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Continuum description of avalanches in granular media

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We develop a continuum description of partially fluidized granular flows. Our theory is based on the hydrodynamic equation for the flow coupled with the order parameter equation, which describes the transition between flowing and static components of the granular system. This theory captures important phenomenology recently observed in experiments with granular flows on rough inclined planes [A. Daerr and S. Douady, Nature (London) **399**, 241 (1999)]: layer bistability, and transition from triangular avalanches propagating downhill at small inclination angles to balloon-shaped avalanches also propagating uphill for larger angles.

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Fundamental understanding of the dynamics of granular media still poses a challenge for physicists [1–3] and engineers [4]. The intrinsic dissipative nature of the interactions between the constituent macroscopic particles sets granular matter apart from conventional gases, liquids, or solids. One of the most interesting phenomena pertinent to the granular systems is the transition from a static equilibrium to a granular flow. The most spectacular manifestation of such a transition occurs during an avalanche. There has been a number of experimental studies of avalanche flows in large sandpiles [5,6], as well as in thin layers of grains on rough inclined surfaces [7–9].

On the theoretical side, a significant progress had been achieved by large-scale molecular dynamics simulations [10,11] and by continuum theory [12–15]. The current continuum approach to the description of avalanche flows in the physics community, was pioneered by Bouchaud, Cates, Ravi Prakash, and Edwards (BCRE) [13], and subsequently developed by de Gennes, Boutreux, and Raphaël [12,14,15]. In their model the granular system is spatially separated into two phases, static and rolling. The interaction between the phases is implemented through certain conversion rates. This model described certain features of thin near-surface granular flows, including avalanches. However, due to its intrinsic assumptions, it only works when the granular material is well separated in a thin surface flow and an immobile bulk. In many practically important situations, this distinction between ‘‘liquid’’ and ‘‘solid’’ phases is more subtle and itself is controlled by the dynamics.

In this Rapid Communication we propose a new continuum model for multiphase granular matter. The underlying idea of our approach is borrowed from the Landau theory of phase transitions [16]. We assume that the shear stresses in a partially fluidized granular matter are composed of two parts: the dynamic part proportional to the shear strain, and the strain-independent (or ‘‘static’’) part. The relative magnitude of the static shear stress is controlled by the order parameter (OP), which varies from 0 in the ‘‘liquid’’ phase to 1 in the ‘‘solid’’ phase. Unlike ordinary matter, the phase transition in granular matter is controlled not by the temperature, but the dynamics stresses themselves. In particular, the Mohr-Coloumb yield failure condition [4] is equivalent to a critical melting temperature of a solid. The OP can be related to the local entropy [17] of the granular material. OP dynamics is then coupled to the hydrodynamic equation for the granular flow. We apply this model to study the transition to flow in thin granular layer on inclined planes with rough bottom. Our model captures important phenomenology observed by Pouliquen [9] and Daerr and Douady [7], including the structure of the stability diagram, the triangular shape of downhill avalanches at small inclination angles, and the balloon shape of uphill avalanches for larger angles.

Model. The continuum description of the granular flow is based on the Navier-Stokes equation

$$\rho_0 Dv_i/Dt = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho_0 g_i, \quad j=1,2,3, \tag{1}$$

where v_i are the components of velocity, $\rho_0=\text{const}$ is the density of material (we set $\rho_0=1$), $\mathbf{g}$ is the acceleration of gravity, and $D/Dt=\partial_t+v_i\partial_{x_i}$ denotes the material derivative. Since the relative density fluctuations are small, the velocity obeys the incompressibility condition $\boldsymbol{\nabla}\times\mathbf{v}=0$.

