Short wavelength spectrum and Hamiltonian stability of vortex rings

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We compare dynamical and energetical stability criteria for vortex rings. It is argued that vortex rings will be intrinsically unstable against perturbations with short wavelengths below a critical wavelength because the canonical vortex Hamiltonian is unbounded from below for these modes. To explicitly demonstrate this behavior, we derive the oscillation spectrum of vortex rings in incompressible, inviscid fluids within a geometrical cutoff procedure for the core. The spectrum develops an anomalous branch of negative group velocity and approaches the zero of energy for wavelengths that are about six times the core diameter. We show the consequences of this dispersion relation for the thermodynamics of vortex rings in superfluid ^{4}He at low temperatures.

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I. INTRODUCTION

The long wavelength oscillation spectrum of large vortex rings in incompressible, inviscid fluids is established since the pioneering work of Thomson (Lord Kelvin) [1], Thomson [2], and Pocklington [3]. The validity of that spectrum is restricted to wave numbers much less than the inverse core size and rings that are large compared to the extension of the core. There are, however, processes for which it is desirable to know the large wave number properties of the spectrum for smaller rings: Vortex ring nucleation, reconnection of vortex filaments, and dissipation in the turbulent energy cascade are believed to occur on very small length scales, reaching down to a few times the vortex core size. Because a line vortex represents a string object, elasticity modes will be excited during the rapid movements executed by the string on small length scales. A fluctuating line should have equilibrium states different from a nonfluctuating one because the quantum or classical statistical fluctuations renormalize the total free energy as compared to the undeformed ring.

In what follows, we shall derive the collective, small amplitude oscillation modes of a vortex ring in an incompressible, inviscid fluid. The dispersion relation is exact within the geometrical cutoff procedure we employ and displays a maximum and an anomalous branch of negative group velocity. The critical wavelength for the spectrum to possess a positive excitation energy corresponds to one oscillation of line within a length about an order of magnitude above a geometrically defined core size. It will be argued that, due to the properties of this spectrum and the structure of the canonical Hamiltonian, a vortex ring is potentially unstable in an intrinsic manner because the Hamiltonian is unbounded from below for short wavelengths. Physically, the energetical instability is caused by the fact that quantities playing the role of ''mass'' and ''spring constant'' in the Hamiltonian simultaneously assume negative values. The energetical instability, taking place for sufficiently large perturbations that are of wavelengths less than the critical wavelength (of about six times the core diameter in our core model), occurs though the ring is dynamically stable for nearly all wavelengths down to the core size. The dynamical instability of a vortex ring, related to the occurrence of imaginary excitation frequencies, is only relevant for certain critical wavelengths. We will thus present in this paper an argument that the relevant stability criterion for a vortex should be that of energetical stability.

Below, in the section to follow, we will first introduce an action principle that gives a transparent representation of the vortex dynamical behavior in incompressible, inviscid fluids, from which the canonical Hamiltonian for small perturbations in Sec. III, representing the vortex eigenmodes, naturally follows. Section IV gives an account of the thermodynamics of vortex rings, related to the excitation spectrum on a quantized vortex in superfluid ^{4}He, where the consequences of the predictions made in this work should be particularly clearly seen. We conclude with some remarks.

II. DERIVATION OF THE OSCILLATION MODES

A. Action principle

The peculiarity of the dynamical behavior of vortices in the incompressibility approximation, the fluid having a constant mass density ρ_0, consists in the fact that configuration space and phase space coincide [4,5]. The momenta are, in this limit, generally expressible as functions of the coordinates and play no independent dynamical role. This fact gives rise to the following action functional of the line configuration $\mathcal{C}$, in terms of the positions $\boldsymbol{R}$ of line elements $d\boldsymbol{R}$ with a constant velocity circulation Γ [6],

$$S[\mathcal{C}]=\int_0^t dt\left(-\frac{\Gamma}{3}\rho_0 \oint_{\mathcal{C}}\langle d\boldsymbol{R}\wedge\boldsymbol{R},\ \partial_t\boldsymbol{R}\rangle - H[\mathcal{C}]\right). \quad (1)$$

