Universal scaling functions for bond percolation on planar-random and square lattices with multiple percolating clusters

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Percolation models with multiple percolating clusters have attracted much attention in recent years. Here we use Monte Carlo simulations to study bond percolation on $L_1 \times L_2$ planar random lattices, duals of random lattices, and square lattices with free and periodic boundary conditions, in vertical and horizontal directions, respectively, and with various aspect ratios L_1/L_2. We calculate the probability for the appearance of n percolating clusters, W_n; the percolating probabilities P; the average fraction of lattice bonds (sites) in the percolating clusters, $\langle c^b \rangle_n$ ($\langle c^s \rangle_n$), and the probability distribution function for the fraction c of lattice bonds (sites), in percolating clusters of subgraphs with n percolating clusters, $f_n(c^b)$ [$f_n(c^s)$]. Using a small number of nonuniversal metric factors, we find that W_n, P, $\langle c^b \rangle_n$ ($\langle c^s \rangle_n$), and $f_n(c^b)$ [$f_n(c^s)$] for random lattices, duals of random lattices, and square lattices have the same universal finite-size scaling functions. We also find that nonuniversal metric factors are independent of boundary conditions and aspect ratios.

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I. INTRODUCTION

Percolation is related to many interesting scientific phenomena [1]. In recent years percolation problems with multiple percolating clusters have attracted much attention [2–19]. Most simulational studies of such problems were restricted to percolation on lattices [20]. However, many physical systems with multiple percolating clusters such as Carbino disks used in the study of quantum Hall effects [2], or oil fields confronted with drilling problems, do not have underlined regular lattice structures. Thus it is of interest to know the relationship between quantities for percolation on regular lattices and quantities for percolation not on regular lattices, such as random lattices. In the present paper, we use Monte Carlo simulations to study bond percolation on $L_1 \times L_2$ planar random lattices, duals of random lattices, and square lattices with free and periodic boundary conditions in vertical and horizontal directions, respectively, and with various aspect ratios L_1/L_2. We calculate the probability for the appearance of n percolating clusters, W_n; the percolating probabilities P; the average fraction of lattice bonds (sites) in percolating clusters, $\langle c^b \rangle_n$ [$\langle c^s \rangle_n$], and the probability distribution function for fraction c of lattice bonds (sites), in percolating clusters of subgraphs with n percolating clusters, $f_n(c^b)$ [$f_n(c^s)$]. Using a small number of nonuniversal metric factors, we find that W_n, P, $\langle c^b \rangle_n$ ($\langle c^s \rangle_n$), and $f_n(c^b)$ [$f_n(c^s)$], for random lattices, duals of random lattices, and square lattices, have the same universal finite-size scaling functions. We also find that nonuniversal metric factors are independent of boundary conditions and aspect ratios. Furthermore, this study is related to recent developments in the universality and scaling of critical phenomena.

Universality and scaling are two important concepts in the modern theory of critical phenomena [21–23], and percolation models are ideal systems for studying critical phenomena [1]. Thus universality and scaling have been actively studied in recent decades, especially for percolation models [24]. In 1992, Langlands *et al.* [25] proposed that for bond and site percolation models on square (sq), planar triangular (pt), and honeycomb (hc) lattices, the critical existence probability (also called the crossing probability or spanning probability) is a universal quantity, when aspect ratios of sq, hc, and pt lattices have relative ratios $1:\sqrt{3}:\sqrt{3}/2$. In 1995 and 1996, Hu, Lin, and Chen (HLC) [26,3] calculated the existence probability E_p, the percolation probability P, and the probability for the appearance of n percolating clusters, W_n, of bond and site percolation models on sq, hc, and pt lattices with aspect ratios $1:\sqrt{3}:\sqrt{3}/2$; they showed that all their scaled data fall on the same universal scaling functions, by selecting a very small numbers of nonuniversal metric factors and maintaining similar nonuniversal metric factors under free and periodic boundary conditions. By using renormalization group theory, in 1996 Hovi and Aharony [27] also pointed out that scaling functions for the spanning probability are universal at the fixed point for every system with the same dimensionality, spanning rule, aspect ratio, and boundary conditions. In 1996 Okabe and Kikuchi [28], extended the work of HLC to a two-dimensional Ising model on planar regular lattices. In 1997, Hu and Wang [11] found that lattice and continuum percolations of hard and soft disks have the same universal scaling functions for W_n. Using the connection between an Ising model and a bond-correlated percolation model [29], in 1999 Tomita, Okabe, and Hu [17] calculated the probability for the appearance of n percolating clusters, W_n, the percolating probabilities, P, the average fraction of lattice sites in percolating clusters, $\langle c \rangle_n$, and the probability distribution function for the fraction c of lattice sites in percolating clusters of subgraphs with n percolating

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clusters, $f_n(c)$, for bond-correlated percolation model on sq, hc, and pt lattices, with aspect ratios of $1:\sqrt{3}:\sqrt{3}/2$. Using a small number of nonuniversal metric factors, they found that W_n, P, $\langle c\rangle_n$, and $f_n(c)$ for sq, hc, and pt lattices have the same universal finite-size scaling functions.