The central conjecture of our theory is that in partially fluidized flows, some of the grains are involved in plastic motion, while others maintain prolonged static contacts with their neighbors. Accordingly, we write the stress tensor σ_{ij} as a sum of the hydrodynamic part proportional to the flow strain rate e_{ij}, and the strain-independent part, σ_{ij}^s, i.e., $\sigma_{ij}=e_{ij}+\sigma_{ij}^s$. We assume that the diagonal elements of the tensor σ_{ii}^s coincide with the corresponding components of the ‘‘true’’ static stress tensor σ_{ii}^0 for the immobile grain configuration in the same geometry, and the shear stresses are reduced by the value of the order parameter ρ, characterizing the ‘‘phase state’’ of granular matter. Thus, we write the stress tensor in the form

$$\sigma_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)+\sigma_{ij}^0[\rho+(1-\rho)\delta_{ij}]. \tag{2}$$

Here, η is the viscosity coefficient. In a static state, $\rho=1$, $\sigma_{ij}=\sigma_{ij}^0$, and $v_i=0$, whereas in a fully fluidized state, $\rho=0$ and the shear stresses are simply proportional to the strain rates as in ordinary fluids.

To close the system we need a set of constitutive relations between static shear and normal stresses, as well as an equation for the order parameter ρ. The issue of constitutive relations in granular materials is complex and not completely understood [4,18]. It appears that in many cases, the constitutive relations are determined by the construction history [19]. Recent studies indicated a fundamental role of the network of the force chains, which carry forces longitudinally [20]. We will assume that for any given problem, the corresponding static constitutive relations have been specified.

For the order parameter ρ, we apply pure dissipative dynamics, which can be derived from the ''free-energy'' type functional $\mathcal{F}$, i.e., $\tau\dot{\rho}=-\delta\mathcal{F}/\delta\rho$. We adopt the standard Landau form for $\mathcal{F}\sim\int d\mathbf{r}[l^2|\nabla\rho|^2+f(\rho,\phi)]$, which includes a ''local potential energy'' and the diffusive spatial coupling. Here, l,τ are characteristic length and time respectively. From dimension arguments, one expects that l is of the order of average grain size and τ of the order of collision time, $\tau\sim\sqrt{l/g}$. The potential energy $f(\rho,\phi)$ should have extrema at $\rho=0$ and $\rho=1$, corresponding to uniform solid and liquid phases. According to the Mohr-Coulomb yield criterion for noncohesive grains [4] or its generalization [20], the static equilibrium failure and transition to flow is controlled by the value of the nondimensional ratio $\phi=\max|\sigma^0_{mn}/\sigma^0_{nn}|$, where the maximum is sought over all possible orthogonal directions n and m, in the bulk of the granular material. We simply use this ratio as a parameter in the potential energy for the OP ρ. Without loss of generality, we write the equation for ρ,

$$\tau\dot{\rho}=l^2\nabla^2\rho-\rho(1-\rho)F(\rho,\phi). \tag{3}$$

Further, according to observations, we assume that the static equilibrium is unstable if $\phi\leqslant\phi_1$, where $\varphi_1=\tan^{-1}\phi_1$ is the internal friction angle for a particular granular material. Additionally, we assume that if $\phi<\phi_0$, the ''dynamic'' phase $\rho=0$ is unstable. Values of ϕ_0 and ϕ_1 do not coincide in general. Typically there is a range in which both static and dynamics phases coexist (this is related to the so-called Bagnold hysteresis [5]). The simplest form of $F(\rho,\phi)$, which satisfies these constraints, is $F(\rho,\phi)=-\rho+\delta$, where $\delta=(\phi-\phi_0)/(\phi_1-\phi_0)$. Rescaling $x\to x/l, t\to t/\tau$, we arrive at

$$\dot{\rho}=\nabla^2\rho+\rho(1-\rho)(\rho-\delta). \tag{4}$$

For $\phi_0<\phi<\phi_1$, both static ($\rho=1$) and dynamic ($\rho=0$) phases are linearly stable, and Eq. (4) possesses a moving front solution that ''connects'' these phases. The speed of the front, in the direction of $\rho=0$, is given by $V=(1-2\delta)/\sqrt{2}$. At $\delta=1/2$ both phases coexist.