The factor $\frac{1}{3}$ in the kinematical term reflects our choice of cartesian co-ordinates in what follows and corresponds to a (co-ordinate) gauge for the vortex momentum [7]. The vortex kinetic energy is given by the Biot-Savart expression

$$H[\mathcal{C}]=\frac{\Gamma^2}{2}\rho_0 \oint_{\mathcal{C}}\oint_{\mathcal{C}}\frac{1}{4\pi}\frac{\langle d\boldsymbol{R},d\boldsymbol{R}'\rangle}{|\boldsymbol{R}-\boldsymbol{R}'|}, \quad (2)$$

where the shorthand notation $\boldsymbol{R}=\boldsymbol{R}(\phi,t)$ and $\boldsymbol{R}'=\boldsymbol{R}(\phi',t)$ is used. This relation for the energy yields the usual asymptotic logarithmic dependence of the stationary vortex energy on the infrared cutoff L (system size, distance to the next vortex, or radius of a vortex ring) and the ultraviolet cutoff ξ_c (the core size) in the form $\ln[L/(C\xi_c)]$, where C is a core model dependent constant. Stationarity of the action for first-order variation of the action (1) after $\boldsymbol{R}$ leads to the local velocity of a line element being perpendicular to the line element, and given by the Biot-Savart nonlocal induction law,

$$d\boldsymbol{R}\wedge\left(\partial_t\boldsymbol{R}-\frac{\Gamma}{4\pi}\oint_{\mathcal{C}}d\boldsymbol{R}'\wedge\frac{\boldsymbol{R}-\boldsymbol{R}'}{|\boldsymbol{R}-\boldsymbol{R}'|^3}\right)=\boldsymbol{0}. \tag{3}$$

The above proves that the action (1) leads to the correct equations of motion familiar from the fluid dynamics literature [8].

Let $\boldsymbol{R}(\phi,t)$, for $0\leqslant\phi\leqslant 2\pi$, describe the instantaneous shape of a moving vortex ring at time t, fluctuating with amplitude $\boldsymbol{u}(\phi,t)$ around its circular equilibrium shape $\boldsymbol{R}_0(\phi,t)$, so that $\boldsymbol{R}(\phi,t)=\boldsymbol{R}_0(\phi,t)+\boldsymbol{u}(\phi,t)$. Then, we parametrize line element position, equilibrium position, and small perturbations around this equilibrium as follows

$$\boldsymbol{R}(\phi,t)=R_\perp(\phi,t)(\hat{\boldsymbol{e}}_x\cos\phi+\hat{\boldsymbol{e}}_y\sin\phi)+\hat{\boldsymbol{e}}_z R_\parallel(\phi,t)$$

$$R_\perp=r_0+u_\perp(\phi,t),\quad R_\parallel=v_0t+u_\parallel(\phi,t). \tag{4}$$

From this choice of co-ordinates and the form of the action (1), the phase space variables for small oscillations $\boldsymbol{u}(\phi,t)$ are concluded to be

$$q(\phi,t)=u_\parallel(\phi,t)$$

$$p(\phi,t)=(\Gamma\rho_0 r_0)u_\perp(\phi,t). \tag{5}$$

These phase space variables are employed for the description of the vortex eigenmodes, which follows.

B. The spectrum

The fundamental cutoff to be introduced for the continuum description to be valid is that the separation of two line elements should always exceed a length ξ_c,

$$|\boldsymbol{R}-\boldsymbol{R}'|>\xi_c. \tag{6}$$

The length ξ_c is thus defined as the *cutoff diameter* of the vortex core. The above prescription is the simplest exact procedure to ensure that the Biot-Savart integrals remain regular. If the Biot-Savart description is refined by, e.g., a density profile in the core, smoothly increasing within a distance $\xi_c/2$ to the constant ρ_0, instead of being cut off to be exactly zero at $\xi_c/2$, this will effectively yield a different ultraviolet cutoff, that is, a different core constant C of order unity, multiplying a (fixed) value of ξ_c. However, the dynamical behavior of the vortex line on a scale well outside the core domain will not be affected by the core model.