However, the studies mentioned above mostly focused on regular lattices, with fixed coordination numbers [20]. In 1999 Hsu and Huang (HH) [30] determined the percolation thresholds and critical exponents, and demonstrated explicitly that the ideas of universal critical exponents and universal scaling function with nonuniversal metric factors can be extended to bond percolation on $L\times L$ periodic planar random lattices, duals of random lattices, and square lattices, for both existence and percolating probabilities and the mean cluster size. This paper will study bond percolation on $L_1\times L_2$ planar random lattices, duals of random lattices, and square lattices in more detail, and consider the case where the lattices have free and periodic boundary conditions in vertical and horizontal directions, respectively, as in Ref. [3]. The percolating probability was defined in Ref. [30] by the ratio of the number of bonds in the percolating clusters to the total number of bonds. Here we consider two different definitions of the percolating probability, in terms of bonds and sites; the latter was also used in Refs. [3] and [17]. We calculate the probability W_n for the appearance of n percolating clusters, the percolating probability P, the average fraction of lattice bonds (sites) in percolating clusters, $\langle c^b\rangle_n$ ($\langle c^s\rangle_n$), and the probability distribution function for the fraction c of lattice bonds (sites) in percolating clusters of subgraphs with n percolating clusters, $f_n(c^b)$ [$f_n(c^s)$], for various values of aspect ratios L_1/L_2, and finally check the universal finite-size scaling behaviors for these quantities. In Ref. [30], HH used two nonuniversal metric factors D_2 and D_3 to fix universal finite-size scaling functions for the percolating probability in terms of bonds. In the present paper, we calculated two nonuniversal metric factors of the percolating probability in terms of sites, and obtained previously known values of nonuniversal metric factors determined by HH, to check whether we have universal scaling functions for W_n, P, $\langle c^b\rangle_n$ ($\langle c^s\rangle_n$), and $f_n(c^b)$ [$f_n(c^s)$] for bond percolation on random lattices, duals of random lattices, and square lattices.

Dirichlet and Voronoi [31] first used the concept of random lattices in condensed matter theory. Christ, Friedberg, and Lee (CFL) [32] used another type of random lattice to formulate quantum field theory. Here we adopt the CFL algorithm, and give a brief review of the construction of planar random lattices and their duals. First we randomly generate N sites in the $L_1\times L_2$ rectangular domain with periodic boundary conditions. Next we arbitrarily choose three nearby sites, and draw a circle to go through the three sites. If there are no lattice sites inside the circle, the three sites are connected by links to form a triangle. A planar random lattice is constructed by repeating the process until all sites are connected by links. The whole rectangular domain is divided into $2N$ nonoverlapping triangles, whose vertices are sites of the random lattice; circle centers of triangles are the sites of dual lattices. Thus there is a one to one correspondence between triangles and dual lattice sites. Because a link of the random lattice is shared by two triangles, the two corresponding dual lattice sites are connected by one dual link. There is a one to one correspondence between links and dual links. The whole rectangular domain is partitioned into N nonoverlapping planar convex polyhedra, which are formed by dual links, and the vertices of N polyhedra are sites for dual lattices. There is also a one to one correspondence between the lattice sites and polyhedra on the dual lattice. An example of a planar random lattice with dual, under periodic boundary conditions, in both vertical and horizontal directions, is shown in Fig. 1(a).

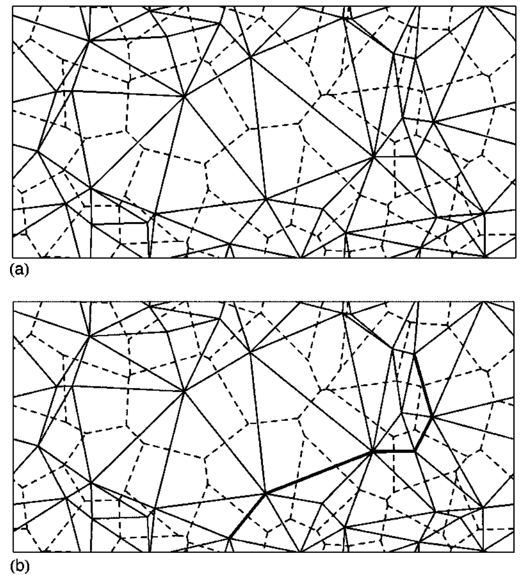

FIG. 1. Examples of (a) an $L_1\times L_2=8\times 4$ planar random lattice (solid lines), with its dual (dashed lines) on an $L_1\times L_2=8\times 4$ rectangular area, with periodic boundary conditions; and (b) a first kind of percolating cluster path, without boundary bonds (bold solid lines) on a random lattice.

This paper is organized as follows: In Sec. II, we present simulational results for W_n, P, $\langle c^b\rangle_n$ ($\langle c^s\rangle_n$), and $f_n(c^b)$ [$f_n(c^s)$] for bond percolation, on $L_1\times L_2$ random lattices, duals of random lattices, and square lattices, under free and periodic boundary conditions in vertical and horizontal directions with $L_1/L_2=4$. The boundary bonds which cross the rectangular domain in the vertical direction on the random lattices, due to periodic boundary conditions, are eliminated because of free boundary conditions in the vertical direction considered in this paper. We adopt the method of HH [30] to find percolating clusters. Only the first kind of percolating cluster paths without boundary bonds in the vertical direction (the clusters extend from top to bottom), should be identified, and an example of this is shown in Fig. 1(b). In Sec. III, we use finite-size scaling theory to check the scaling behaviors of various quantities, and to show that such quantities have universal finite-size scaling functions for regular lattices and random lattices. A summary is provided in Sec. IV.

