

## Noise properties of stochastic processes and entropy production

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Based on a Fokker-Planck description of external Ornstein-Uhlenbeck noise and cross-correlated noise processes driving a dynamical system we examine the interplay of the properties of noise processes and the dissipative characteristic of the dynamical system in the steady state entropy production and flux. Our analysis is illustrated with appropriate examples.

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### I. INTRODUCTION

A dynamical system in contact with a reservoir has been a subject of wide attention in dissipative dynamics and irreversible thermodynamics. The focal theme lies on the possible link between the rate of phase space volume contraction and the thermodynamically inspired quantities like entropy production, entropy flux, and Onsager coefficients etc. [1–11]. While on the other hand it has been argued that the entropy production is related to the intrinsic properties of phase space structure of the dynamical systems through the Lyapunov exponents [6–11], the traditional wisdom asserts that entropy production in a class of thermostatted Hamiltonian system is defined [3] as the work per unit time (in the leading order) done on the system by an external constraint under nonequilibrium steady state condition. Recently based on a Markovian description of a stochastic process, Daems and Nicolis [12], have critically analyzed the two aspects from the consideration of an information entropy balance equation.

The object of the present paper is to extend the treatment to color [13] and cross-correlated noise processes [14,15] and to search for an appropriate signature of an intrinsic interplay between the noise properties of these processes and the dissipative characteristics of the dynamical system in the steady state entropy production and flux. We specifically consider the overall system to be open, i.e., the noises are of external origin such that they do not, in general, satisfy fluctuation-dissipation relations. Whenever possible we allow ourselves to make a fair comparison with the standard results for closed systems.

The organization of the paper is as follows: In Sec. II we consider two types of external stationary and Gaussian noise processes, namely, the Ornstein-Uhlenbeck and cross-correlated noise processes in terms of a Fokker-Planck description and setup an entropy balance equation to identify the drift term, which reveals that in addition to dissipation constant it contains the essential properties of noise processes. Section III is devoted to explicit examples to calculate the entropy production. The paper is concluded in Sec. IV.

### II. THE NOISE PROCESSES AND ENTROPY PRODUCTION

#### A. Fokker-Planck description

##### 1. External Ornstein-Uhlenbeck noise processes

The Langevin equations of motion in phase space for an  $N$ -degrees-of-freedom system that is driven by the external color noise process  $\eta_i$  can be written as

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} = p_i, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} - \gamma_i p_i + \eta_i, \quad i=1, \dots, N, \end{aligned} \quad (1)$$

where  $N$  is the number of degrees of freedom of the system,  $\gamma_i$  is the damping constant for  $i$ th degree of freedom, and  $q_i, p_i$  are the corresponding coordinate and the momentum, respectively. While the presence of  $\gamma_i$  imparts a dissipative character in the dynamics, the stochastic forcing  $\eta_i$  ensures a canonical distribution at equilibrium when the fluctuation-dissipation relation is satisfied.  $H$  is Hamiltonian of an initially the conservative system and is given by

$$H = \sum_{i=1}^N \frac{p_i^2}{2} + V(\{q_i\}, t). \quad (2)$$

The masses of all the degrees of freedom have been set to unity.  $V(\{q_i\}, t)$  is the potential of the Hamiltonian system.

The term  $\eta_i$  in Eq. (1) refers to an external, Gaussian color noise for the  $i$ th degree of freedom and follows the two time correlation function

$$\langle \eta_i(t) \eta_i(t') \rangle = \frac{D_i^0}{\tau_i} e^{-|t-t'|/\tau_i}, \quad (3)$$

where  $\tau_i$  is the correlation time and  $D_i^0$  is the noise strength. The time evolution of  $\eta_i$  can be conveniently expressed in terms of the white noise process  $\zeta_i(t)$  for the  $i$ th component

$$\dot{\eta}_i = -\frac{\eta_i}{\tau_i} + \frac{\sqrt{D_i^0}}{\tau_i} \zeta_i, \quad (4)$$

$$\langle \zeta_i(t) \zeta_i(t') \rangle = 2\delta(t-t'),$$

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$$\langle \zeta_i \rangle = 0.$$

In case there exists no fluctuation-dissipation relation between  $\gamma_i$  and  $\eta_i$  the system described by the Eq. (1) is sometimes termed as thermodynamically open [16].