To leading order in the fluctuations, the condition (6) is equivalent to

$$\left|\sin\frac{\phi-\phi'}{2}\right|>\frac{\xi_c}{2r_0+u_\perp+u'_\perp}. \tag{7}$$

Introducing the above condition in the integrals determining the velocity in Eq. (3) by means of a Heaviside step function, we obtain the equilibrium velocity of the ring,

$$v_0=\frac{\Gamma}{4\pi}\frac{1}{4r_0}\int_\delta^{2\pi-\delta}d\phi''\frac{1}{\sin\dfrac{\phi''}{2}}=\frac{\Gamma}{4\pi r_0}\ln\left(\frac{2r_0+2\sqrt{r_0^2-\xi_c^2}}{\xi_c}\right)$$

$$=\frac{\Gamma}{4\pi r_0}\ln[\cot(\delta/4)], \tag{8}$$

where the cutoff angle is determined by the parameter

$$\delta=2\arcsin\frac{\xi_c}{2r_0}. \tag{9}$$

To obtain the ring oscillation modes, we expand the small quantities $u_\parallel$ and $u_\perp$ in a Fourier series, $u_\perp(\phi,t)=\Sigma_n u_{\perp,n}(t)\,e^{in\phi}$, $u_\parallel(\phi,t)=\Sigma_n u_{\parallel,n}(t)\,e^{in\phi}$, and use the above described cutoff procedure of Eqs. (6) and (7), respectively. We then obtain the linearized equations of motion for parallel and perpendicular oscillations of the filament,

$$\partial_t u_{\parallel,n}=b_n\,u_{\perp,n},$$

$$\partial_t u_{\perp,n}=-a_n\,u_{\parallel,n}. \tag{10}$$

The coefficients in this linearized version of Eq. (3),

$$a_n=\frac{\Gamma}{4\pi}\frac{1}{r_0^2}\left[n^2\ln\cot\frac{\delta}{4}-I_{\parallel,n}\right], \tag{11}$$

$$b_n=\frac{\Gamma}{4\pi}\frac{1}{r_0^2}\left[\frac{1+\cos(n\delta)}{2\cos\dfrac{\delta}{2}}-(1-n^2)\ln\cot\frac{\delta}{4}-I_{\perp,n}\right],$$

are given in terms of integrals containing, due to our parametrization (4) of the ring oscillations, trigonometric functions only,

$$I_{\perp,n}=\frac{1}{8}\int_\delta^{2\pi-\delta}d\phi\left|\sin\frac{\phi}{2}\right|^{-3}\Big[\{1-\cos(n\phi)\}\frac{1}{2}(1+\cos\phi)$$

$$-n\sin(n\phi)\sin\phi+n^2(1-\cos\phi)\Big]$$

$$=-\frac{1}{2}\left[\frac{n\sin(n\delta)}{\sin\left(\dfrac{\delta}{2}\right)}-\frac{\sin^2\left(n\dfrac{\delta}{2}\right)\cos\left(\dfrac{\delta}{2}\right)}{\sin^2\left(\dfrac{\delta}{2}\right)}\right]$$

$$+\left(2n^2-\frac{1}{2}\right)\sum_{j=1}^{n}\frac{\cos\left[\left(j-\dfrac{1}{2}\right)\delta\right]}{2j-1},$$

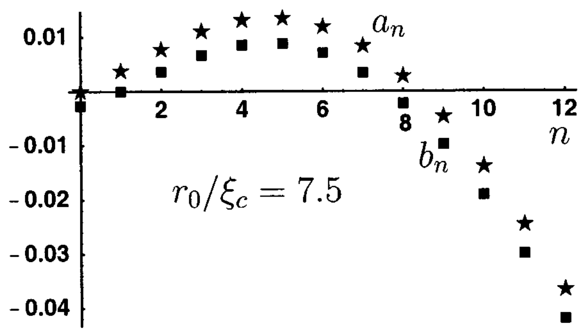

FIG. 1. The coefficients a_n (stars) and b_n (boxes) in the equations of motion (10) in units of ω_c, defined in Eq. (14), for $r_0/\xi_c = 7.5$ up to $n=12$. There occurs a dynamically unstable mode of imaginary frequency for $n=8$.

$$I_{\|,n}=I_{\perp,n}-\frac{1}{8}\int_{\delta}^{2\pi-\delta}d\phi\frac{1-\cos(n\phi)}{\sin\dfrac{\phi}{2}}$$

$$=I_{\perp,n}-\sum_{j=1}^{n}\frac{\cos\left[\left(j-\frac{1}{2}\right)\delta\right]}{2j-1}. \tag{12}$$

In Fig. 1, the coefficients a_n and b_n for $r_0/\xi_c=7.5$ are shown.