Chute flow. Let us now apply this formulation to a specific problem of the chute flow. We consider a layer of dry cohesionless grains on an inclined rough surface (see Fig. 1). In the static equilibrium, one has,

$$\begin{aligned}\sigma^0_{zz,z}+\sigma^0_{xz,x}&=-g\cos\varphi,\\ \sigma^0_{xz,z}+\sigma^0_{xx,x}&=g\sin\varphi,\end{aligned} \tag{5}$$

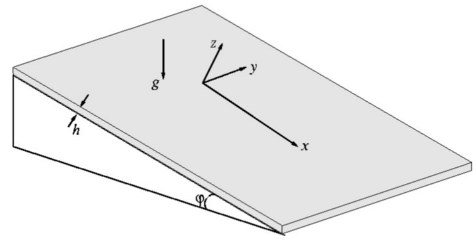

FIG. 1. Schematic representation of a chute geometry.

where the subscripts after commas mean partial derivatives. The solution to Eqs. (5), in the absence of lateral stresses $\sigma^0_{yy}=\sigma^0_{yx}=\sigma^0_{yz}=0$, is given by

$$\begin{aligned}\sigma^0_{zz}&=-g\cos\varphi z,\\ \sigma^0_{xz}&=g\sin\varphi z,\quad \sigma^0_{xx,x}=0.\end{aligned} \tag{6}$$

In a static equilibrium there is a simple relation between shear and normal stresses, $\sigma^0_{xz}=-\tan\varphi\sigma^0_{zz}$. According to our conjecture, this relation between the static components of the stress is maintained in the flowing regime as well. For the chute flow geometry, the value of parameter ϕ in Eq. (4) can also be easily specified. In this case, the most ''unstable'' yield direction is parallel to the inclined plane, so we can simply write $\phi=|\sigma^0_{xz}/\sigma^0_{zz}|$.

Stationary solutions of Eq. (4), for the confined chute geometry in Fig. 1, are subject to the following boundary conditions (BC): no-flux condition $\rho_z=0$ at the free surface $z=0$, and $\rho=1$ at the bottom of the chute, $z=-h$ (a granular medium is assumed to be in a solid phase near the rough surface). There always exists a stationary solution to Eq. (4) $\rho=1$, corresponding to a static equilibrium. For $\delta>1$, it is stable at small h, but loses stability at a certain threshold $h_s>1$. The most ''dangerous'' mode of instability satisfying the above boundary conditions, is $a\cos(\pi z/2h)$. The eigenvalue of this mode is $\lambda(h)=\delta-1-\pi^2/4h^2$, thus the neutral curve $\lambda=0$, for the linear stability of the solution $\rho=1$, is given by

$$h_s=\frac{\pi}{2\sqrt{\delta-1}}. \tag{7}$$

For $h>h_s(\delta)$, grains spontaneously start to roll, and a granular flow ensues. In addition to the trivial state $\rho=1$, for $h>h_c(\delta)$, there exists a unique nontrivial stationary solution satisfying the above BC. The value of h_c can be found as a minimum of the following integral, as a function of ρ_0, the value of ρ at the surface $z=0$:

$$h_c=\min\int_{\rho_0}^{1}\frac{d\rho}{\sqrt{\dfrac{\rho^4}{2}-\dfrac{2(\delta+1)\rho^3}{3}+\delta\rho^2-c(\rho_0)}}, \tag{8}$$

where $c(\rho_0)=\rho_0^4/2-2(\delta+1)\rho_0^3/3+\delta\rho_0^2$. This integral can be calculated analytically for $\delta\to\infty$ and $\delta\to1/2$. It is easy to show that for large δ, the critical solution of Eq. (4) has a