II. $W_n(L_1,L_2,p)$, $f_n(C)$, AND $\langle c\rangle_n$

We look at the bond percolation on a lattice G, with linear dimensions L_1 and L_2 in horizontal and vertical directions, respectively; the probability for the appearance of n top-to-

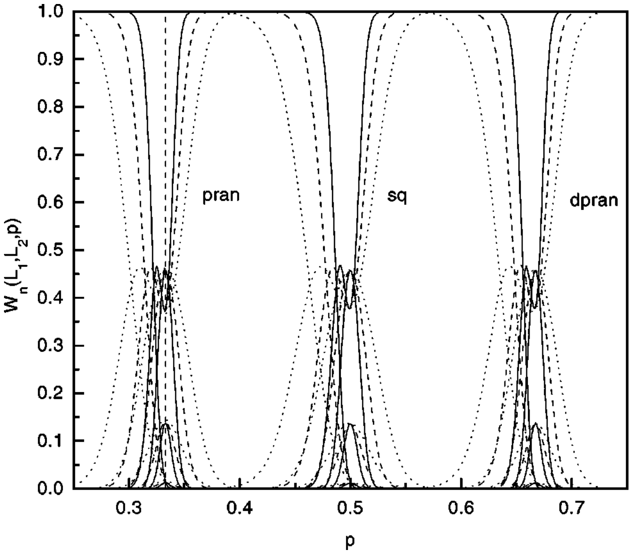

FIG. 2. $W_n(L_1,L_2,p)$ on square lattices, planar random lattices, and their duals of sizes 128×32, 256×64, and 512×128, which are represented by dotted, dashed, and solid lines, respectively.

bottom percolating clusters, $W_n(L_1,L_2,p)$, is defined by [3]

$$W_n(L_1,L_2,p)=\sum_{G_n'\subseteq G} p^{b(G_n')}(1-p)^{E-b(G_n')}. \tag{1}$$

Here the percolating cluster is defined as a cluster extending from top to bottom in G, G_n' denotes a percolating subgraph with n percolating clusters, $b(G_n')$ is the number of occupied bonds in G_n', and E is the total number of links in G. The existence probability E_p can be obviously expressed as

$$E_p=\sum_{n=1}^{\infty} W_n, \tag{2}$$

with $W_0=1-E_p$.

To obtain more detailed information about the contents of the percolating cluster, following Tomita *et al.* [17], we decompose W_n as

$$W_n=\int_0^1 f_n(c)dc, \tag{3}$$

where $n=1,\ldots,\infty$, c denotes the fraction of lattice bonds (sites) in percolating clusters, and $f_n(c)$ is the probability distribution function of c in subgraphs with n percolating clusters. The probability distribution function of c in all subgraphs is the overall summation of $f_n(c)$, i.e.,

$$f(c)=\sum_{n=1}^{\infty} f_n(c). \tag{4}$$

In terms of $f_n(c)$, the average fraction of lattice bonds (sites) in subgraphs with n percolating clusters can be expressed as

$$\langle c\rangle_n=\int_0^1 cf_n(c)dc, \tag{5}$$

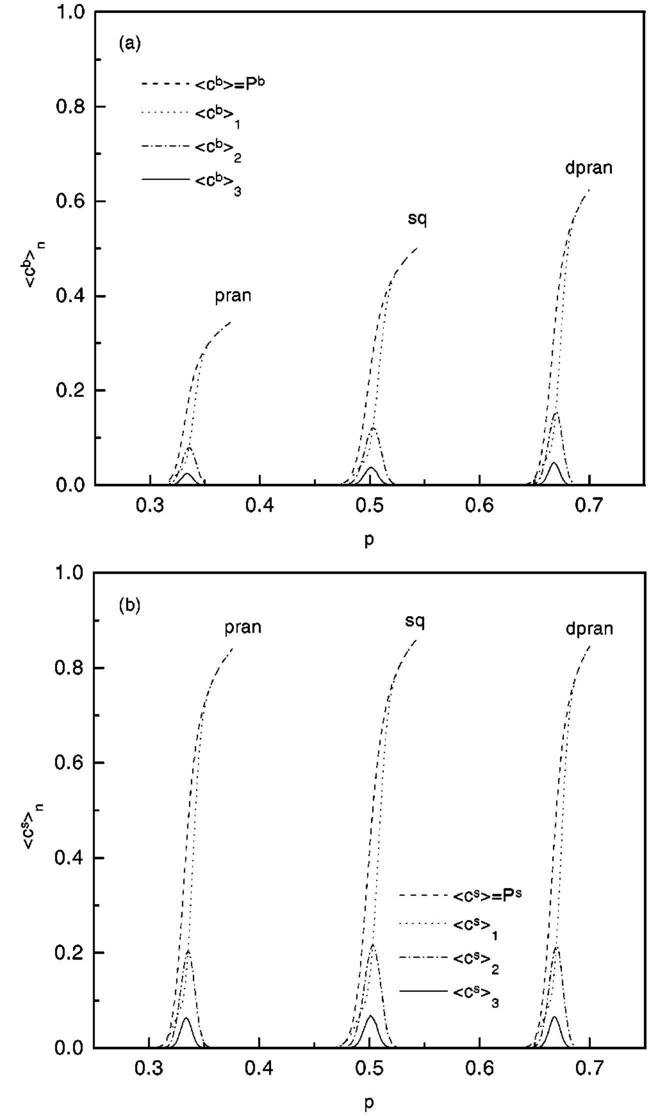

FIG. 3. (a) P^b and $\langle c^b\rangle_n$ and (b) P^s and $\langle c^s\rangle_n$ on a square lattice, planar random lattice, and the dual of a planar random lattice of size 512×128.

where $n=1,\ldots,\infty$, and the percolating probability P can be written as

$$\langle c\rangle=\sum_{n=1}^{\infty}\langle c\rangle_n=\int_0^{\infty} cf(c)dc=P. \tag{6}$$

To generate subgraphs, we use a random bond occupation process with equal occupation probabilities for each link. The simulations are performed on 128×32, 256×64, and 512×128 planar random (pran) lattices, and their duals (dpran), with free and periodic boundary conditions in the vertical and horizontal directions, respectively. To compare the results with regular lattices, we also perform simulations on square (sq) lattices of the same sizes. On each lattice, we take 60 occupation probabilities around the critical percolation threshold for every 0.002 increment, and use the random bond occupation process to generate 10^5–10^6 configurations for each occupied probability, p. We calculate