According to Eq. (10), the frequencies of oscillation of a vortex ring are evaluated from

$$\omega_n^2=a_n b_n. \tag{13}$$

We scale the coefficients a_n, b_n in Figs. 1 and 2 and frequencies ω_n in Fig. 3 below in units of the fundamental cyclotron frequency of the vortex core,

$$\omega_c=\frac{4|\Gamma|}{\pi\xi_c^2}, \tag{14}$$

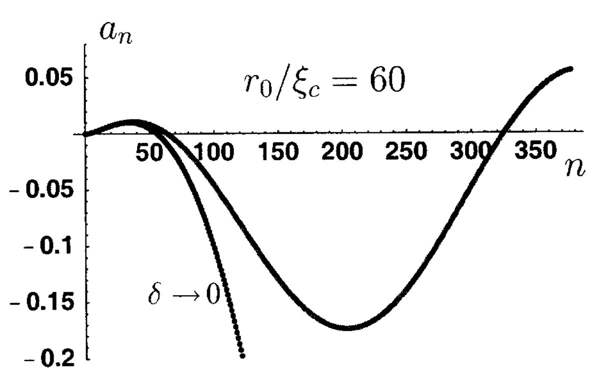

FIG. 2. The coefficients a_n given in Eq. (11) up to $n=2\pi\delta^{-1} \simeq 2\pi r_0/\xi_c$ for $r_0/\xi_c=60$ in units of ω_c defined in Eq. (14). For these small values of δ, a_n and b_n are indistinguishable within the figure's resolution. The dotted curve is the asymptotic result for $\delta \to 0$ in Eq. (15).

the frequency with which a point on the core revolves around the line designated by $\boldsymbol{R}(\phi,t)$ (the name stemming in the magnetic analogy from the role of Γ as a flux).

The frequencies (13) are exactly zero for both $n=0$ and $n=1$ to any order in δ. In the first case, $a_0=0$ and $b_0=(\Gamma/4\pi r_0^2)[\cos(\delta/2)]^{-1}-v_0/r_0$ whereas in the latter case, $a_1=v_0/r_0$ and $b_1=0$. The first case of symmetry corresponds to $u_{\perp,0}=$const, and tells us that the radius of the ring as a function of δ is determined up to the (constant) value of $u_{\perp,0}$. In the second case, in turn, $u_{\|,1}$ is a constant. The resulting line deformation resembles, for $\delta\to 0$, a translation of the ring as a whole. Except for radii r_0 that are just about an order of magnitude above ξ_c, the coefficients a_n and b_n are practically the same over the range of allowed values of n, so that for $\delta \ll 1$ the frequency squared ω_n^2 is essentially equal to either a_n^2 or b_n^2 and the waves around the ring are circularly polarized, like for ordinary Kelvin waves. However, for r_0 getting closer to ξ_c, a_n becomes increasingly different from b_n and the waves become elliptically polarized, the absolute ratio of amplitudes in the ring plane and out of the plane being given by $|u_{\perp,n}/u_{\|,n}|=\sqrt{|a_n/b_n|}$. A small ring thus oscillates more in the ring plane than out of the ring plane. Let us also stress that for the calculation of the oscillation modes, the nonlocality in the Biot-Savart integrals of Eqs. (2) and (3) is fully taken into account.

For direct comparison with the dispersion of Kelvin waves on rings, we consider, for fixed n, the limes $\delta\to 0$ in Eqs. (11) and (12) to obtain

$$a_n=\frac{\Gamma}{4\pi}\frac{1}{r_0^2}\left\{n^2\left(\ln\left[\frac{4r_0}{\xi_c}\right]-2S_n+\frac{1}{2}\right)+\frac{3}{2}S_n\right\},$$

$$b_n=\frac{\Gamma}{4\pi}\frac{1}{r_0^2}\left\{(n^2-1)\left(\ln\left[\frac{4r_0}{\xi_c}\right]-2S_n+\frac{1}{2}\right)-\frac{3}{2}(S_n-1)\right\}$$