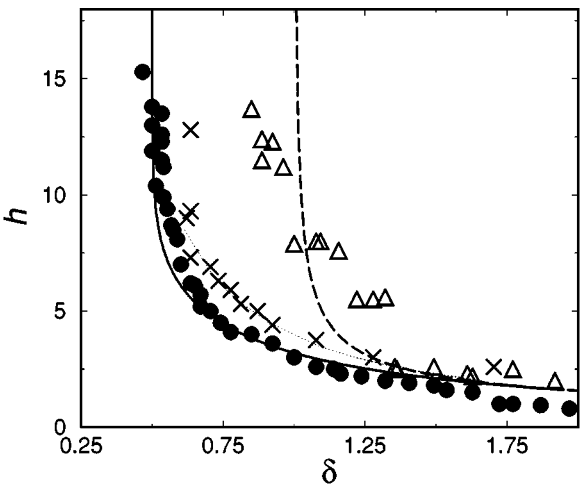

FIG. 2. Stability diagram. Dashed line shows the neutral curve [Eq. (7)], solid line shows the existence limit of fluidized state [Eq. (8)], dotted line shows the transition from triangular to up-hill avalanches for $\beta=3.15$ and $\alpha=0.025$. Symbols show experimental data from Ref. [7].

form $\rho=1+a\cos(kz)$, with $a\ll1$ and $k=(\delta-1)^{1/2}$, and therefore, $h_c(\delta)\to h_s(\delta)$. For $\delta\to1/2$, the critical phase trajectory comes close to two saddle points, $\rho=0$ and $\rho=1$, and an asymptotic evaluation of Eq. (8) gives $h_c=-\sqrt{2}\log(\delta-1/2)+\text{const}$. This expression agrees with the empirical formula $\phi-\phi_0\sim\exp[-h_c/h_0]$ proposed in Ref. [7].

The neutral stability curve, $h_s(\delta)$, and the critical line, $h_c(\delta)$, limiting the region of existence of nontrivial granular flow solutions, are shown in Fig. 2. They divide the parameter plane (δ,h) into three regions. At $h<h_c(\delta)$, the trivial static equilibrium $\rho=1$ is the only stationary solution of Eq. (4) for chosen BC. For $h_c(\delta)<h<h_s(\delta)$, there is a bistable regime, the static equilibrium state coexists with the stationary flow. For $h>h_s(\delta)$, the static regime is linearly unstable, and the only stable regime corresponds to the granular flow. This qualitative picture completely agrees with the recent experimental findings [7,9]. Moreover, if we rescale the experimental phase diagram obtained by Daerr and Douady [7] using the limiting values $\phi_{0,1}$ [22], and choose the characteristic length scale l to be equal to the particle size, we obtain excellent agreement with our theoretical phase diagram (see Fig. 2).

The velocity profile corresponding to a stationary profile of $\rho(z)$, can be easily found from Eq. (2),

$$\eta\frac{\partial v_x}{\partial z}=g\sin\varphi z-\rho\sigma_{xz}^0=g\sin\varphi(1-\rho)z. \tag{9}$$

The flux of grains in the stationary flow J is given by

$$J=\int_{-h}^{0}v_y(z)dz=\frac{g\sin\varphi}{\eta}\int_{-h}^{0}\int_{-h}^{z}[1-\rho(z')]z'dz'dz. \tag{10}$$

For a deep chute $(h\gg1)$, the stationary solution of Eq. (4) can be found analytically (cf. Ref. [21]). However, in this case the slope of the free surface may not be equal to the slope of the inclined plane, but is itself determined by the amount of sand which is poured on the surface up stream. Thus, the closure of the problem will be provided by the constraint $J=\text{const}$.

Avalanches in shallow chute. In the vicinity of the neutral curve [Eq. (7)] Eqs. (1) and (3) can be simplified. We look for the solution in the form

$$\rho=1-A\cos\left(\frac{\pi}{2h}z\right)+(\text{higher-order terms}), \tag{11}$$

where $A\ll1$ is a slowly varying function of t, x, and y. Substituting ansatz [Eq. (11)] into Eq. (3) and applying orthogonality conditions, we obtain

$$A_t=\lambda(h)A+\nabla_\perp^2A+\frac{8(2-\delta)}{3\pi}A^2-\frac{3}{4}A^3, \tag{12}$$

