Solution of the modified Helmholtz equation in a triangular domain and an application to diffusion-limited coalescence

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A new transform method for solving boundary value problems for linear and integrable nonlinear partial differential equations recently introduced in the literature is used here to obtain the solution of the modified Helmholtz equation $q_{xx}(x,y)+q_{yy}(x,y)-4\beta^2 q(x,y)=0$ in the triangular domain $0\leqslant x\leqslant L-y\leqslant L$, with mixed boundary conditions. This solution is applied to the problem of diffusion-limited coalescence, $A+A\rightleftharpoons A$, in the segment $(-L/2,L/2)$, with traps at the edges.

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I. INTRODUCTION

A new method for solving boundary value problems for linear and for integrable nonlinear partial differential equations (PDEs) has been introduced recently [1]. Here we apply this method to the equation

$$E_{xx}+E_{yy}+\gamma(-E_x+E_y)=0, \tag{1.1}$$

in the triangular domain $-L/2\leqslant x\leqslant y\leqslant L/2$, where $E(x,y)$ is a scalar function and γ is a positive constant. A solution of Eq. (1.1) in the semi-infinite wedge $0\leqslant x\leqslant y$ has been presented in [2]. Using the substitution $E(x,y)=1-e^{-\gamma/2(y-x)}q(x,y)$, Eq. (1.1) becomes the modified Helmholtz equation

$$q_{xx}+q_{yy}-4\beta^2 q=0, \quad \beta=\frac{\gamma}{\sqrt{8}}. \tag{1.2}$$

Equation (1.1) with $\gamma=v/D$ represents the steady state of the diffusion-limited reaction $A+A\rightleftharpoons A$ on the line, where the A particles diffuse with diffusion constant D, they merge immediately upon encounter, and split into two particles (the back reaction) at rate v [3–5]. $E(x,y)$ represents the probability that the interval (x,y) is empty. The concentration profile of the particles is related to $E(x,y)$ through $c(x)=-E_y(x,x)$. Suppose that we limit ourselves to the segment $-L/2\leqslant x\leqslant L/2$, then the domain of Eq. (1.1) is $-L/2\leqslant x\leqslant y\leqslant L/2$. The forward reaction is described by the boundary condition (BC) $E(x,x)=1$. If there are perfect traps at the edges, $x=\pm L/2$, one gets the BCs $E_x(-L/2,y)=0$, $E_y(x,L/2)=0$. These BCs transform into the following BCs for Eq. (1.2):

$$q(x,x)=0, \quad -\frac{L}{2}\leqslant x\leqslant\frac{L}{2},$$

$$\frac{\gamma}{2}q\left(-\frac{L}{2},y\right)+q_x\left(-\frac{L}{2},y\right)=0, \quad -\frac{L}{2}\leqslant y\leqslant\frac{L}{2},$$

$$-\frac{\gamma}{2}q\left(x,\frac{L}{2}\right)+q_x\left(x,\frac{L}{2}\right)=0, \quad -\frac{L}{2}\leqslant x\leqslant\frac{L}{2}.$$

We rotate and translate the (x,y) axes, with the mapping $(x,y)\mapsto(-y+L/2,x+L/2)$. Equation (1.2) remains invariant, but the domain is now $0\leqslant x\leqslant L-y\leqslant L$—the isosceles right triangle with vertices at (0,0), $(0,L)$, $(L,0)$—and the BCs become

$$q(x,L-x)=0, \quad 0\leqslant x\leqslant L, \tag{1.3a}$$

$$\frac{\gamma}{2}q(x,0)+q_y(x,0)=0, \quad 0\leqslant x\leqslant L, \tag{1.3b}$$

$$\frac{\gamma}{2}q(0,y)+q_x(0,y)=0, \quad 0\leqslant y\leqslant L. \tag{1.3c}$$

For the sake of generality, instead of the BC (1.3b), (1.3c) we consider

$$\frac{\gamma}{2}q(x,0)+q_y(x,0)=f(x), \quad 0\leqslant x\leqslant L, \tag{1.3b$'$}$$

$$\frac{\gamma}{2}q(0,y)+q_x(0,y)=f(y), \quad 0\leqslant y\leqslant L, \tag{1.3c$'$}$$

where $f(\cdot)$ is an arbitrary smooth function.