$$(n \ll 1/2\delta) \tag{15}$$

where $S_n=\sum_{j=1}^{n}(2j-1)^{-1}$. There is an important difference between the dispersion relation (13) (which is exact within our hollow core model), with a_n and b_n from Eqs. (11) and the usually quoted asymptotic results of Lord Kelvin [1], Thomson [2], and Grant [9,10] (also cf. the work of Pismen and Nepomnyashchy [12], who found the same result as Grant, but within a much simpler scheme similar to ours). These results correspond to the relations (15) for the coefficients a_n and b_n in the limit of $\delta\to 0$ for fixed n (save for different core structure constants). The important difference consists in the fact that the geometric cutoff prescription (6), which ensures that a core of diameter ξ_c is always excluded in the evaluation of Eq. (3), is taken care of in relations (11) exactly for any admissible value of the ratio of ring radius and core diameter r_0/ξ_c, such that $n\delta \sim O(1)$ can be consistently realized. The anomalous branch also occurs, shifted to smaller mode numbers, as a consequence of relations (15). However, the minimum resides at values of $n \sim \delta^{-1}$ that are beyond the applicability of Eqs. (15). We have depicted the difference between the exact and asymptotic results in Fig. 2 for the whole range of n up to the value $n=2\pi\delta^{-1}$

$\simeq 2\pi r_0/\xi_c$ corresponding to $k \simeq 2\pi/\xi_c$ ($\lambda \simeq \xi_c$). For the value $r_0/\xi_c = 60$ used, the coefficient b_n is essentially identical to a_n within the resolution of the figure (cf. Fig. 1, which has $r_0/\xi_c = 7.5$ and where the difference between a_n and b_n is clearly discernible).

For completeness, we state the solution of the equations of motion (10). If the vortex ring undergoes at time $t=0$ a deformation represented by

$$u_\parallel(\phi,0) = \mathrm{Re}\sum_n u^0_{\parallel,n} e^{in\phi},$$

$$u_\perp(\phi,0) = \mathrm{Re}\sum_n u^0_{\perp,n} e^{in\phi}, \tag{16}$$

the solution at a later time t takes the form

$$u_\parallel(\phi,t) = \mathrm{Re}\sum_n \left[u^0_{\parallel,n}\cos(\omega_n t) + u^0_{\perp,n}\frac{b_n}{\omega_n}\sin(\omega_n t)\right] e^{in\phi},$$

$$u_\perp(\phi,t) = \mathrm{Re}\sum_n \left[u^0_{\perp,n}\cos(\omega_n t) - u^0_{\parallel,n}\frac{a_n}{\omega_n}\sin(\omega_n t)\right] e^{in\phi}, \tag{17}$$

where $\omega_n = \sqrt{a_n b_n}$.

III. HAMILTONIAN

The Hamiltonian corresponding to the equations of motion (10) assumes the form

$$\mathcal{H} = E_0 + \sum_{2 \leq n \leq n_c} \frac{\Gamma \rho_0 r_0}{2}\left[a_n u^2_{\parallel,n} + b_n u^2_{\perp,n}\right], \tag{18}$$

where the stationary energy of the ring is given by

$$E_0 = \frac{\Gamma^2 \rho_0 r_0}{2}\left(\ln\left[\cot\left(\frac{\delta}{4}\right)\right] - 2\cos\left[\frac{\delta}{2}\right]\right). \tag{19}$$

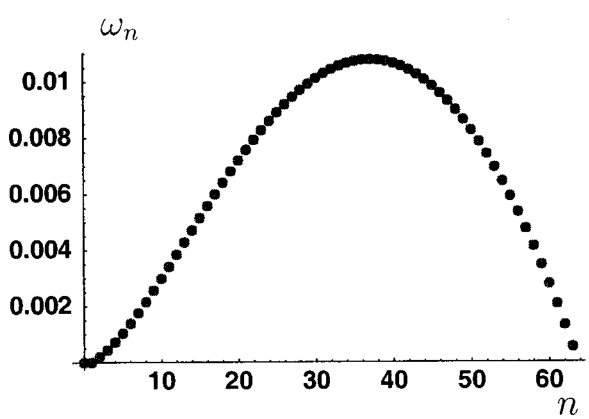

FIG. 3. Stable oscillation frequencies of a vortex ring in units of the cyclotron frequency ω_c as a function of the mode number n for the ratio $r_0/\xi_c = 60 \simeq \delta^{-1}$, up to $n_c \simeq \pi/(3\delta)$. For this ratio of radius and core diameter, the exact value is $n_c = 63$. The maximum is, essentially independent of the value of δ, situated at $\max[\omega_n] \simeq 0.011\,\omega_c$ with $n \simeq n_c/2$.

