Multifractal random walk

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We introduce a class of multifractal processes, referred to as multifractal random walks (MRWs). To our knowledge, it is the first multifractal process with continuous dilation invariance properties and stationary increments. MRWs are very attractive alternative processes to classical cascadelike multifractal models since they do not involve any particular scale ratio. The MRWs are indexed by four parameters that are shown to control in a very direct way the multifractal spectrum and the correlation structure of the increments. We briefly explain how, in the same way, one can build stationary multifractal processes or positive random measures.

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Multifractal models have been used to account for scale invariance properties of various objects in very different domains ranging from the energy dissipation or the velocity field in turbulent flows [1] to financial data [2]. The scale invariance properties of a deterministic fractal function $f(t)$ are generally characterized by the exponents ζ_q, which govern the power-law scaling of the absolute moments of its fluctuations, i.e.,

$$m(q,l)=K_q l^{\zeta_q}, \tag{1}$$

where, for instance, one can choose $m(q,l)=\Sigma_t|f(t+l)-f(t)|^q$. When the exponents ζ_q are linear in q, a single scaling exponent H is involved. One has $\zeta_q=qH$ and $f(t)$ is said to be *monofractal*. If the function ζ_q is no longer linear in q, $f(t)$ is said to be *multifractal*. In the case of a stochastic process $X(t)$ with stationary increments, these definitions are naturally extended using

$$m(q,l)=E(|\delta_l X(t)|^q)=E(|X(t+l)-X(t)|^q), \tag{2}$$

where $E(\cdot)$ stands for the expectation. Some very popular monofractal stochastic processes are the so-called *self-similar processes* [3]. They are defined as processes $X(t)$ that have stationary increments and that verify (in law)

$$\delta_{\lambda l}X(t)=\lambda^H \delta_l X(t), \quad \forall l,\lambda>0. \tag{3}$$

Widely used examples of such processes are fractional Brownian motions (fBm) and Levy walks. One reason for their success is that, as it is generally the case in experimental time series, they do not involve any particular scale ratio [i.e., there is no constraint on l or λ in Eq. (3)]. In the same spirit, one can try to build multifractal processes that do not involve any particular scale ratio. A common approach originally proposed in the field of fully developed turbulence [1,4–7] has been to describe such processes in terms of stochastic equations, in the scale domain, describing the cascading process that determines how the fluctuations evolve when going from coarse to fine scales. One can state that the fluctuations at scales l and λl $(\lambda<1)$ are related (for fixed t) through the cascading rule

$$\delta_{\lambda l}X(t)=W_\lambda \delta_l X(t), \tag{4}$$

where $\ln(W_\lambda)$ is a random variable which law G_λ depends only on λ. Let us note that this latter equation can be seen as a generalization of Eq. (3) with H being stochastic. Since Eq. (4) can be iterated, it implicitly imposes the random variable W_λ to have a log infinitely divisible law [8]. It is then easy to show that the iterative rule satisfied by W_λ implies that the Fourier transform of G_λ can be written as $\hat{G}_\lambda(k)=\hat{G}^{\ln\lambda}(k)$. It follows that the q-order absolute moments at scale l scale like

$$m(q,l)=\hat{G}_{l/L}(-iq)m(q,L)=m(q,L)\left(\frac{l}{L}\right)^{F(-iq)}, \tag{5}$$

where $F=\ln\hat{G}$ refers to the cumulant generating function of $\ln W_\lambda$ [7,10]. Thus, identifying this latter equation with Eq. (1), one finds $\zeta(q)=F(-iq)$. In the case of self-similar processes of exponent H, $\ln(W_\lambda)$ is nonstochastic and has a Dirac function law $G_\lambda(u)=\delta(u-H\ln(\lambda))$ leading to $\zeta_q=qH$. The simplest nonlinear (i.e., multifractal) case is the so-called log-normal model that corresponds to a normal shape for G_λ and thus to a parabolic ζ_q spectrum.