Equation (4) implies that  $\eta_i$  can be treated as a phase space variable on the same footing as  $q_i, p_i$ . Thus the original  $2N$  dimensional stochastic system (1,3) now becomes a  $3N$  dimensional Markovian process where Eq. (1) and (4) are written in a compact form

$$\dot{X}_i = F_i(X) + \zeta_i \quad (5)$$

where

$$X_i = \begin{cases} q_i & \text{for } i = 1, \dots, N \\ p_i & \text{for } i = N+1, \dots, 2N \\ \eta_i & \text{for } i = 2N+1, \dots, 3N \end{cases}$$

$$F_i = X_{i+N} \text{ for } i = 1, \dots, N,$$

$$F_i = -\frac{\partial V(X_1, \dots, X_N)}{\partial X_i} - \gamma_i X_i + X_{i+N} \text{ for } i = N+1, \dots, 2N,$$

$$F_i = -\frac{X_i}{\tau_i} + \frac{\sqrt{D_i^0}}{\tau_i} \zeta_i \text{ for } i = 2N+1, \dots, 3N,$$

and

$$\begin{aligned} \langle \zeta_i(t) \zeta_i(t') \rangle &= 0 \quad \text{for } i = 1, \dots, 2N \\ \langle \zeta_i(t) \zeta_i(t') \rangle &= 2\delta(t-t') \quad \text{for } i = 2N+1, \dots, 3N. \end{aligned} \quad (6)$$

The Fokker-Planck equation [13] corresponding to Langevin Eq. (5) can be written as

$$\frac{\partial P(X, t)}{\partial t} = -\sum_{i=1}^{3N} \frac{\partial}{\partial X_i} (F_i P) + \sum_{i=2N+1}^{3N} D_i \frac{\partial^2 P}{\partial X_i^2}, \quad (7)$$

where  $D_i = D_i^0 / \tau_i^2$ .

$P(X, t)$  is the extended phase space probability distribution function. The extension is due to the inclusion of  $N$  noise variables due to the external agency as phase variables. We conclude by pointing out that the above formulation contains the thermodynamically closed system as a special case where the internal noise strength  $D_i^0$  is related to dissipation  $\gamma_i$  through the relation  $D_i^0 = \gamma_i kT$ , where  $T$  refers to the equilibrium temperature of the reservoir.

## 2. Cross-correlated noise processes

Next we consider a dynamical system driven by both additive and multiplicative noise processes  $\eta_i$  and  $\zeta_i$ , respectively. The Langevin equation for this process, in general, can be written as

$$\dot{X}_i = L_i(\{X_j\}, t) + g_i(X_i) \zeta_i + \eta_i \quad i = 1, \dots, N, \quad (8)$$

where  $L_i$  contains the dissipative term as well as the external applied deterministic force, if any,  $g_i(X_i)$  is the coupling between the system and the multiplicative process,  $\zeta_i$  and  $\eta_i$  are white Gaussian noise processes with the following correlation between them;

$$\langle \zeta_i(t) \zeta_j(t') \rangle = 2D'_{ij} \delta(t-t') \delta_{ij},$$

$$\langle \eta_i(t) \eta_j(t') \rangle = 2\alpha_{ij} \delta(t-t') \delta_{ij},$$

$$\langle \zeta_i(t) \eta_j(t') \rangle = \langle \zeta_i(t') \eta_j(t) \rangle = 2\lambda_{ij} \sqrt{D'_{ij} \alpha_{ij}} \delta(t-t') \delta_{ij}, \quad (9)$$

where  $D'_{ij}$  and  $\alpha_{ij}$  correspond to the strength of multiplicative and additive noises, respectively,  $\lambda$  represents the cross correlation between them with the limit  $0 \leq \lambda \leq 1$ . The cross correlation between these noise processes is known to cause symmetry breaking leading to nonequilibrium phase transitions [14] in spatially extended systems and generate interesting ratchet motion [15] in systems with symmetric potential under isothermal condition.