We will show that: (a) Eq. (1.2) with the BCs (1.3a), (1.3b′), (1.3c′) has a unique solution that can be expressed in closed form. (b) Equation (1.2) with the homogeneous BCs (1.3a), (1.3b), (1.3c) has only the trivial solution $q(x,y)=0$, i.e., the only steady state of the process $A+A\rightleftharpoons A$, in a segment demarcated by traps, is the vacuum—when there are no particles left—regardless of the magnitude of v, the rate of the back reaction $A\rightarrow A+A$. (c) For large back reaction rates, $\gamma L\gg 1$, the characteristic relaxation time to the empty, absorbing state grows exponentially as $(D/2v^2)e^{vL/2D}$.

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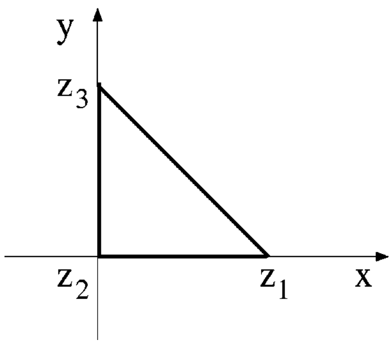

FIG. 1. Domain of the modified Helmholtz equation, Eq. (1.2).

Let $z=x+iy$, let a bar denote the complex conjugate ($\bar{z}=x-iy$), and let z_j denote the corners of the domain $0\leqslant x\leqslant L-y\leqslant L$ (see Fig. 1);

$$z_1=L,\quad z_2=0,\quad z_3=iL. \tag{1.4}$$

II. ANALYSIS OF THE INHOMOGENEOUS PROBLEM

It is shown in Ref. [6] that the general solution of the modified Helmholtz equation in the above domain can be represented as

$$q(x,y)=\frac{1}{2\pi i}\sum_{j=1}^{3}\int_{\ell_j}e^{ikz-i(\beta^2/k)\bar{z}}\rho_j(k)\frac{dk}{k},$$

$$0\leqslant x\leqslant L-y\leqslant L, \tag{2.1}$$

where ℓ_1, ℓ_2, ℓ_3, are the rays on the complex k plane defined by $\arg k=0$, $\pi/2$, $5\pi/4$, and oriented from zero to infinity (see Fig. 2), while the functions $\rho_j(k)$ are defined by

$$\rho_j(k)=\int_{z_{j+1}}^{z_j}e^{-ikz+i(\beta^2/k)\bar{z}}\left[\frac{1}{2}(q_x-iq_y)dz+i\frac{\beta^2}{k}qd\bar{z}\right],$$

$$k\in\mathbf{C},\quad j=1,2,3,\quad z_4=z_1. \tag{2.2}$$

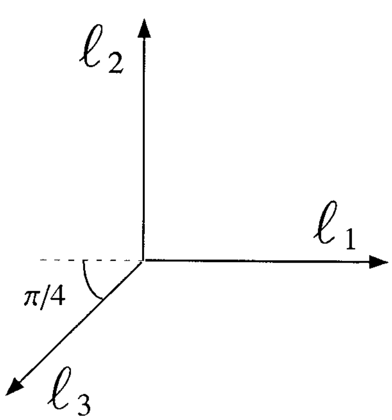

FIG. 2. The rays ℓ_j, in the complex plane, along which $q(x,y)$ is computed [Eq. (2.1)].

Using the boundary conditions (1.3) to simplify the expressions for $\rho_j(k)$, we find the following:

$$\rho_1(k)=-\frac{1}{2}q(0,0)+i\alpha(k)\psi_1(-ik)-iF(-ik),\quad k\in\mathbf{C}, \tag{2.3a}$$

$$\rho_2(k)=\frac{1}{2}q(0,0)+i\alpha(-ik)\psi_2(k)-iF(k),\quad k\in\mathbf{C}, \tag{2.3b}$$

$$\rho_3(k)=iE(k)\psi_3(-ke^{i(\pi/4)}),\quad k\in\mathbf{C}, \tag{2.3c}$$

where

$$\alpha(k)=\frac{1}{2}\left(\frac{\beta^2}{k}+k+\frac{\gamma}{2}\right),\quad E(k)=e^{(k+\beta^2/k)L},$$

$$F(k)=\tfrac{1}{2}\int_0^L e^{(k+\beta^2/k)y}f(y)dy, \tag{2.4}$$

and the unknown functions ψ_1, ψ_2, ψ_3 are defined by

$$\psi_1(k)=\int_0^L e^{(k+\beta^2/k)x}q(x,0)dx,$$

$$\psi_2(k)=\int_0^L e^{(k+\beta^2/k)y}q(0,y)dy,$$

$$\psi_3(k)=\int_0^{\sqrt{2}L}e^{(k+\beta^2/k)s}q_s\left(\frac{s}{\sqrt{2}},L-\frac{s}{\sqrt{2}}\right)ds. \tag{2.5}$$