However, the previous "top to bottom" cascade construction is, to a large extent, only formal. To the best of our knowledge, nobody has succeeded in building effectively such processes yet, mainly because of the strong constraints in the time-scale half-plane. Indeed, the variables $\delta_\lambda X(t)$ cannot be chosen freely because they must satisfy $\delta_\lambda X(t)=\delta_\gamma X(t)+\delta_{\lambda-\gamma}X(t+\gamma)$ for all $\gamma\leq\lambda$. Multiplicative cascading processes [11–14] that consist in writing Eq. (4) starting from some "coarse" scale L and then iterating it towards finer scales have been constructed exclusively using an arbitrary fixed scale ratio (e.g., $\lambda=\frac{1}{2}$). Such processes possess neither stationary increments nor continuous dilation invariance properties. Since they involve a particular arbitrary scale ratio, Eq. (1) holds only for the discrete scales $l_n = \lambda^n L$.

The goal of this paper is to build a multifractal process $X(t)$, referred to as a multifractal random walk (MRW), with stationary increments such that Eq. (1) holds for all $l \leqslant L$. We first build a discretized version $X_{\Delta t}(t=k\Delta t)$ of this process. Let us note that the theoretical issue of whether the limit process $X(t)=\lim_{\Delta t\to 0}X_{\Delta t}(t)$ is well defined will be addressed in a forthcoming paper. In this paper, we explain how it is built and prove that different quantities (q-order moments, increment correlation, etc.) converge when $\Delta t \to 0$.

Writing Eq. (4) for a log-normal cascade [1] at the smallest scale suggests that a good candidate might be such that $\delta_{\Delta t}X_{\Delta t}(k\Delta t)=\epsilon_{\Delta t}[k]W_{\Delta t}[k]$, where $\epsilon_{\Delta t}$ is a Gaussian variable and $W_{\Delta t}[k]=e^{\omega_{\Delta t}[k]}$ is a log-normal variable, i.e.,

$$X_{\Delta t}(t)=\sum_{k=1}^{t/\Delta t}\delta_{\Delta t}X_{\Delta t}(t)=\sum_{k=1}^{t/\Delta t}\epsilon_{\Delta t}[k]e^{\omega_{\Delta t}[k]}, \tag{6}$$

with $X_{\Delta t}(0)=0$ and $t=K\Delta t$. Moreover, we will choose $\epsilon_{\Delta t}$ and $\omega_{\Delta t}$ to be decorrelated and $\epsilon_{\Delta t}$ to be a white noise of variance $\sigma^2\Delta t$. Obviously, we need to correlate the $\omega_{\Delta t}[k]$'s otherwise $X_{\Delta t}$ would simply converge towards a Brownian motion. Since, in the case of cascadelike processes, it has been shown [14–16] that the covariance of the logarithm of the increments decreases logarithmically, it seems natural to choose

$$\mathrm{cov}(\omega_{\Delta t}[k_1],\omega_{\Delta t}[k_2])=\lambda^2\ln\rho_{\Delta t}[|k_1-k_2|], \tag{7}$$

with

$$\rho_{\Delta t}[k]=\begin{cases}\dfrac{L}{(|k|+1)\Delta t} & \text{for } |k|\leqslant L/\Delta t-1\\ 1 & \text{otherwise,}\end{cases} \tag{8}$$

i.e., the $\omega_{\Delta t}[k]$'s are correlated up to a distance of L and their variance $\lambda^2\ln(L/\Delta t)$ goes to $+\infty$ when Δt goes to 0 [18]. For the variance of $X_{\Delta t}$ to converge, a quick computation shows that we need to choose

$$E(\omega_{\Delta t}[k])=-r\,\mathrm{var}(\omega_{\Delta t}[k])=-r\lambda^2\ln(L/\Delta t), \tag{9}$$

with $r=1$ (this value will be changed later), for which we find $\mathrm{var}(X(t))=\sigma^2 t$.