The (constant) variables $u_{\perp,0}$ and $u_{\parallel,1}$ do not appear in Eq. (18) because of our choice of parametrization (4), which corresponds to a transformation to the rest frame of a ring of radius r_0 moving with velocity v_0. The above expression for $\mathcal{H}$ then represents the rest frame Hamiltonian of the vortex. The phase space variables may, for example, be chosen to be $q_n = u_{\parallel,n}$ and $p_n = \Gamma\rho_0 r_0 u_{\perp,n}$, like in (5), so that the mass $M_n = \Gamma\rho_0 r_0/b_n$ and elasticity (spring) constant $D_n = \Gamma\rho_0 r_0 a_n$. Equally well, we may choose the option $q_n = u_{\perp,n}$ and $p_n = -\Gamma\rho_0 r_0 u_{\parallel,n}$, which reverses the role of mass and elasticity coefficients in conventional Hamiltonian language (replaces a_n by b_n and b_n by a_n in M_n and D_n). The identity of phase space and configuration space [4–7] implies that both options are viable.

From the Hamiltonian (18), we gather that stable oscillation modes are those that have a_n and b_n *both* positive. They contribute positive energy to the Hamiltonian. Energetically unstable, though giving a real frequency, are the modes that have a_n and b_n both negative, because they contribute negative energy to the Hamiltonian. A different sign of a_n and b_n leads to dynamically unstable modes, which have imaginary frequencies and amplitudes $\boldsymbol{u}$ exponentially growing (or decaying) in time. For mode numbers above

$$n = n_c \simeq \frac{\pi}{3\delta} \tag{20}$$

and up to $n \simeq 5\delta^{-1}$, both a_n and b_n become negative such that the energy contribution corresponding to these modes is *negative*. To quadratic order in the oscillation amplitude, oscillations with wave lengths smaller than $\lambda \sim 6\xi_c$ thus imply that vortex modes of such small wave number are unstable. Hence, the stable spectrum is restricted to mode numbers of magnitude less than $n_c \simeq \delta^{-1}$, by definition the last mode number for which the oscillation energy is positive semidefinite, before entering the negative energy domain seen in Fig. 2. We have plotted the dispersion relation of the stable modes in Fig. 3, for $r_0/\xi_c = 60$.

With regard to the validity of the assumption of an incompressible fluid, we note that the frequency at the maximum in Fig. 3, situated at $n \simeq n_c/2 \simeq (2\delta)^{-1}$ for all values of r_0/ξ_c not too close to unity, scales as $\omega_n \simeq 0.011\,\omega_c$, with the cyclotron frequency ω_c defined in Eq. (14) [$\omega_c \sim 10^{12}$ sec^{-1} in superfluid ^{4}He (helium II)]. Oscillation velocities thus remain, for moderate oscillation amplitudes of the order of a few ξ_c, well below the speed of sound even in the superfluid helium II, where ξ_c is of atomic size and $c_s\xi_c \sim \Gamma = \pi\omega_c(\xi_c/2)^2$, so that the incompressibility approximation holds. This is more questionable for the second, much larger, frequency maximum at $n \simeq 3\delta^{-1}$, corresponding to the maximum negative value of the coefficient a_n in Fig. 2 given by $a_n \simeq -0.18\,\omega_c$.

Consider, for a physical interpretation of the energetical instability around $n = n_c$, Fig. 4 where we show the shape of deformation of the vortex core for a small wavelength of order $\lambda_c \simeq 6\xi_c$, corresponding to the crossover to the unstable oscillations regime. We may infer that the negative oscillation energy, occurring at a smaller wavelength than that shown in Fig. 4, is due to a volume exclusion effect. The

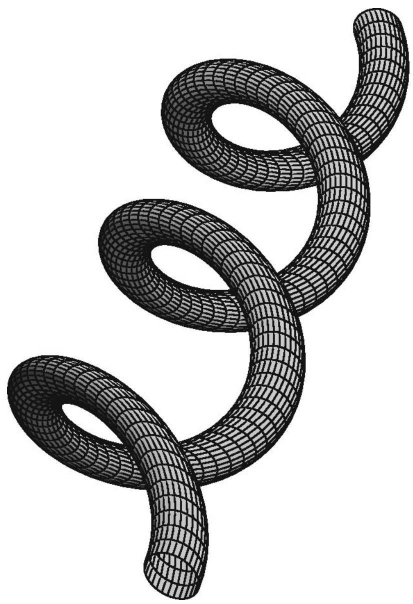

FIG. 4. Shape of wave traveling along the ring for mode numbers near the critical mode number n_c showing the helical vortex core displacement.