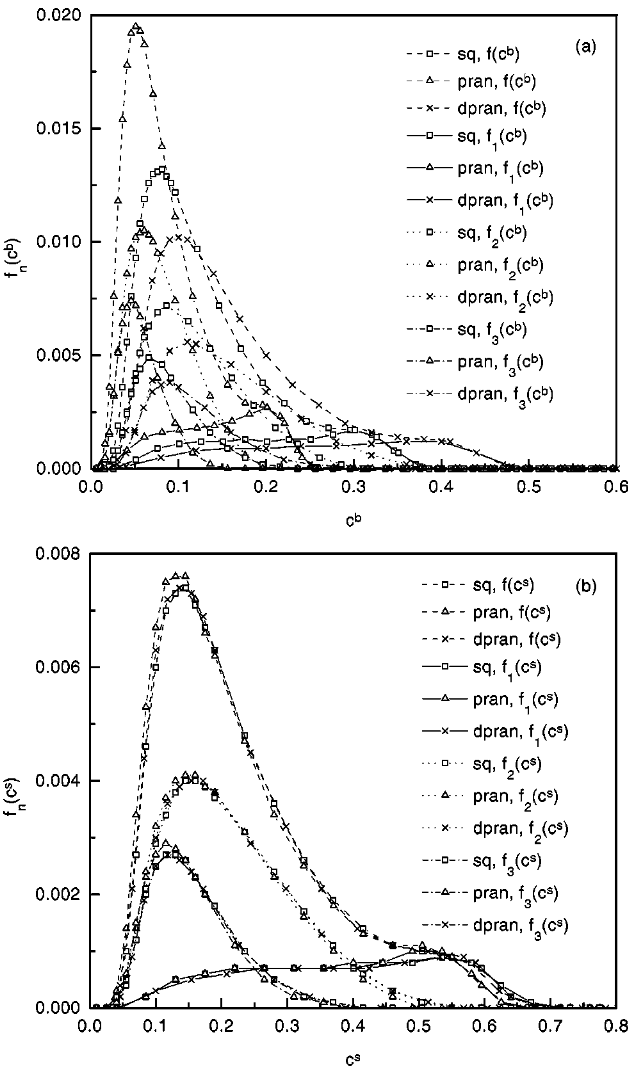

FIG. 4. At $p=p_c$, (a) $f(c^b)$ and $f_n(c^b)$ and (b) $f(c^s)$ and $f_n(c^s)$ on a square lattice, a planar random lattice, and the dual of a planar random lattice of size 512×128.

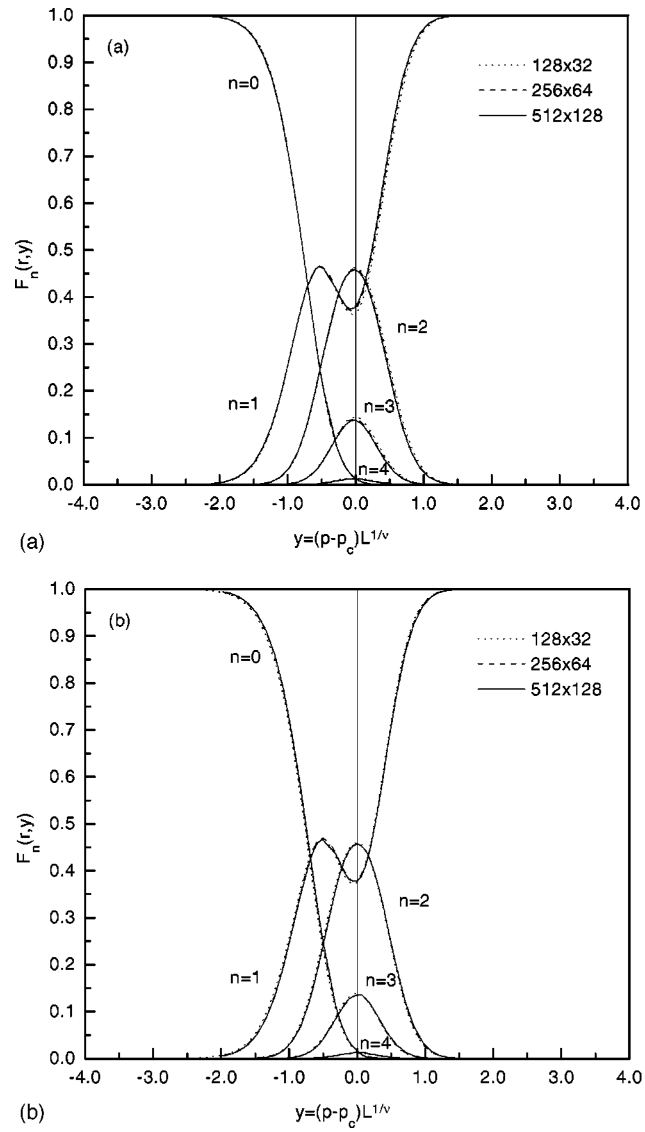

FIG. 5. The scaled results of $F_n(r,y)=W_n(L_1,L_2,p)$ as a function of $y=(p-p_c)L^{1/\nu}$ for (a) planar random lattices and (b) their duals of sizes 128×32, 256×64, and 512×128. The monotonic decreasing function is for $F_0(r,y)$. The S shaped curve is for $F_1(r,y)$. The bell shaped curves from top to bottom are for $F_n(r,y)$, with $n=2$, 3, and 4, respectively.