Let us compute the moments of the increments of the MRW $X(t)$. Using the definition of $X_{\Delta t}(t)$, one gets

$$E(X_{\Delta t}(t_1)\cdots X_{\Delta t}(t_m)) =\sum_{k_1=1}^{t_1/\Delta t}\cdots\sum_{k_m=1}^{t_m/\Delta t}E(\epsilon_{\Delta t}[k_1]\cdots\epsilon_{\Delta t}[k_m]) \times E(e^{\omega_{\Delta t}[k_1]+\cdots+\omega_{\Delta t}[k_m]}).$$

Since $\epsilon_{\Delta t}$ is a zero mean Gaussian process, this expression is 0 if m is odd. Let $m=2p$. Since the $\epsilon_{\Delta t}[k]$'s are δ-correlated Gaussian variables, one shows, using the Wick theorem, that the preceding expression reduces to

$$\frac{\sigma^{2p}}{2^p p!}\sum_{\mathcal{S}\in S_{2p}}\sum_{k_1=1}^{(t_{\mathcal{S}(1)}\wedge t_{\mathcal{S}(2)})/\Delta t}\cdots\sum_{k_p=1}^{(t_{\mathcal{S}(2p-1)}\wedge t_{\mathcal{S}(2p)})/\Delta t} \times E(e^{2\Sigma_{j=1}^{p}\omega_{\Delta t}[k_j]})\Delta t^p,$$

where $a\wedge b$ refers to the minimum of a and b and S_{2p} to the set of the permutations on $\{1,\ldots,2p\}$. On the other hand, we have $E(e^{2\Sigma_{j=1}^{p}\omega_{\Delta t}[k_j]})=\Pi_{i<j}\rho[k_i-k_j]^{4\lambda^2}$. Then, when $\Delta t\to 0$, the general expression of the moments is

$$E(X(t_1)\cdots X(t_{2p})) =\frac{\sigma^{2p}}{2^p p!}\sum_{\mathcal{S}\in S_{2p}}\int_0^{t_{\mathcal{S}(1)}\wedge t_{\mathcal{S}(2)}}du_1\cdots\int_0^{t_{\mathcal{S}(2p-1)}\wedge t_{\mathcal{S}(2p)}}du_p \times\prod_{i<j}\rho(u_i-u_j)^{4\lambda^2}, \tag{10}$$

where $\rho(t)=\lim_{\Delta t\to 0}\rho_{\Delta t}[t/\Delta t]$. In the special case $t_1=t_2=\cdots=t_{2p}=l$, a simple scaling argument leads to the continuous dilation invariance property

$$m(2p,l)=K_{2p}\left(\frac{l}{L}\right)^{p-2p(p-1)\lambda^2},\quad \forall l\leqslant L, \tag{11}$$

where we have denoted the prefactor

$$K_{2p}=L^p\sigma^{2p}(2p-1)!!\int_0^1 du_1\cdots\int_0^1 du_p\prod_{i<j}|u_i-u_j|^{-4\lambda^2}.$$

By analytical continuation, we thus obtain the following ζ_q spectrum:

$$\zeta_q=[q-q(q-2)\lambda^2]/2. \tag{12}$$

We have illustrated this scaling behavior in Fig. 1. Thus, the MRW $X(t)$ is a multifractal process with stationary increments and with continuous dilation invariance properties up to the scale L. Let us note that above this scale ($l\geqslant L$), one gets from Eq. (10) that $\zeta_q=q/2$, i.e., the process scales like a simple Brownian motion, as if the $\omega_{\Delta t}[k]$'s were not correlated, though, of course, $X(t)$ is not Gaussian. Indeed, K_{2p} is merely the moment of order $2p$ of the random variable $X(L)$ that is infinite for $p>1+(1/2\lambda^2)$. Consequently, the pdf of $X(L)$ has fat tails. In Fig. 2, we illustrate that the cascade picture of Eq. (4) accounts very well for the evolution of the probability density function of the increments at different scales. One shows that the smaller the scale l, the fatter the tails of the pdf of $\delta_l X(t)$.