Indeed, for the derivation of Eq. (2.3a) we use $z=x$, and we note that the boundary condition (1.3b′) implies

$$\frac{1}{2}(q_x(x,0)-iq_y(x,0))+i\frac{\beta^2}{k}q(x,0)$$
$$=\frac{1}{2}q_x(x,0)+i\left(\frac{\beta^2}{k}+\frac{\gamma}{4}\right)q(x,0)-\frac{i}{2}f(x);$$

integrating by parts the terms involving q_x we find Eq. (2.3a). The derivation of Eq. (2.3b) is similar, where we use the condition (1.3c′). For the derivation of Eq. (2.3c) we use $z=iL+x-ix$, and we note that the boundary condition $q(x,L-x)=0$ implies $q_x(x,L-x)-q_y(x,L-x)=0$.

In order to simplify the analysis, we have assumed that the *same* function f appears in the BCs (1.3b′) and (1.3c′). This implies that the PDE (1.2), the triangular domain, and the BCs (1.3a), (1.3b′), (1.3c′) are invariant under the reflection $x\leftrightarrow y$, thus $q(x,y)=q(y,x)$. Hence, $\psi_1(k)=\psi_2(k)$.

We introduce the following notations:

$$\psi_1(k)=\psi_2(k)=\varphi(k),\quad \psi_3(-ke^{-i\pi/4})=\psi(-),$$

$$\psi_3(-ke^{i\pi/4})=\psi(+),\quad e(k,z,\bar{z})=e^{ikz-i(\beta^2/k)\bar{z}}. \tag{2.6}$$

A. The analysis of the global relation

Equations (2.3) express $\rho_j(k)$ in terms of the unknown functions $\varphi(-ik)$, $\varphi(k)$, and $\psi(+)$. These functions satisfy the *global condition*: $\Sigma_{j=1}^{3}\rho_j(k)=0$ [6]. This equation, and its complex conjugate, are

$$\alpha(k)\varphi(-ik)+\alpha(-ik)\varphi(k)+E(k)\psi(+)=F(k)+F(-ik),\quad k\in\mathbf{C},\tag{2.7}$$

$$\alpha(k)\varphi(ik)+\alpha(ik)\varphi(k)+E(k)\psi(-)=F(k)+F(ik),\quad k\in\mathbf{C}.\tag{2.8}$$

Following Ref. [6] we supplement these equations with the equations obtained from Eqs. (2.7) and (2.8) by using the transformations in the complex k plane, which invariantly leave the *pairs* $\{\varphi(-ik),\varphi(ik)\}$ and $\{\psi(+),\psi(-)\}$. The first pair is invariant under $k\mapsto -k$, and the second pair is invariant under $\{k\mapsto -ik, k\mapsto ik\}$. Using the latter transformations, Eqs. (2.7) and (2.8) yield

$$\alpha(-ik)\varphi(-k)+\alpha(-k)\varphi(-ik)+E(-ik)\psi(-)=F(-ik)+F(-k),\quad k\in\mathbf{C},\tag{2.9}$$

$$\alpha(ik)\varphi(-k)+\alpha(-k)\varphi(ik)+E(ik)\psi(+)=F(ik)+F(-k),\quad k\in\mathbf{C}.\tag{2.10}$$

Equations (2.7)–(2.10) are invariant under $k\mapsto -k$, thus we do not obtain any additional equations using this transformation. Equations (2.7)–(2.10) are the basic equations needed for the determination of the unknown functions $\varphi(k)$, $\varphi(-ik)$, $\psi(+)$. The analysis of the basic equations leads to a matrix Riemann-Hilbert problem. However, in what follows we will show that this problem can be bypassed, and that $q(x,y)$ can be obtained using only *algebraic manipulations* of the basic equations.

Equations (2.7)–(2.10) imply that $\varphi(-ik)$, $\varphi(k)$, $\psi(+)$ can be expressed in terms of $\varphi(ik)$ and $\psi(-)$:

$$\varphi(-ik)=A(-ik)\varphi(ik)+\frac{E(k)}{\alpha(-k)\Delta(k)}[A(-ik)^2E(ik)-E(-ik)]\psi(-)+G_1(k),\tag{2.11a}$$

$$\varphi(k)=-\frac{\alpha(k)}{\alpha(ik)}\varphi(ik)-\frac{E(k)}{\alpha(ik)}\psi(-)+G_2(k),\tag{2.11b}$$

$$\psi(+)=\frac{A(-ik)E(k)+A(k)E(-ik)}{\Delta(k)}\psi(-)+G_3(k),\tag{2.11c}$$

where

$$A(k)=\frac{\alpha(k)}{\alpha(-k)},\quad \Delta(k)=E(k)+A(k)A(-ik)E(ik),\tag{2.12}$$