$W_n(L_1,L_2,p)$, $\langle c^b\rangle_n$, and $\langle c^s\rangle_n$, where c^b denotes the fraction of bonds in percolating clusters, and c^s denotes the fraction of sites in percolating clusters; the results are shown in Figs. 2 and 3. The calculated results of the percolating probabilities in terms of bonds, P^b, and in terms of sites, P^s, are also shown in Fig. 3. We calculate $f_n(c^b)$ and $f_n(c^s)$ at $p=p_c$ and take $p_c=0.3333$ for planar random lattices and $p_c=0.6667$ for dual lattice [30]. The results are shown in Fig. 4. The differences between bond and site contents in percolating clusters are shown in Figs. 3 and 4; here, for clarity of presentation, only the results for 512×128 lattices are plotted in the figures.

III. UNIVERSAL FINITE-SIZE SCALING FUNCTIONS

The finite-size scaling theory was first formulated by Fisher in 1971 [22]. According to the theory, for a physical quantity X, which scales as $X(t)\sim t^{\rho}$ in a thermodynamic system near a critical point $t=0$, the same quantity in a finite system with a linear dimension L, $X_L(t)$, should obey the general law

$$X_L(t)\sim L^{-\rho/\nu}F(tL^{1/\nu}). \tag{7}$$

Here $F(x)$ with $x=tL^{1/\nu}$ is labeled as a scaling function, with ν as a correlation length exponent. In 1984, Privman and Fisher [23] considered universal finite-size scaling functions and nonuniversal metric factors, and proposed that the singular part of the free energy of a critical system can be written as

$$f_L^s(t)\sim L^{-d}Y(DtL^{1/\nu}), \tag{8}$$

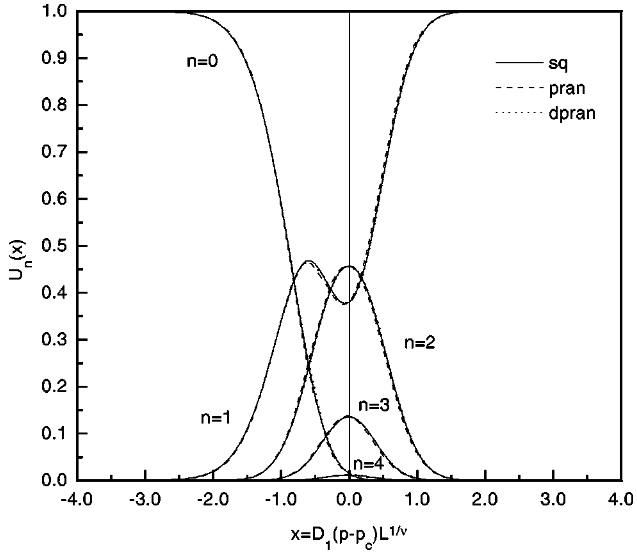

FIG. 6. The scaled results of $U_n(x)=W_n(L_1,L_2,p)$, as a function of $x=D_1(p-p_c)L^{1/\nu}$ for a square lattice (solid curves), a planar random lattice (dashed curves), and the dual of a planar random lattice (dotted curves) of size 512×128.

where d is the spatial dimensionality of the lattice, Y is a universal scaling function, and D is a nonuniversal metric factor.

At a critical point $p=p_c$, there also exists a finite-size scaling form for the distribution function of $X_L(t)$ [17]:

$$Q(X_L(t=0))\sim L^{\rho/\nu}Y(X_L(t=0)L^{\rho/\nu}). \tag{9}$$

In Refs. [26,30], three nonuniversal metric factors D_1, D_2, and D_3 were used for regular lattices and random lattices, to describe the universal scaling functions of existence probability E_p and the percolating probability P, i.e.,

$$E_p(p,L)=F(x), \tag{10}$$

with $x=D_1(p-p_c)L^{1/\nu}$, and

$$D_3P(p,L)=L^{-\beta/\nu}S_p(z), \tag{11}$$

with $z=D_2(p-p_c)L^{1/\nu}$.

Following Hu and Lin [3], in Figs. 5(a) and 5(b), respectively, we use the evaluated percolation threshold p_c [30] and the exact value of the critical exponent, $\nu=4/3$, to plot W_n, as a function of $y=(p-p_c)L^{1/\nu}$, for planar random lattices and their duals. We can see from these results that the

TABLE I. The values of metric factors D_1, D_2^b, D_3^b, D_2^s and D_3^s, for square lattices, random lattices, and their duals, with free and periodic boundary conditions in vertical and horizontal directions, respectively.

Lattices	Square	Planar random	Dual of planar random
D_1	1	1.166 ± 0.020	1.177 ± 0.016
D_2^b	1	1.164 ± 0.014	1.176 ± 0.015
D_3^b	1	1.512 ± 0.008	0.778 ± 0.002
D_2^s	1	1.186 ± 0.012	1.180 ± 0.014
D_3^s	1	1.062 ± 0.001	1.005 ± 0.002