The Fokker-Planck equation corresponding to Langevin Eq. (8) can be written as

$$\frac{\partial P(X)}{\partial t} = -\sum_{i=1}^N \frac{\partial}{\partial X_i} (F_i P) + \sum_{i=1}^N D_i \frac{\partial^2 P}{\partial X_i^2}, \quad (10)$$

where the drift for the  $i$ th component  $F_i$  is

$$\begin{aligned} F_i &= L_i(\{X_j\}, t) + \nu \left[ D'_{ii} \frac{\partial g_i(X_i)}{\partial X_i} g_i(X_i) + \lambda_{ii} \sqrt{\alpha_{ii} D'_{ii}} \right] \\ &+ D'_{ii} \frac{\partial g_i^2(X_i)}{\partial X_i} + 2\lambda_{ii} \sqrt{\alpha_{ii} D'_{ii}} \frac{\partial g_i(X_i)}{\partial X_i}, \end{aligned} \quad (11)$$

where  $\nu = 1$  stands for the Stratonovich and  $\nu = 0$  for the Ito convention, respectively. Diffusion coefficient  $D_i$  within small noise approximation can be written as

$$D_i = \alpha_{ii} + D'_{ii} g_i^2(X_{ie}) + 2\lambda_{ii} \sqrt{\alpha_{ii} D'_{ii}} g_i(X_{ie}), \quad (12)$$

where  $e$  in  $X_{ie}$  refers to the steady state value of  $X_i$ , i.e.,  $X_{ie}$  is a solution of

$$F_i(\{X_j\}) = 0 \quad i = 1, \dots, N \quad (13)$$

The choice of specific forms of nonlinearity in  $L_i(\{X_j\}, t)$  results in typical features of nonequilibrium phase transitions in model systems. For the present purpose, however, we retain a general structure for the rest of the treatment.

## B. Information entropy production

Information entropy  $S$  is formally defined in terms of the phase space distribution function  $P(X, t)$  through the well-known relation

$$S = - \int dX P(X, t) \ln P(X, t). \quad (14)$$

The above definition allows us to have an evolution equation for entropy. To this end we observe from Eqs. (10) [(7)] and (14) that

$$\frac{dS}{dt} = - \int dX \left[ - \sum_i \frac{\partial}{\partial X_i} (F_i P) + \sum_i D_i \frac{\partial^2 P}{\partial X_i^2} \right] \ln P. \quad (15)$$

Performing partial integration of the right hand side of the above Eq. (15) and then dropping boundary terms (since the probability density tends to zero as  $|X| \rightarrow \infty$ ), one obtains the following form of information entropy balance:

$$\frac{dS}{dt} = \int dX P \nabla_X \cdot F + \sum_i D_i \int \frac{1}{P} \left( \frac{\partial P}{\partial X_i} \right)^2. \quad (16)$$

The first term in Eq. (16) has no definite sign while the second term is positive definitely, because of positive definiteness of  $D_i$ . Therefore the second one can be identified as the entropy production ( $\dot{S}_0$ ) [12],

$$\dot{S}_0 = \sum_i D_i \int \frac{1}{P_s} \left( \frac{\partial P_s}{\partial X_i} \right)^2 dX, \quad (17)$$

in the steady state. The subscript  $s$  of  $P_s$  refers to steady state. It is therefore evident from Eq. (17) that

$$\begin{aligned} \dot{S}_{flux} &= \int dX P_s(X) \nabla_X \cdot F = \overline{\nabla_X \cdot F}, \\ \dot{S}_0 &= -\dot{S}_{flux}. \end{aligned} \quad (18)$$

Note that since we consider the system to be dissipative,  $\overline{\nabla_X \cdot F}$  is negative and therefore  $\dot{S}_0$  turns out to be positive.