Let us study the correlation structure of the increments of $X(t)$. Since $\zeta_2=1$, one can prove that they are decorrelated (though not independent). Let

$$C_{2p}(l,\tau)=\langle|\delta_\tau X(l)|^{2p}|\delta_\tau X(0)|^{2p}\rangle, \tag{13}$$

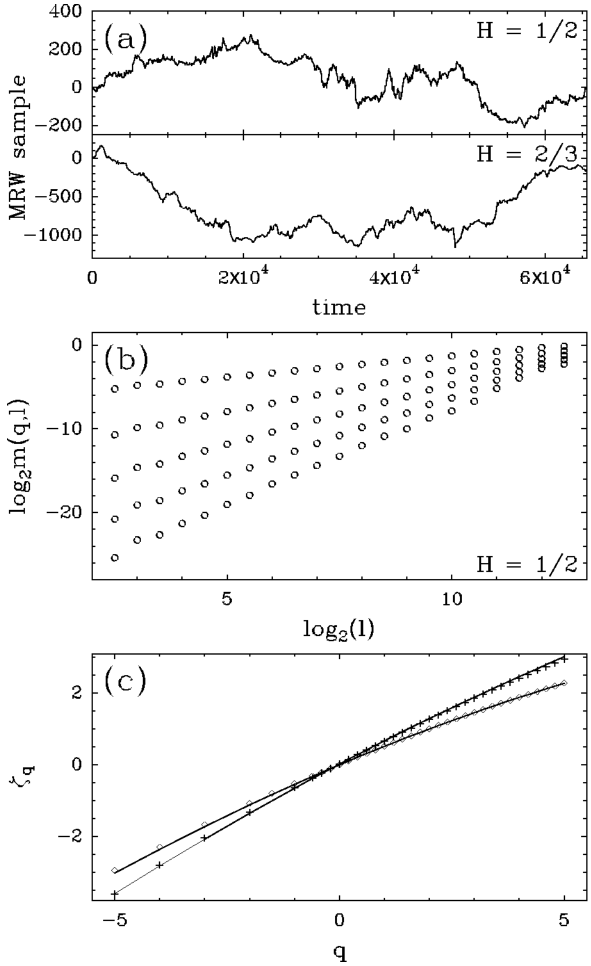

FIG. 1. (a) Plot of two realizations of 2^{17} samples of two MRWs with $\lambda^2=0.03$, $L=2048$, and where $\epsilon_{\Delta t}$ is (top plot) a white noise or (bottom plot) a fGn [Eq. (16)] with $H=\frac{2}{3}$. (b) Log-log plots of $m(q,l)$ of the MRW plotted in (a) (top plot) versus l for $q=1,2,3,4,5$. (c) (○) [(+)]: ζ_q spectrum estimation of the MRW plotted at the top (bottom) in (a). These estimations (obtained using the WTMM method [17]) are in perfect agreement with the theoretical predictions (——) given by Eq. (12) [Eq. (17)].

with $\tau<l$. Using the same kind of arguments as above, one can show that

$$C_{2p}(l,\tau)=[\sigma^{2p}(2p-1)!!]^2 \times \int_l^{l+\tau} du_1 \cdots \int_l^{l+\tau} du_p \int_0^{\tau} du_{p+1} \cdots \int_0^{\tau} du_{2p} \times \prod_{1\leqslant i<j\leqslant 2p} \rho(u_i-u_j)^{4\lambda^2}. \tag{14}$$