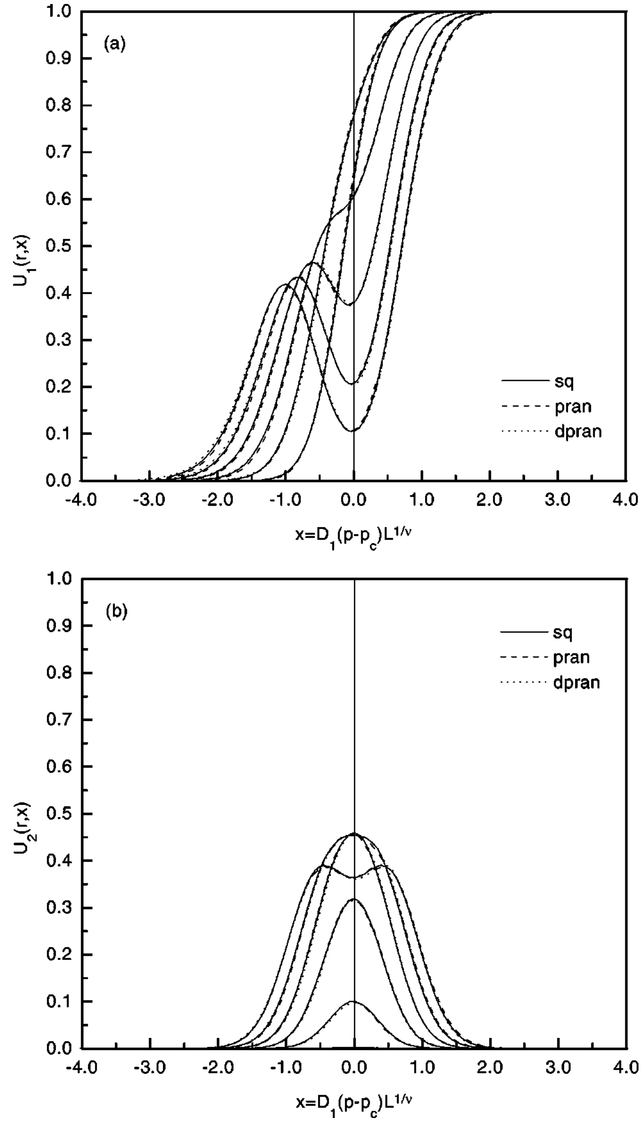

FIG. 7. $U_n(r,x)$ for a square lattice (solid curves), a planar random lattice (dashed curves), and the dual of a planar random lattice (dotted curves). (a) $n=1$; the intersection of the curves on the $x=0$ axis, from top to bottom, are for $r=L_1/L_2=1,2,3,\ldots,6$. (b) The width of curves from small to large are for $r=L_1/L_2=1,2,3,\ldots,6$.

scaled data for W_n can be described by a single scaling function $F_n(r,y)$ with $r=L_1/L_2$, and $F_n(r,y)$ for $n\geqslant2$ as a symmetric function of y. In Fig. 6 we plot $W_n(L_1,L_2,p)$ as a function of x for bond percolation on a 512×128 random lattice, the dual of a random lattice, and a square lattice, where $x=D_1(p-p_c)L^{1/\nu}$, with D_1 taken from Table I and $L=(L_1\times L_2)^{1/2}$. Figure 6 shows that the calculated results for each n can be well described by a single universal scaling function $U_n(x)$.

In Hu and Lin's paper [3], the scaling functions $F_n(r,y)$ were calculated for bond percolation on a square lattice for various values of the aspect ratio r. We will examine whether the same nonuniversal metric factors D_1 can be extended to different aspect ratios. We calculate W_n for $L_1\times L_2$ random lattices, the duals of random lattices, and square lattices with

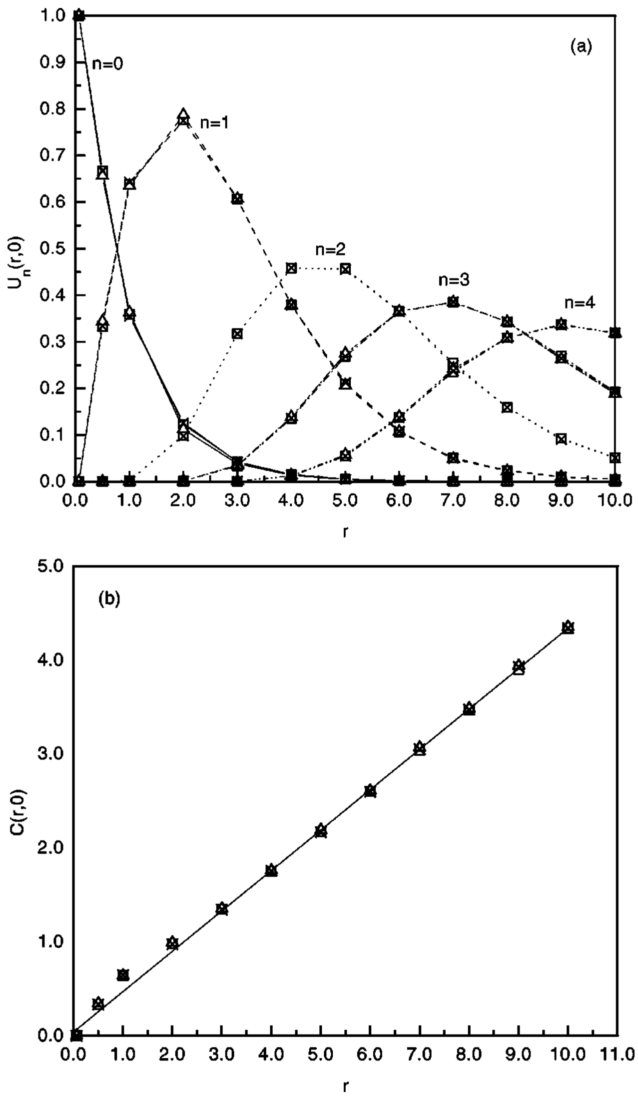

FIG. 8. (a) $U_n(r,0)$ as a function of $r=L_1/L_2$, for a number of percolating clusters from 0 to 4, and (b) $C(r,0)$ as a function of $r=L_1/L_2$, with the slope of the fitting line 0.43. The square lattice (□), the planar random lattice (△), and the dual of the planar random lattice (×), all have horizontal periodic boundary conditions.