### C. Influence of external perturbation

It is now interesting to examine the entropy production, when the dissipative system is thrown away from the steady state due to an additional weak applied force. To this end we consider the drift  $F_1$  due to external force so that the total drift  $F$  has now two contributions:

$$F(X) = F_0(X) + hF_1(X). \quad (19)$$

When  $h=0$ ,  $P=P_s$ , the deviation of  $P$  from  $P_s$  in presence of nonzero small  $h$  can be explicitly taken into account once we make use of the identity for the diffusion term [12]

$$\frac{\partial^2 P}{\partial X_i^2} = \frac{\partial}{\partial X_i} \left[ P \frac{\partial \ln P_s}{\partial X_i} \right] + \frac{\partial}{\partial X_i} \left[ P_s \frac{\partial}{\partial X_i} \frac{P}{P_s} \right]. \quad (20)$$

When  $P=P_s$  the second term in Eq. (20) vanishes. In presence of additional forcing the Eq. (10) becomes,

$$\frac{\partial P}{\partial t} = -\nabla_X \cdot \psi P - h \nabla_X \cdot F_1 P + \sum_i D_i \frac{\partial}{\partial X_i} \left( P_s \frac{\partial}{\partial X_i} \frac{P}{P_s} \right), \quad (21)$$

where  $\psi$  is defined as

$$\psi = F_0 - \sum_i D_i \frac{\partial \ln P_s}{\partial X_i}. \quad (22)$$

Here we have assumed for simplicity that  $D_i$  is not affected by the additional forcing. The leading order influence is taken into account through the additional drift term in Eq. (21).

Under steady state condition ( $P=P_s$ ) and  $h=0$ , the second and the third terms in Eq. (21) vanish yielding

$$\nabla_X \cdot \psi P_s = 0. \quad (23)$$

It is immediately apparent that  $\psi P_s$  refers to a current  $\mathcal{J}$  where  $\mathcal{J} = \psi P_s$ . The steady state condition therefore reduces to an equilibrium condition ( $\mathcal{J}=0$ ) if

$$\psi = 0. \quad (24)$$

(In the next section we shall consider two explicit examples to show that  $\psi=0$ ). This suggests a formal relation between  $F_0$  and  $D_i$  as

$$F_0 = \sum_i D_i \frac{\partial \ln P_s}{\partial X_i}, \quad (25)$$

where  $P_s$  may now be referred to as the *equilibrium* density function in phase space.  $F_0$  contains dissipation constant  $\gamma$ . Depending on the problem it also depends on the correlation time  $\tau_i$  of the color noise or on the cross correlation  $\lambda_{ii}$  between the noise processes.

To consider the information entropy balance equation in presence of external forcing we first differentiate Eq. (14) with respect to time and use Eq. (21). Following Ref. [12] one can show that in the new steady state (in presence of  $h \neq 0$ ), the entropy production ( $\dot{S}_h$ ), and the flux ( $\Delta S_{flux}$ ) like terms balance each other as follows:

$$\dot{S}_h = -\Delta S_{flux} \quad (26)$$

with

$$\dot{S}_h = \sum_{i,j} \mathcal{D}_{ij} \int dX P \left( \frac{\partial}{\partial X_i} \ln \frac{P}{P_s} \right)^2 \quad (27)$$

and

$$\Delta S_{flux} = h^2 \int dX \delta P \nabla_X \cdot F_1 + h^2 \int dX \left( \sum_i F_{1i} \frac{\partial \ln P_s}{\partial X_i} \right) \delta P, \quad (28)$$

where we have put  $h \delta P = P - P_s$ .

In the following section we shall work out the specific cases to provide explicit expressions for the entropy production and some related quantities due to external forcing for different kinds of open systems mentioned in the last section.

### III. APPLICATIONS

#### A. Entropy production in a system driven by an external color noise

To illustrate the theory we now consider a damped harmonic oscillator driven by an external, Gaussian Ornstein-Uhlenbeck noise,  $\eta_1$ . The noise correlation of  $\eta_1$  is given by Eq. (30).

$$\dot{q}_1 = p_1,$$

$$\dot{p}_1 = -\omega_0^2 q_1 - \gamma p_1 + \eta_1, \quad (29)$$

$$\langle \eta_1(t) \eta_1(t') \rangle = \frac{D^0}{\tau} e^{-|t-t'|/\tau}, \quad (30)$$

where  $\omega_0$  is the frequency of the oscillator.

To make notation consistent with Eq. (5) we would like to let  $X_1$ ,  $X_2$ , and  $X_3$  correspond to  $q_1$ ,  $p_1$ , and  $\eta_1$  respectively.