and the known functions $G_j(k)$, $j=1,2,3$, are defined in terms of f as follows:

$$G_1(k)=\frac{1}{\Delta(k)\alpha(-k)}\{[E(k)+A(-ik)E(ik)]\times[F(-ik)-A(-ik)F(ik)]+[1-A(-ik)][A(-ik)E(ik)F(k)+E(k)F(-k)]\}.\tag{2.13a}$$

$$G_2(k)=\frac{F(k)+F(ik)}{\alpha(ik)},\tag{2.13b}$$

$$G_3(k)=\frac{1}{\Delta(k)}\{[1-A(k)][F(-ik)-A(-ik)F(ik)]+[1-A(-ik)][F(k)-A(k)F(-k)]\}.\tag{2.13c}$$

Indeed, Eq. (2.11b) is Eq. (2.8). Eliminating $\varphi(-k)$ from Eqs. (2.9), (2.10), we find

$$\alpha(-k)\varphi(-ik)+E(-ik)\psi(-)-A(-ik)[\alpha(-k)\varphi(ik)+E(ik)\psi(+)]=F(-ik)+F(-k)-A(-ik)[F(ik)+F(-k)].\tag{2.14}$$

Replacing in this equation $\varphi(ik)$ by Eq. (2.8) and comparing with Eq. (2.7), we find Eq. (2.11c). Replacing $\psi(+)$ in terms of $\psi(-)$ in Eq. (2.14), using Eq. (2.11c), we find Eq. (2.11a).

Equation (2.1) expresses $q(x,y)$ in terms of $\rho_j(k)$, and Eqs. (2.3) and (2.11) express $\rho_j(k)$ in terms of the *unknown* functions $\varphi(ik)$, $\psi(-)$, and the known functions $G_j(k)$. The known functions give rise to the contribution

$$G(x,y)=\frac{1}{2\pi}\int_{\ell_1}e(k,z,\bar{z})[\alpha(k)G_1(k)-F(-ik)]\frac{dk}{k}+\frac{1}{2\pi}\int_{\ell_2}e(k,z,\bar{z})[\alpha(-ik)G_2(k)-F(k)]\frac{dk}{k}+\frac{1}{2\pi}\int_{\ell_3}e(k,z,\bar{z})E(k)G_3(k)\frac{dk}{k}.\tag{2.15}$$

In what follows we will show that, by using appropriate contour rotations, the integrals involving the functions $\varphi(ik)$, $\psi(-)$ can be evaluated in terms of residues. Furthermore, these residues can be computed in terms of the functions $G_j(k)$. For the justification of these rotations we use the following facts (see Fig. 3).

(i) $e(k,z,\bar{z})$, $e(k,z,\bar{z})E(k)$, $e(k,z,\bar{z})E(-ik)$, are bounded for $0<\arg k<\pi/2$, $\pi/2<\arg k<5\pi/4$, $5\pi/4<\arg k<2\pi$, respectively.

(ii) $E(-k)E(ik)$ and $\psi(-)$ are bounded for $-\pi/4<\arg k<3\pi/4$, while $E(k)E(-ik)\psi(-)$ is bounded for $3\pi/4<\arg k<7\pi/4$.

(iii) $\Delta(k)\sim E(k)$, $k\to 0$ and $k\to\infty$, in $-\pi/4<\arg k<3\pi/4$; $\Delta(k)\sim E(ik)$, $k\to 0$ and $k\to\infty$, in $3\pi/4<\arg k<7\pi/4$.

Indeed, since $x\geqslant 0$ and $y\geqslant 0$, $e(k,z,\bar{z})$ is bounded both at $k=0$ and $k=\infty$ in the first quadrant of the complex k plane. Since $-\pi/2<\arg(z-z_3)<-\pi/4$, it follows that if $\pi/2<\arg k<5\pi/4$, then $0<\arg k(z-z_3)<\pi$. Hence $\exp[ik(z-z_3)-i\beta^2/k(\bar{z}-\bar{z}_3)]$ is bounded both at $k=0$ and $k=\infty$; using $z_3=iL$, this exponential equals $e(k,z,\bar{z})E(k)$. Similar considerations apply to $e(k,z,\bar{z})E(-ik)$.

$\Delta(k)=E(ik)[E(k)E(-ik)+A(k)A(-ik)]$. If $-\pi/4<\arg k<3\pi/4$, $E(k)E(-ik)$ is exponentially large at $k=0$ and $k=\infty$, and $\Delta(k)\sim E(k)$. Similarly, if $3\pi/4<\arg k<7\pi/4$, $E(k)E(-ik)$ is exponentially small, and $\Delta(k)\sim E(ik)A(k)A(-ik)\sim E(ik)$.