$r=L_1/L_2=1,2,\ldots,6$, and determine the universal scaling functions $U_n(r,x)$, where $x=D_1(p-p_c)L^{1/\nu}$, and where D_1 is taken from Table I. The results for $n=1$ and 2 are shown in Figs. 7(a) and 7(b), respectively. We can see that the scaled data for each r can be described very well by a single universal scaling function. The results of $U_n(r,x)$ as a function of r, for $n=0,1,\ldots,4$ at the critical point $p=p_c$, are presented in Fig. 8(a), which shows that the three lattices provide similar results. We also calculate the average number of percolating clusters $C(r,x)$, defined by

$$C(r,x)=\sum_{n=1}^{\infty} U_n(r,x)n. \tag{12}$$

$C(r,0)$ for a random lattice, the dual of a random lattice, and a square lattice, as a functions of r, are shown in Fig. 8(b).

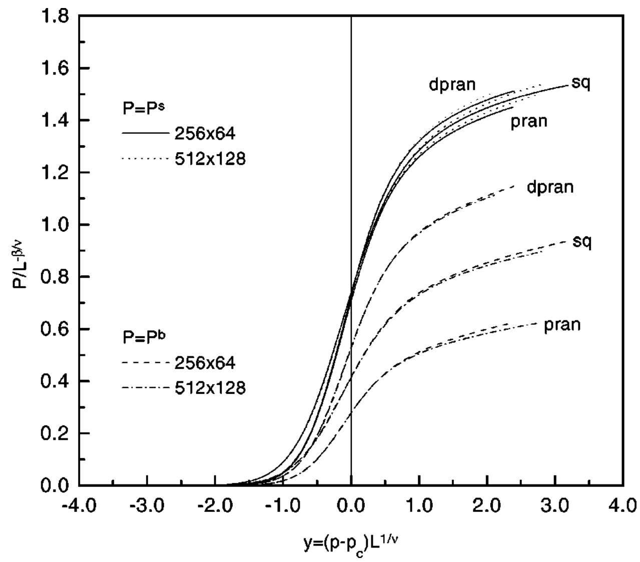

FIG. 9. The scaled results of $P(p,L)/L^{-\beta/\nu}$, in terms of bonds ($P=P^b$) and sites ($P=P^s$), for different types of lattice sizes 256×64 and 512×128, as a function of $y=(p-p_c)L^{1/\nu}$.

Figure 8(b) shows that $C(r,0)$ increases linearly with an increasingly large r, and that different lattices have the same slope of approximately 0.43.

Tomita *et al.* [17] obtained universal finite-size scaling functions of $\langle c\rangle_n$, $\langle c\rangle$, $f_n(c)$, and $f(c)$ for a bond-correlated percolation model, corresponding to an Ising model on planar regular lattices. It is of interest to extend such a study to bond random percolation on random lattices. From Eqs. (5), (6), and (11), the universal scaling functions of $\langle c\rangle$ and $\langle c\rangle_n$ can be expressed as

$$D_3\langle c(p,L)\rangle=L^{-\beta/\nu}G(z) \tag{13}$$

and

$$D_3\langle c(p,L)\rangle_n=L^{-\beta/\nu}G_n(z), \tag{14}$$

with $z=D_2(p-p_c)L^{1/\nu}$. At $p=p_c$, the universal scaling functions of $f(c)$ and $f_n(c)$ are expressed as

$$D_3^{-1}f(c)=L^{\beta/\nu}H(z') \tag{15}$$

and

$$D_3^{-1}f_n(c)=L^{\beta/\nu}H_n(z'), \tag{16}$$

with $z'=D_3cL^{\beta/\nu}$. To check the finite-size scaling and universality of these quantities, we use simulation results for 256×64 and 512×128 square lattices, planar random lattices, and their duals. In Ref. [30], the percolating probability P is defined in terms of the bond number in percolating clusters, and the nonuniversal metric factors $D_2=D_2^b$ and $D_3=D_3^b$ are used. To evaluate factors D_2^s and D_3^s, we adopt the same procedure as in Ref. [30], plotting $P^b/L^{-\beta/\nu}$ and $P^s/L^{-\beta/\nu}$ as functions of $y=(p-p_c)L^{1/\nu}$, as shown in Fig. 9. All the nonuniversal metric factors for the different types of lattices used in this paper are listed in Table I.

In Figs. 10(a) and 10(b), we plot $D_3P/L^{-\beta/\nu}$ and $D_3\langle c\rangle_n/L^{-\beta/\nu}$ as functions of $z=D_2(p-p_c)L^{1/\nu}$ for bond content and site content, respectively, with D_2 and D_3 taken