One shows that

$$K_{2p}^2 \frac{(\tau/L)^{2\zeta_{2p}}}{[(l+\tau)/L]^{4\lambda^2 p^2}} \leqslant C_{2p}(l,\tau) \leqslant K_{2p}^2 \frac{(\tau/L)^{2\zeta_{2p}}}{[(l-\tau)/L]^{4\lambda^2 p^2}},$$

and consequently for $\tau \ll l$ fixed, using analytical continuation one expects $C_q(l,\tau)$ to scale like

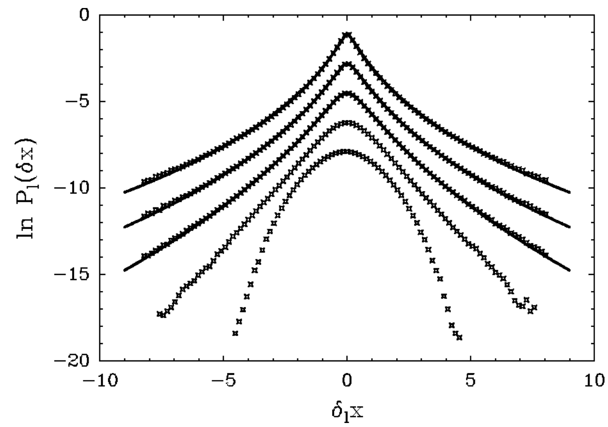

FIG. 2. (x) Standardized estimated pdf's of $\ln \delta_l X(t)$ for $l=4$, 32, 256, 2048, and 4096 (from top to bottom). These estimations have been made on 500 realizations of 2^{17} samples each of a MRW with $\lambda^2=0.06$ and $L=2048$. Plots have been arbitrarily shifted for illustration purposes. (——) Theoretical prediction from the estimated pdf at the largest scale ($l=2048$) using the cascade equation (4).

$$C_q(l,\tau) \sim K_q^2 \left(\frac{\tau}{L}\right)^{2\zeta_q} \left(\frac{l}{L}\right)^{-\lambda^2 q^2}. \tag{15}$$

This behavior is illustrated in Fig. 3.

From the behavior of $C_q(l,\tau)$ when $q\to 0$, we can obtain using Eq. (15) that the covariance of the logarithm of the increments at scale τ and lag l behaves (for $\tau \ll l$) like $-\lambda^2 \ln(l/L)$. Thus, this correlation reflects the correlation of the $\omega_{\Delta t}$ process and is the same as observed in Refs. [14–16] for the cascade models. This behavior is checked in Fig. 4.

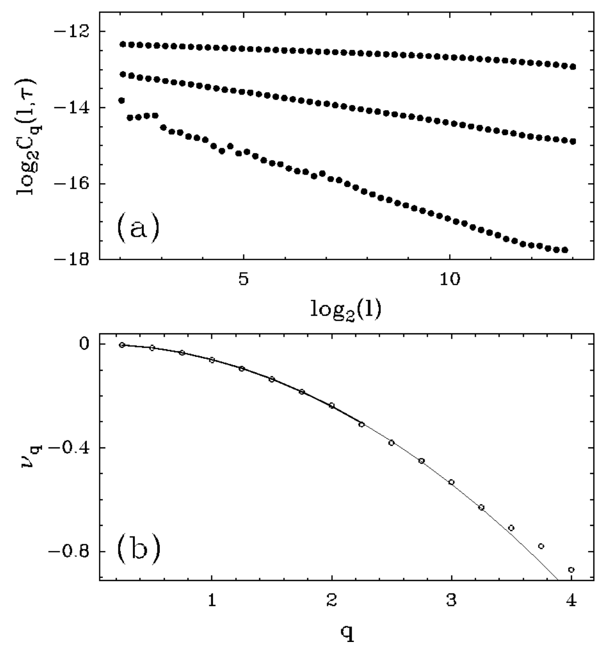

FIG. 3. (a) Log-log plots of $C_q(l,\tau)$ versus l for $q=1,2,3$. (b) Estimation (○) of the power-law exponent $C_q(l,\tau) \sim l^{\nu_q}$. It is in perfect agreement with the theoretical prediction [Eq. (15)] $\nu_q=-\lambda^2 q^2$ (——).