excluded core volume kinetic energy is large enough such that there is only small energetical cost of exciting a perturbation on the filament for the stable modes with mode numbers slightly below n_c, and an energetical gain for the unstable ones. In the Hamiltonian (18), we neglect the tail of positive excitation energies corresponding to positive values of a_n, b_n seen in Fig. 2 because it is very close to the limit of core elements touching themselves, at which point our core model certainly becomes invalid because it is then meaningless to speak about helical oscillations of a hollow torus. That the coefficients a_n and b_n, and thus the excitation energy, do increase again after $n \simeq 3\delta^{-1}$, can be traced back to the fact that line elements having like circulation approach each other closely if we further compress the spiral of Fig. 4 along its axis. The volume energy exclusion effect we just described is then counterbalanced for these very short wavelengths by the resulting strong repulsion of adjacent elements of the same circulation.

The frequency (13) is imaginary if a_n and b_n have different signs and a *dynamical* instability results [11,12]. We stress, however, that the unboundedness of the Hamiltonian (18) from below leads to the energetical instability of the ring for mode numbers beyond n_c. This instability will exist for any value of δ respectively of r_0/ξ_c. The change induced by choosing some different, more regular and differentiable core structure than our prescription (6) is the numerical value of the slope of the negative group velocity branch within a number of order unity, and the mode number for which the excitation energy becomes negative. The fact that around n_c dynamical instabilities can take place has also been recognized in [12], where the dynamical instability was investigated using the Gross-Pitaevskiĭ model of a superfluid and it was indeed found that $n_c \xi_c / r_0 = O(1)$. However, what has been missed in this work (and others in the conventional fluid mechanical framework [8,11]), is that an energetical instability will take place right after we have crossed the dynamical instability region, independent of the precise value of the critical n_c as a function of r_0/ξ_c. It is apparent from Fig. 4 that the instability will persist for any (regularizing) model taken of the core region as long as the energy density of the core is significantly less than that of the surrounding bulk fluid, i.e., as long as it is still significantly reduced compared to the bulk if we, for example, turn on interactions (take into account compressibility) inside the core. For smaller energy density differences between core and bulk, the value of n_c will be shifted upwards (for a given value of r_0/ξ_c) but the energetical instability will still exist.

IV. LOW TEMPERATURE THERMODYNAMICS OF VORTEX OSCILLATIONS IN HELIUM II

Up to this point, our considerations have been in terms of a classical vortex. Consider now the quantum mechanical zero point fluctuations of a vortex line in the quantum fluid helium II, each with a contribution $\frac{1}{2}\hbar\omega_n$ to the vortex (ground state) energy. If we sum up these contributions to the limiting value n_c, we get $E_{\rm fl} \equiv \Sigma_{n=2}^{n_c} \frac{1}{2}\hbar\omega_n \simeq 0.0035\hbar\omega_c\,\delta^{-1}$ (cf. the area under the dispersion curve in Fig. 3). Comparing this with the stationary energy E_0, see Eq. (19) of an undeformed ring in helium II, we obtain

$$E_{\rm fl} \simeq 0.014\left(\frac{d}{\pi^{2/3}\xi_c}\right)^3 \frac{E_0}{\ln(r_0/\xi_c)}, \tag{21}$$

where the interparticle distance $d = (\rho_0/m)^{-1/3}$ ($\sim \xi_c$ in helium II). The total quantum-mechanical fluctuation energy in the stable modes (at zero temperature) is thus much less than the stationary energy E_0, of the order of a few percent of E_0. This need not be the case if we take into account thermal fluctuations as well. The vortex free energy may be written as

$$\mathcal{F}(T,\delta) = E_0 + \beta^{-1}\sum_{n=2}^{n_c} \ln(2\sinh[\beta\hbar\omega_n/2]). \tag{22}$$

The entropic part of the free energy, due to ring oscillations, plays an important role if the temperature is a significant fraction of the cyclotron energy of the core. We stress that the temperatures for fluctuations to become important are significantly higher if the cutoff is chosen well below n_c. We have also convinced ourselves that the absolute ratio of the oscillation free energy part over the stationary ring energy, $|\mathcal{F} - E_0|/E_0$, is larger for smaller radius r_0, i.e., the oscillations play an increasingly important thermodynamic role for smaller rings.