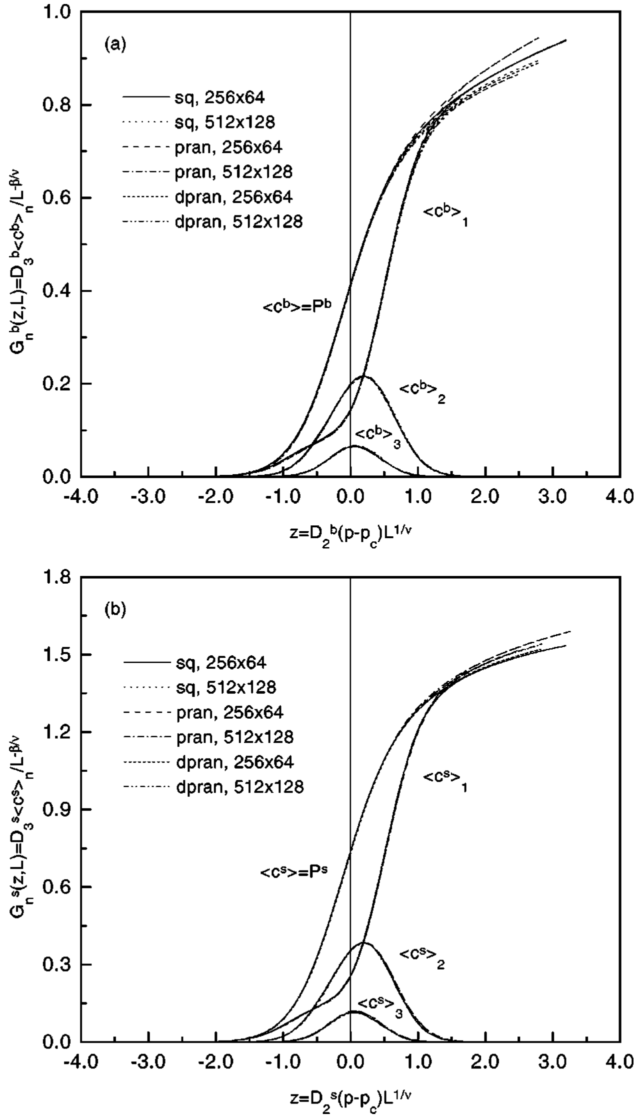

FIG. 10. The scaled results of $G_n(z,L)=D_3<c(p,L)>_n/L^{-\beta/\nu}$, as a function of $z=D_2(p-p_c)L^{1/\nu}$, for square lattices, planar random lattices, and their duals of sizes 256×64 and 512×128. (a) $c=c^b$, $D_2=D_2^b$, and $D_3=D_3^b$; (b) $c=c^s$, $D_2=D_2^s$, and $D_3=D_3^s$.

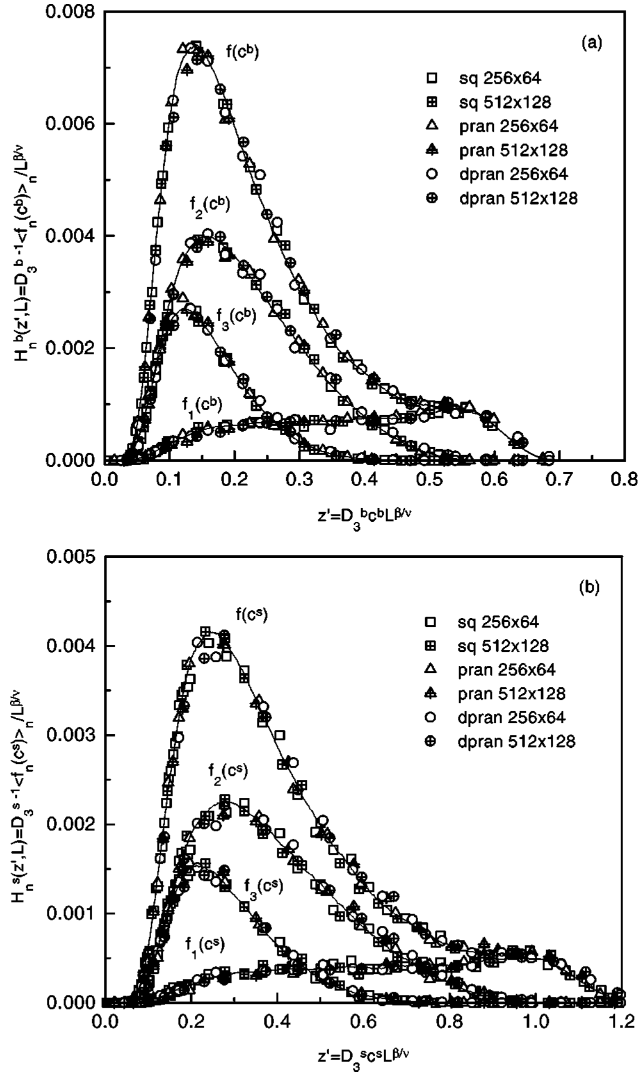

FIG. 11. The scaled results of $H_n(z',L)=D_3^{-1}f_n(c)/L^{\beta/\nu}$, as a function of $z'=D_3(p-p_c)L^{\beta/\nu}$ for square lattices, planar random lattices, and their duals of sizes 256×64 and 512×128, with fitting results (solid curves): (a) $c=c^b$ and $D_3=D_3^b$; (b) $c=c^s$ and $D_3=D_3^s$.

from Table I. At $p=p_c$, the scaled data $D_3^{-1}f(c)/L^{\beta/\nu}$ and $D_3^{-1}f_n(c)/L^{\beta/\nu}$, as functions of $z'=D_3cL^{\beta/\nu}$, are presented in Figs. 11(a) and 11(b), respectively, for the bond content and site content. Figures 10 and 11 show that the bond percolation processes on square lattices, random lattices, and their duals have universal finite-size scaling functions.

IV. SUMMARY AND DISCUSSION

Having used nonuniversal metric factors from Table I in this paper, we found that universal finite-size scaling functions for W_n (Figs. 6 and 7), $\langle c^b\rangle_n$ and P^b [Fig. 10(a)], $\langle c^s\rangle_n$ and P^s [Fig. 10(b)], $f_n(c^b)$ [Fig. 11(a)], and $f_n(c^s)$ [Fig. 11(b)]. Figure 7 includes results for different aspect ratios r, i.e., $6\geqslant r\geqslant 1$. The values of nonuniversal metric factors, D_1, D_2^b, and D_3^b of Table I are consistent with the corresponding values of Ref. [30], where the boundary conditions are different from the boundary conditions of the present paper. These results suggest that, in random lattices, the nonuniversal metric factors are also independent of the boundary conditions and aspect ratios, as in the case of regular lattices [3,26]. Please also note that D_1, D_2^b, and D_2^s in Table I are consistent within numerical errors, but D_3^b is not equal to D_3^s.

Many interesting problems are related to the properties of multiple percolating clusters. It is of interest to extend the study of the present paper to higher spatial dimensions. In particular, a further study of multiple percolating clusters in three dimensions could be related to an oil drilling problem.

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