The relevant equations of motion are therefore as follows

$$\dot{X}_1 = F_1 = X_2,$$

$$\dot{X}_2 = F_2 = -\omega_0^2 X_1 - \gamma X_2 + X_3,$$

$$\dot{X}_3 = F_3 = -\frac{X_3}{\tau} + \frac{\sqrt{D^0}}{\tau} \zeta_3, \quad (31)$$

where  $\zeta_3$  is a  $\delta$ -correlated noise

$$\langle \zeta_3(t) \zeta_3(t') \rangle = 2\delta(t-t').$$

Therefore, for the Langevin Eq. (31) the Fokker-Planck Eq. (7) becomes

$$\begin{aligned} \frac{\partial P}{\partial t} = & -X_2 \frac{\partial P}{\partial X_1} + \frac{\partial}{\partial X_2} (\omega_0^2 X_1 + \gamma X_2 - X_3) P \\ & + \frac{1}{\tau} \frac{\partial}{\partial X_3} (X_3 P) + \frac{D^0}{\tau^2} \frac{\partial^2 P}{\partial X_3^2}. \end{aligned} \quad (32)$$

We now use the following transformation

$$U = aX_1 + bX_2 + X_3, \quad (33)$$

where  $a$  and  $b$  are constants to be determined.

Then under steady state condition Eq. (32) reduces to the following form:

$$\frac{\partial}{\partial U} (\Gamma U) P_s + D_s \frac{\partial^2 P_s}{\partial U^2} = 0, \quad (34)$$

where

$$D_s = \frac{D^0}{\tau}, \quad (35)$$

and

$$\Gamma U = -aX_2 + b\omega^2 X_1 + b\gamma X_2 - bX_3 + \frac{X_3}{\tau}. \quad (36)$$

Here  $\Gamma$  is again a constant to be determined. Putting Eq. (33) in Eq. (36) and comparing the coefficients of  $X_1$ ,  $X_2$ , and  $X_3$  we find

$$\Gamma a = -\omega_0^2 b, \quad \Gamma b = -a + b\gamma,$$

and

$$\Gamma = -b + \frac{1}{\tau}. \quad (37)$$

The physically allowed solutions for  $a$ ,  $b$ , and  $\Gamma$  are as follows:

$$a = \frac{1}{2} \left( -\frac{\gamma}{2} + \frac{1}{\tau} - \frac{1}{2} \sqrt{\gamma^2 - 4\omega_0^2} \right) (+\gamma - \sqrt{\gamma^2 - 4\omega_0^2}),$$

$$b = -\frac{\gamma}{2} + \frac{1}{\tau} - \frac{1}{2} \sqrt{\gamma^2 - 4\omega_0^2},$$

and

$$\Gamma = \frac{\gamma}{2} + \frac{1}{2} \sqrt{\gamma^2 - 4\omega_0^2}. \quad (38)$$

$P_s$ , the stationary solution of Eq. (34) is then given by

$$P_s = N_s e^{-\Gamma U^2/2D_s}. \quad (39)$$

Here  $N_s$  is the normalization constant. By virtue of Eq. (39)  $\psi$  corresponding to Eq. (22) is therefore

$$\psi = \Gamma U - D_s \frac{\partial \ln P_s}{\partial U} = 0. \quad (40)$$

Since  $\psi P_s$  defines a current,  $P_s$  defines a zero current situation or an equilibrium condition. The equilibrium solution  $P_s$  from Eq. (39) can now be used to calculate the steady state entropy production as given by Eq. (17). We thus have

$$\dot{S}_0 = D_s \int_{-\infty}^{\infty} \frac{1}{P_s} \left( \frac{\partial P_s}{\partial U} \right)^2 dU. \quad (41)$$

Explicit evaluation shows

$$\dot{S}_0 = \Gamma, \quad (42)$$

where  $\Gamma$  is given by Eq. (38). Thus at equilibrium the entropy production is inversely proportional to relaxation time of the process.