where $\nabla_\perp^2=\partial_x^2+\partial_y^2$, $\lambda(h)=\delta-1-\pi^2/4h^2$. Deriving this equation, we assumed that $(2-\delta)A^2$ and A^3 are of the same order, i.e., $\delta\approx2$, however, a qualitatively similar equation, with a different nonlinearity, can be obtained for any δ and h. Equation (12) must be coupled to the mass conservation equations, which reads as (here we neglect contribution from the flux along the y axis $J_y\sim\partial_y h\ll J$),

$$\frac{\partial h}{\partial t}=-\frac{\partial J}{\partial x}=-\alpha\frac{\partial h^3A}{\partial x}, \tag{13}$$

where J was calculated from Eq. (10) and $\alpha=2(\pi^2-8)g\sin\varphi/\eta\pi^3$. Taking into account that variations in h also change local surface slope, we adopt $\delta=\delta_0-\beta h_x$ with $\beta=1/(\phi_1-\phi_0)$.

We studied Eqs. (12) and (13) numerically. The simulations were performed in fairly large systems, 400 dimensionless units in the x direction (downhill), and 200 units in the y direction, with the number of grid points 1200×600, respectively. As initial conditions, we used uniform static layer: $h=h_0, A=0$. We triggered avalanches by a localized perturbation introduced near the point $(y,z)=(L_y/4,L_z/2)$. Close to the solid line in Fig. 2, we indeed observed avalanches propagating only downhill, with the shape very similar to the experimental one. The avalanche (see Fig. 3) leaves a triangular trace, with the opening angle ψ, in which the layer thickness h is decreased with respect to original value h_0. At the front of the avalanche, the layer depth is increased with respect to h.

For larger values of δ or h we observed avalanches of the second type (see Fig. 4). The avalanche propagates also uphill, and contrary to the previous case, the whole avalanche zone is in motion, as new rolling particles constantly arrive from the upper boundary of the avalanche zone. Sometimes, we observed small secondary avalanches in the wake of a large primary avalanche [see Fig. 4(c)].

The transition from triangular to up-hill avalanches occurs at the dotted line in Fig. 2. At large h, this line approaches the solid line $h_c(\delta)$, limiting the region of existence of granular flow. Thus, in deep layers, there are only up-hill

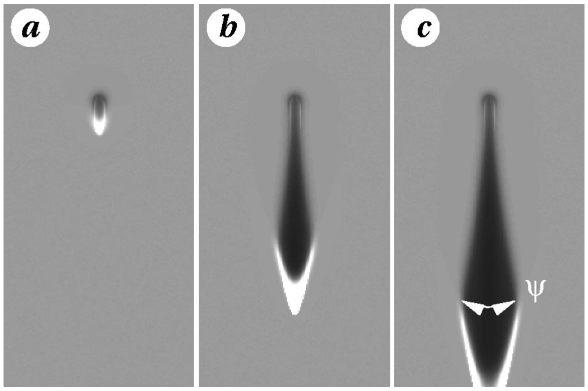

FIG. 3. Gray-coded images demonstrating evolution of triangular avalanche for $t=50$ (a), $t=200$ (b), and 250 (c). White shade corresponds to maximum height of the layer, and black to minimum height. Parameters of Eqs. (12) and (13) are, $\alpha=0.15$, $\beta=0.25$, $\delta=1.2$, and $h_0=3$.

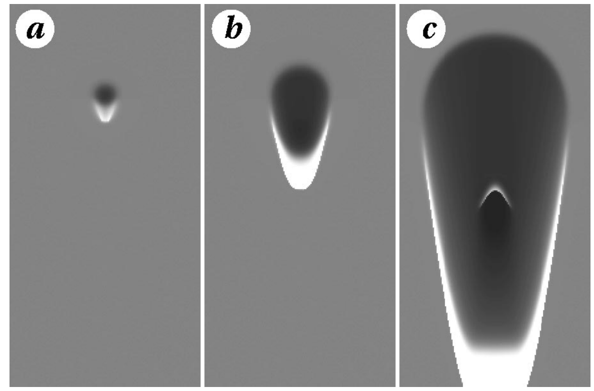

FIG. 4. Images of an up-hill avalanche for $t=40$ (a), $t=100$ (b), and 180 (c). Parameters of Eqs. (12) and (13) are, $\alpha=0.05$, $\beta=0.25$, $\delta=1.07$, and $h_0=5.5$. A small secondary avalanche is seen on the image (c).

avalanches. The transition line in Fig. 2 is plotted for $\alpha=0.025$. At this value of α, this line agrees well with experimental data by Daerr and Douady [7].