For low temperatures, fulfilling $k_B T \ll \max[\hbar\omega_n] \simeq 0.011\hbar\omega_c$, Kelvin modes of small wave number and the modes with approximately linear dispersion around n_c (cf. Fig. 3) are populated [13]. In the superfluid helium II, where $\hbar\omega_c$ is of the order of ten kelvins, the condition on the temperature leads to the requirement $T \ll 100$ mK, which is feasible in experimental practice. For vortex rings of a given size and orientation, we thus expect two contributions to the specific heat at low temperatures, coming from the aforementioned two asymptotic branches of excitations on the filament. For the indicated range of mode numbers, we may approximate the dispersion by the Kelvin-like form,

$$\omega_K = \gamma_1 n^2 \ln[n_c^*/n] \quad (1 < n \ll n_c/2), \tag{23}$$

where the parameters γ_1 and n_c^* are

$$\gamma_1 = \frac{|\Gamma|}{4\pi r_0^2} = \omega_c \left(\frac{\xi_c}{4r_0}\right)^2, \quad n_c^* = 8\sqrt{e}\frac{r_0}{\xi_c}. \tag{24}$$

Near n_c, a linear law obtains

$$\omega_A \simeq \gamma_2 (n_c - n) \quad (n_c - n \ll n_c/2), \tag{25}$$

where, numerically,

$$\gamma_2 \simeq 0.045\, \omega_c\, \xi_c / r_0. \tag{26}$$

Both of these approximate dispersion relations are valid for large r_0/ξ_c (small δ). The density of states for the ω_K branch, within logarithmic accuracy, may be written $N_K(E) \simeq (4\hbar\gamma_1 \ln[n_c^*]E)^{-1/2}$; for the ω_A branch it is independent of the energy E, $N_A(E) = (\hbar\gamma_2)^{-1}$. The asymptotical behavior of the vortex specific heat for low temperatures then assumes the form

$$\frac{C_v}{k_B} \simeq \frac{2\pi r_0}{\xi_c}\left(\frac{1.1}{\sqrt{\ln n_c^*}}\sqrt{\frac{k_B T}{\hbar\omega_c}} + 11.6\frac{k_B T}{\hbar\omega_c}\right). \tag{27}$$

It is, as expected, proportional to the ''volume,'' i.e., the circumference of the ring, and has a contribution proportional to $\sqrt{T}$ from the Kelvin-like modes and a new contribution proportional to T stemming from the linear dispersion, large wave number branch. This last term gives a dependence of C_L proportional to the area $2\pi r_0 \xi_c$. For a randomized ensemble of vortex rings, with different orientations and radii, we expect the indicated dependence on temperature to hold for a dilute system of effectively noninteracting rings. For a dense vortex tangle, coupling of the vortex rings by mutual induction will modify the spectrum and the above thermodynamic behavior, a problem which is left for future work.

V. CONCLUSION

We have derived the oscillation modes on vortex rings using the canonical phase space structure of small ring oscillations in an incompressible, inviscid fluid and a geometrical cutoff procedure for the core region. Beyond a critical wave number, the excitation energy becomes negative indicating that the vortex ring is energetically unstable for perturbations on scales of short wavelengths. The instability relies on the energy exclusion effect of the helically displaced core and may be interpreted in conventional Hamiltonian language as being due to the fact that the classical or quantum particle representing the excitation has both a kinetic energy with negative mass and a potential with negative spring constant.

The existence and peculiar anomalous dispersion of propagating modes with very small wavelengths should have important implications for the dynamics of vortex reconnection events [14] as well as the final stages of the energy cascade process in superfluid turbulence [15]. In addition, we expect that scattering cross sections of the elementary roton excitation in helium II with vortices, and thus the coefficients of mutual friction between superfluid and normal components [16], will be influenced by the presence of low energy modes with wave numbers of the order of the inverse core size.

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