We now introduce an additional weak forcing in the dynamics. This may be achieved by adding a constant external force field  $f_c$  in the dynamics. Eq. (31) then becomes

$$\dot{X}_1 = X_2, \quad (43)$$

$$\dot{X}_2 = -\omega_0^2 X_1 - \gamma X_2 + f_c + X_3,$$

$$\dot{X}_3 = -\frac{X_3}{\tau} + \frac{\sqrt{D^0}}{\tau} \zeta_3.$$

Then the nonequilibrium situation (due to additional forcing,  $F_{12}=f_c$ ) corresponding to Eq. (43) is governed by

$$\begin{aligned} \frac{\partial P}{\partial t} = & -X_2 \frac{\partial P}{\partial X_1} + \frac{\partial}{\partial X_2} (\omega_0^2 X_1 + \gamma X_2 - X_3) P - \frac{\partial}{\partial X_2} (f_c P) \\ & + \frac{1}{\tau} \frac{\partial}{\partial X_3} (X_3 P) + \frac{D^0}{\tau^2} \frac{\partial^2 P}{\partial X_3^2}. \end{aligned} \quad (44)$$

Using the transformation (33) again in Eq. (44) we have

$$\frac{\partial}{\partial U} (\Gamma U) P_s - \frac{\partial}{\partial U} F_u P + D_s \frac{\partial^2 P_s}{\partial U^2} = 0, \quad (45)$$

Here,  $\Gamma$  and other constants are given by Eq. (38). Here  $F_u$  is

$$F_u = b f_c. \quad (46)$$

The now stationary solution of Eq. (45) in presence of external forcing is now given by,

$$P'_s = N' \exp\left(-\frac{\Gamma}{2D_s} \left[U^2 - \frac{2F_u U}{\Gamma}\right]\right), \quad (47)$$

where  $N'$  is the normalization constant.

We are now in a position to calculate the steady state entropy flux ( $\Delta S_{flux}$ ) due to external forcing ( $h \neq 0$ ) from Eq. (28)

$$\Delta S_{flux} = \int dX \delta P \nabla \cdot F_1 + \int dX \left( \sum_i F_{1i} \frac{d \ln P_s}{dX_i} \right) \delta P, \quad (48)$$

putting  $h = 1$ .

The components of  $F_1$  in  $U$  space can be identified as

$$F_{11} = F_u \text{ and } \nabla_U \cdot F_1 = 0. \quad (49)$$

$\delta P = P'_s - P_s$  denotes the deviation from the initial equilibrium state due to external forcing. For normalized probability functions  $P'_s$  and  $P_s$  the first integral in Eq. (48) vanishes. Thus the entropy production at steady state due to weak forcing is given by

$$\dot{S}_h = -\Delta S_{flux} = \frac{\Gamma}{D_s} \int F_u U \delta P dU.$$

Making use of the definition of  $\delta P$  and integrating explicitly we obtain

$$\dot{S}_h = \frac{b^2 f_c^2}{D_s}. \quad (50)$$

Putting  $D_s$  from Eq. (35) and  $b$  from Eq. (38) we obtain

$$\dot{S}_h = \frac{[4 - 4\gamma\tau + \gamma^2\tau^2 + \tau^2(\gamma^2 - 4\omega_0^2) - 2\tau\sqrt{\gamma^2 - 4\omega_0^2}(2 - \gamma\tau)]f_c^2}{4D^0} \quad (51)$$

We now examine specifically the following two limits:

(i) In the Markovian limit  $\tau \rightarrow 0$  the above expression reduces to the following form:

$$\dot{S}_h = \frac{f_c^2}{D^0} \quad (52)$$

For the closed thermodynamic system  $D^0 = \gamma kT$  that reduces the above expression to the standard result for entropy production of irreversible thermodynamics for Brownian oscillator.

(ii) Next we consider an interesting limiting case  $\omega_0 \rightarrow 0$ , which implies that for a free Brownian particle we have from Eqs. (37)

$$a = 0, \quad b = \frac{1 - \gamma\tau}{\tau}, \text{ and } \Gamma = \gamma,$$

$$\dot{S}_h = \frac{(1 - \gamma\tau)^2 f_c^2}{D^0}. \quad (53)$$

The above expression depicts an interplay of the dissipation constant  $\gamma$  of the system and the correlation time  $\tau$  of the noise in determining the entropy production. Two different cases are noteworthy;

(a)  $\gamma\tau < 1$  or  $\tau < 1/\gamma$ :

When relaxation time of the system greater than correlation time of external noise the entropy production  $\dot{S}_h$  decreases with increase of  $\tau$  until  $\tau \approx 1/\gamma$ .