In conclusion, we developed a continuum description of partially fluidized granular flows. Our order-parameter model captures important aspects of the phenomenology of chute flows observed in recent experiments [7–9]. The parameters of our model are established from comparison with experiment.

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[1] H.M. Jaeger, S.R. Nagel, and R.P. Behringer, Phys. Today **49**, 32 (1996); Rev. Mod. Phys. **68**, 1259 (1996).
[2] L. Kadanoff, Rev. Mod. Phys. **71**, 435 (1999).
[3] P.G. de Gennes, Rev. Mod. Phys. **71**, S374 (1999).
[4] R.M. Nedderman, *Statics and Kinematics of Granular Materials* (Cambridge University, Cambridge, England, 1992).
[5] R.A. Bagnold, Proc. R. Soc. London, Ser. A **225**, 49 (1954); **295**, 219 (1966).
[6] J. Rajchenbach, in *Physics of Dry Granular Media*, edited by H. Hermann, J.-P. Hovi, and S. Luding (Kluwer, Dordrecht, 1998), p. 421; D. McClung, *Avalanche Handbook* (Mountaineers, Seattle, 1993).
[7] A. Daerr and S. Douady, Nature (London) **399**, 241 (1999).
[8] A. Daerr, Phys. Fluids **13**, 2115 (2001).
[9] O. Pouliquen, Phys. Fluids **11**, 542 (1999).
[10] D. Ertas *et al.*, e-print cond-mat/0005051.
[11] O.R. Walton, Mech. Mater. **16**, 239 (1993); T. Pöshel, J. Phys. II **3**, 27 (1993); X.M. Zheng and J.M. Hill, Powder Technol. **86**, 219 (1996); O. Pouliquen and N. Renaut, J. Phys. II **6**, 923 (1993).
[12] P.G. de Gennes, in *Powders & Grains*, edited by R. Behringer and J. Jenkins (Balkema, Rotterdam, 1997), p. 3.
[13] J.-P. Bouchaud *et al.*, J. Phys. I **4**, 1383 (1994).
[14] T. Boutreux, E. Raphaël, and P.G. de Gennes, Phys. Rev. E **58**, 4692 (1998).
[15] T. Boutreux and E. Raphaël, Phys. Rev. E **58**, 7645 (1998).
[16] L.D. Landau and E.M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1980).
[17] S.F. Edwards and R.B.S. Oakeshott, Physica A **157**, 1080 (1989); S.F. Edwards and D.V. Grinev, Chaos **9**, 551 (1999).
[18] E. Cantelaube and J.D. Goddard, *Powders & Grains*, edited by R. Behringer and J. Jenkins (Balkema, Rotterdam, 1997), p. 231; O. Narayan and S.R. Nagel, Physica A **264**, 75 (1999).
[19] J.P. Wittmer, M.E. Cates, and P.J. Claudine, J. Phys. II **7**, 39 (1997); L. Vanel *et al.*, Phys. Rev. Lett. **84**, 1439 (2000).
[20] M.E. Cates *et al.*, Phys. Rev. Lett. **81**, 1841 (1998).
[21] I.S. Aranson, V.A. Kalatsky, and V.M. Vinokur, Phys. Rev. Lett. **84**, 118 (2000).
[22] Asymptotic values of angles $\varphi_1\approx 32°$ and $\varphi_c\approx 25°$, limiting the bistability region in experiment, Ref. [7], at large h, correspond to our $\delta=1$ and $\delta=1/2$, respectively. Based on this, we obtain $\beta=1/(\phi_1-\phi_0)=3.15$.