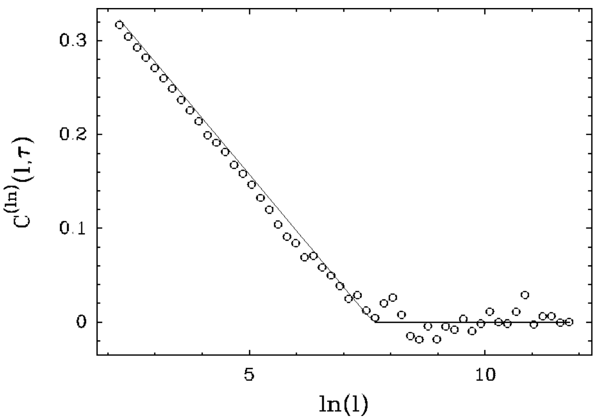

FIG. 4. Estimation (○) of the covariance $C^{(\ln)}(l,\tau)$ of the logarithm of the increments of an MRW. It is in perfect agreement with the analytical expression $-\lambda^2\ln(l/L)$ (——).

Finally, let us note that one can build MRWs with correlated increments by just replacing the white noise $\epsilon_{\Delta t}$ by a fractional Gaussian noise (fGn),

$$\epsilon_{\Delta t}^{(H)}[k]=B_H[(k+1)\Delta t]-B_H(k\Delta t), \tag{16}$$

where $B_H(t)$ is a fBm with the scaling exponent H and of variance $\sigma^2 t^{2H}$, and choosing $r=\frac{1}{2}$ in Eq. (9). Then, one can show that the spectrum of the MRW $X^{(H)}(t)$ is

$$\zeta_q^{(H)}=qH-q(q-1)\lambda^2/2 \tag{17}$$

($\zeta_q^{(H)}=qH$ at scales $\gg L$), and consequently the MRW has correlated increments. Such a construction is illustrated in Fig. 1 with $H=\frac{2}{3}$. Since $H>\frac{1}{2}$, it leads to a process that is more regular than the one previously built.

To summarize, we have built the MRWs, a class of multifractal processes, with stationary increments and continuous dilation invariance. Such processes have been shown to satisfy, in a weak sense, the cascade equation (4). We do believe that they should be very helpful in all the fields where multiscaling is observed. Their construction, using an aggregation of random variables, makes them very interesting for the modeling of such dynamical processes as turbulence or financial time series [19]. From a theoretical point of view, MRW can be seen as the simplest model that contains the main ingredients for multifractality, i.e., the logarithms of amplitude fluctuations are merely but a $1/f$ noise. Moreover, they involve four parameters, the correlation length L, the intermittency parameter λ^2, the variance σ^2, and the roughness exponent H. They all can be easily estimated using the ζ_q spectrum and the increment correlations. The construction of MRWs can be used as a general framework in which one can easily build other classes of processes in order to match some specific experimental situations. For instance, a stationary MRW can be obtained by just adding a friction $\gamma>0$, i.e., $X_{\Delta_t}[k]=(1-\gamma)X_{\Delta t}[k-1]+\epsilon_{\Delta t}[k]e^{\omega_{\Delta t}[k]}$. One can build a strictly increasing MRW (and consequently a stochastic positive multifractal measure) by just setting $\epsilon_{\Delta t}=\Delta t$ in Eq. (6) and use it as a multifractal time for subordinating a monofractal process (such as an fBm). One can also use laws other than the (log-)normal for ϵ and/or ω. Another interesting point concerns the problem of the existence of a limit ($\Delta t\to 0$) stochastic process on the development of a new stochastic calculus associated with this process. All these prospects will be addressed in forthcoming studies.

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