(b)  $\gamma\tau > 1$  or  $\tau > 1/\gamma$ :

The entropy production  $\dot{S}_h$  increases with increase of  $\tau$  until  $\tau > 1/\gamma$ . It is interesting to note that in the limit  $\gamma\tau = 1$  entropy production is zero. A plot of entropy production in the steady state vs correlation time therefore exhibits a minimum (see Fig. 1). It is thus apparent that in presence of the nonequilibrium constraint the properties of noise pro-

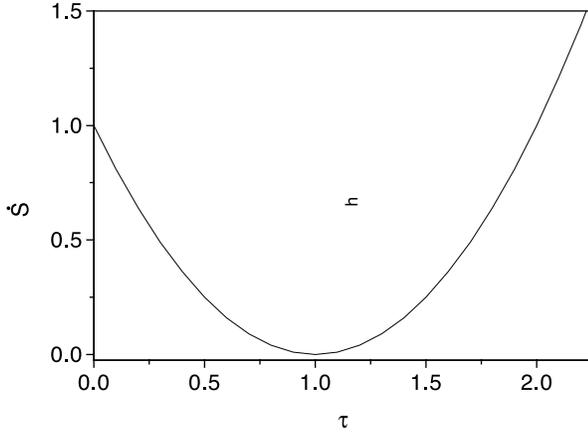


FIG. 1. Plot of entropy production  $\dot{S}_h$  vs correlation time  $\tau$  for the Eq. (53) using  $\gamma=1.0$ ,  $f_c=1.0$ , and  $D^0=1$  (units are arbitrary).

cesses as well as the dynamic characteristic of the system are important for entropy production.

### B. Entropy production in a cross-correlated noise driven system

We now turn to the second case where a simple dissipative system is driven by both additive and multiplicative noises.

$$\dot{X}_1 = -\gamma X_1 - \zeta_1 X_1 + \eta_1. \quad (54)$$

Here  $L_1$  in Eq. (8) corresponds to  $-\gamma X_1$ . The correlation between the noise processes are given by,

$$\langle \zeta_1(t) \zeta_1(t') \rangle = 2D'_{11} \delta(t-t'),$$

$$\langle \eta_1(t) \eta_1(t') \rangle = 2\alpha_{11} \delta(t-t'),$$

$$\begin{aligned} \langle \zeta_1(t) \eta_1(t') \rangle &= \langle \zeta_1(t') \eta_1(t) \rangle \\ &= 2\lambda_{11} \sqrt{D'_{11} \alpha_{11}} \delta(t-t'), 0 \leq \lambda_{11} \leq 1, \end{aligned} \quad (55)$$

where  $\lambda_{11}$  denotes the cross correlation between the two noise processes.

Equation (10) for this system reduces to

$$\frac{\partial P(X_1)}{\partial t} = -\frac{\partial(F_1 P)}{\partial X_1} + D_1 \frac{\partial^2 P}{\partial X_1^2}, \quad (56)$$

where the drift term is

$$F_1 = -(\gamma + 2D'_{11} - \nu)X_1 + (2 - \nu)\lambda_{11} \sqrt{D'_{11} \alpha_{11}} \quad (57)$$

and

$$D_1 = D'_{11} X_{1e}^2 - 2\lambda_{11} \sqrt{D'_{11} \alpha_{11}} X_{1e} + \alpha_{11}, \quad (58)$$

where

$$X_{1e} = \frac{(2 - \nu)\lambda_{11} \sqrt{D'_{11} \alpha_{11}}}{\gamma + 2D'_{11} - \nu}. \quad (59)$$

Making use of steady state value of  $X_1$ , i.e.,  $X_{1e}$  in Eq. (58) we obtain the following constant diffusion coefficient in the weak noise limit

$$D_1 = \frac{[\alpha_{11} \gamma^2 + (2 - \nu) D'_{11} \alpha_{11} \{(2 - \nu) D'_{11} + 2\gamma - 2\gamma \lambda_{11}^2 - \lambda_{11}^2 (2 - \nu) D'_{11}\}]}{\Gamma'^2}, \quad (60)$$

where

$$\Gamma' = \gamma + 2D'_{11} - \nu. \quad (61)$$

Now the stationary solution of Eq. (56) is given by

$$P_s = N_1 e^{-\Gamma'/2D_1 [X_1^2 - 2(2X_1/\Gamma')]}, \quad (62)$$

where  $N_1$  is the normalization constant.

