the coupling of modes, their nonindependent time evolution. Divergences are found in the case of vanishing $f'(y)$, while for $f'(y)\neq 0$, uniform expansions always pertain.

The singularity of the perturbative analysis is renormalizable by means of the renormalization group method. Asymptotic solutions and the results of the renormalized perturbative expansions are in full agreement.

ACKNOWLEDGMENTS

F.V. is grateful to Shuichi Oosaki for drawing his attention to Ref. [5] and to Daniel Gomez for discussions and suggestions. S.M.M.'s work was supported by the U.S. Department of Energy Contract No. DE-FG03-96ER-54346. F.V. and Z.Y.'s work was supported by Grant No. 98-3909 from Toray Science Foundation and Grant in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture No. 10044130.

APPENDIX

Our goal is to solve Eq. (11) for a special class of localized perturbations, therefore we need to evaluate the integral

$$I=\int_{-\infty}^{+\infty}e^{-k|\overline{y}-y|}Q(\overline{y},t)e^{ikt[f(\overline{y})-f(y)]}d\overline{y}. \tag{A1}$$

By the change of variable $z=\overline{y}-y$, I becomes

$$I=\int_{-\infty}^{+\infty}Q(y+z,t)e^{-k|z|}e^{ikt[f(y+z)-f(y)]}dz. \tag{A2}$$

We assume:

(i) $Q(y+z)$ is supported in Ω (bounded) and is $C^\infty(\mathbf{R})$,

(ii) $|\partial_z^{n+1}Q(y+z,t)|<B(t)\quad\forall n,\forall t$, where $B(t)$ is a continuous of t $(t\geqslant 0)$; and

(iii) $f(y+z)=f(y)+f'(y)z$ in Ω with $f'(y)\neq 0$.

Using (iii) above and splitting in two the domain of integration we have

$$I=I_1+I_2, \tag{A3}$$

where

$$I_1=\int_0^{+\infty}Q(y+z,t)e^{-kz}e^{iktf'(y)z}dz \tag{A4}$$

and

$$I_2=\int_{-\infty}^{0}Q(y+z,t)e^{kz}e^{iktf'(y)z}dz. \tag{A5}$$

Integrating $n+1$ times by parts I_1 we obtain

$$I_1=\tilde{I}_1^{(n)}+R_1^{(n)}, \tag{A6}$$

where

$$\tilde{I}_1^{(n)}=\sum_{m=0}^{n}\frac{\partial_y^m Q(y,t)}{[k(1-itf'(y))]^{m+1}} \tag{A7}$$

and

$$R_1^{(n)}=\frac{1}{[k(1-itf'(y))]^{n+1}}\times\int_0^{+\infty}\partial_z^{n+1}Q(y+z,t)e^{-kz(1-itf'(y))}dz. \tag{A8}$$

Analogously, integrating $n+1$ times by parts I_2 we have

$$I_2=\tilde{I}_2^{(n)}+R_2^{(n)}, \tag{A9}$$

where

$$\tilde{I}_2^{(n)}=\sum_{m=0}^{n}(-1)^m\frac{\partial_y^m Q(y,t)}{[k(1+itf'(y))]^{m+1}} \tag{A10}$$

and

$$R_2^{(n)}=\frac{(-1)^{n+1}}{[k(1+itf'(y))]^{n+1}}\times\int_{-\infty}^{0}\partial_z^{n+1}Q(y+z,t)e^{kz(1+itf'(y))}dz. \tag{A11}$$

Now if $k>1$ and using the assumption (ii), we have for all t

$$\lim_{n\to+\infty}|I_1-\tilde{I}_1^{(n)}|=\lim_{n\to+\infty}|R_1^{(n)}|=0, \tag{A12}$$

$$\lim_{n\to+\infty}|I_2-\tilde{I}_2^{(n)}|=\lim_{n\to+\infty}|R_2^{(n)}|=0. \tag{A13}$$

Therefore we obtain for I_1, I_2 the series representations

$$I_1=\sum_{m=0}^{+\infty}\frac{\partial_y^m Q(y,t)}{k^{m+1}(1-itf'(y))^{m+1}}, \tag{A14}$$

$$I_2=\sum_{m=0}^{+\infty}(-1)^m\frac{\partial_y^m Q(y,t)}{k^{m+1}(1+itf'(y))^{m+1}}. \tag{A15}$$

Finally we have for I

$$I=\frac{2Q}{k(1+t^2f'(y)^2)}+\frac{4if'(y)t\partial_y Q}{k^2(1+t^2f'(y)^2)^2}+O(t^{-4}). \tag{A16}$$

Keeping the first two terms in the limit $t\to\infty$ and substituting I in Eq. (11) we again obtain Eq. (15). Therefore the same results previously obtained for large k and $f'(y)\neq 0$ will hold.

[1] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, 1981).
[2] A. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973).
[3] C. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
[4] L. Y. Chen, N. G. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996).
[5] Y. Oono, Butsuri **52**, 501 (1997).
[6] Lord Kelvin, Philos. Mag. **24**, 188 (1887).
[7] W. O. Criminale and P. G. Drazin, Stud. Appl. Math. **83**, 123 (1990).
[8] G. D. Chagelishvili, T. S. Hristov, R. G. Chanishvili, and J. G. Lominadze, Phys. Rev. E **47**, 366 (1993).
[9] G. D. Chagelishvili, A. G. Tevzadze, G. Bodo, and S. S. Moiseev, Phys. Rev. Lett. **79**, 3178 (1997).
[10] G. D. Chagelishvili, A. D. Rogava, and D. T. Tsiklauri, Phys. Plasmas **4**, 1182 (1997).
[11] G. D. Chagelishvili, G. R. Khujadze, J. G. Lominadze, and A. D. Rogava, Phys. Fluids **9**, 1955 (1997).
[12] A. D. Rogava and S. M. Mahajan, Phys. Rev. E **55**, 1185 (1997).
[13] A. G. Tevzadze, Phys. Plasmas **5**, 1557 (1998).
[14] F. Volponi, Z. Yoshida, and T. Tatsuno, Phys. Plasmas **7**, 2314 (2000).
[15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1994).