$l$  is given by

$$l = (2 - \nu)\lambda_{11} \sqrt{D'_{11} \alpha_{11}}. \quad (63)$$

Putting Eq. (62) in Eq. (22) one may show as before that

$$\psi = 0. \quad (64)$$

Thus  $P_s$  is an equilibrium probability distribution function.

Using Eq. (62) in Eq. (17) we obtain the standard expression for entropy production at equilibrium

$$\dot{S}_0 = \Gamma'. \quad (65)$$

As before  $\Gamma'$  is a negative divergence of the drift term in Eq. (61). Equation (65) carries same message as in Eq. (42) but for a different system. It is apparent that the cross correlated diffusion coefficient  $D'_{11}$  between the noise processes is as important as the dissipation factor  $\gamma$  that determines the steady state entropy production.

To study the effect of additional weak forcing on the stationary system we again add a constant field of force  $f_e$  in the Eq. (54). Due to the additional forcing ( $F_{11} = f_e$ ) in Eq. (54) we have

$$\dot{X}_1 = \gamma X_1 - \zeta_1 + \eta_1 + f_e. \quad (66)$$

Then the nonequilibrium situation corresponding to Eq. (66) is given by

$$\frac{\partial P}{\partial t} = \frac{\partial \Gamma' X_1}{\partial X_1} - F'_c \frac{\partial P}{\partial X_1} + D_1 \frac{\partial^2 P}{\partial X_1^2}, \quad (67)$$

where

$$F'_c = f_e + l. \quad (68)$$

Using the stationary solution of Eq. (67) in Eq. (28) as in the preceding section we obtain the expression for entropy production in the steady state

$$\dot{S}_h = \frac{[\gamma^2 + (2 - \nu)^2 D_{11}'^2 + 2\gamma(2 - \nu)D_{11}'] f_e^2}{[\alpha_{11} \gamma^2 + (2 - \nu) D_{11}' \alpha_{11} \{(2 - \nu) D_1 + 2\gamma - 2\gamma \lambda_{11}^2 - \lambda_{11}^2 (2 - \nu) D_{11}'\}]}. \quad (69)$$

One may recover the standard results for a closed system by switching off the multiplicative noise ( $D_{11}' = 0$ ) and implementing fluctuation-dissipation relation  $\alpha_{11} = \gamma kT$  in Eq. (69). We then obtain

$$\dot{S}_h = \frac{f_e^2}{\alpha_{11}} = \frac{f_e^2}{\gamma kT}. \quad (70)$$

Equation (69) implies that for finite  $D_{11}'$  entropy production is an increasing function of the cross correlation (i.e.,  $\lambda_{11}$ ) between the two noise processes.

#### IV. CONCLUSIONS

In this paper we have examined the role of noise properties of stochastic processes in entropy production under a steady state condition. As specific cases we have considered Ornstein-Uhlenbeck noise with finite correlation time and cross-correlated noises driving the dynamical system. Based on an information entropy balance equation we have shown that the entropy production and fluxlike terms not only

depend on the dissipative characteristics of the dynamics of the phase space of the dynamical system, particularly, the rate of phase space volume contraction, but also on the correlation time and strength of cross correlation of the noises. Since the steady state entropy production is identified as a drift term in the Fokker-Planck description in the present formalism and the correlation time or the strength of cross-correlated noises make their presence felt in this term, it is not difficult to trace the origin of the role of interplay of dissipation and the properties of the noise processes. In view of the fact that the Ornstein-Uhlenbeck noise processes or the cross-correlated noise processes are commonly occurring situations in condensed matter physics and chemistry, we hope that the present analysis will be useful in irreversible thermodynamics in relation to dynamical systems, in general.

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