$\psi(-)$ involves $-[ke^{-i\pi/4}+(\beta^2/k)e^{i\pi/4}]$, thus it is bounded for $-\pi/4<\arg k<3\pi/4$. Similarly for $E(k)E(-ik)\psi(-)$.

The contribution of the integral along ℓ_3, due to the terms involving $\psi(-)$ [see Eq. (2.11c)], gives rise to two integrals: one involving $e(k,z,\bar{z})A(k)E(-ik)E(k)\psi(-)/k\Delta(k)$, and one involving $e(k,z,\bar{z})A(-ik)E(k)^2\psi(-)/k\Delta(k)$. The first integral is bounded in $5\pi/4<\arg k<2\pi$, while the second integral is bounded in $\pi/2<\arg k<5\pi/4$. Indeed, the integrand of the first integral is dominated by

$$[e(k,z,\bar{z})E(-ik)][E(k)E(-ik)\psi(-)],$$

$$\frac{5\pi}{4}<\arg k<\frac{7\pi}{4};\quad [e(k,z,\bar{z})E(-ik)][\psi(-)],$$

$$\frac{7\pi}{4}<\arg k<2\pi,$$

and each of the brackets is bounded. Similarly, the integrand of the second integral is dominated by

$$[e(k,z,\bar{z})E(k)][\psi(-)],\quad \frac{\pi}{2}<\arg k<\frac{3\pi}{4};$$

$$[e(k,z,\bar{z})E(k)][E(k)E(-ik)\psi(-)],\quad \frac{3\pi}{4}<\arg k<\frac{5\pi}{4},$$

and each of the brackets is bounded.

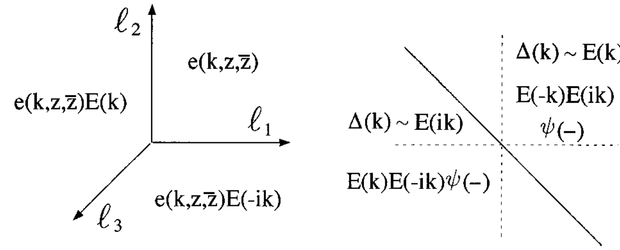

FIG. 3. Regions where $e(k,z,\bar{z})$, $e(k,z,\bar{z})E(k)$, $e(k,z,\bar{z})E(-ik)$, $\psi(-)$, $E(k)E(-ik)\psi(-)$ are bounded, and dominant behavior of $\Delta(k)$.

Hence, the integral along ℓ_3, due to the terms involving $\psi(-)$, equals an integral along ℓ_1 involving $e(k,z,\bar{z})A(k)E(-ik)E(k)\psi(-)/k\Delta(k)$, an integral along ℓ_2 involving $e(k,z,\bar{z})A(-ik)E(k)^2/k\Delta(k)$, and a contribution due to residues, which will be computed below [see Eqs. (2.19b) and (2.19c)]. Combining these integrals with the integrals due to $\varphi(-ik)$ and to $\varphi(k)$ [see Eqs. (2.11a) and (2.11b)], we find

$$J_1(x,y)=\frac{1}{2\pi}\int_{-\ell_2\cup\ell_1} e(k,z,\bar{z})\times\left[\frac{i}{2}q(0,0)+\alpha(k)A(-ik)\varphi(ik)\right]\frac{dk}{k},\tag{2.16}$$

$$J_2(x,y)=\frac{1}{2\pi}\int_{-\ell_2\cup\ell_1}\frac{e(k,z,\bar{z})}{\Delta(k)}A(k)A(-ik)^2E(k)\times E(ik)\psi(-)\frac{dk}{k}.\tag{2.17}$$

For k in the first quadrant of the complex k plane, $E(k)/\Delta(k)$ is dominated by 1, and each of the terms $e(k,z,\bar{z})$, $\varphi(ik)$, $E(ik)$, $\psi(-)$ is bounded. Thus, both J_1 and J_2 can be computed in terms of residues.

The definition of $A(k)$ implies

$$A(-ik)=-\frac{(k+\Lambda_1)(k+\Lambda_2)}{(k-\Lambda_1)(k-\Lambda_2)},$$

$$A(k)=-\frac{(k+\Lambda_1)(k-\Lambda_2)}{(k-\Lambda_1)(k+\Lambda_2)},\quad \Lambda_1=\frac{\gamma}{4}(1+i),$$

$$\Lambda_2=\frac{\gamma}{4}(-1+i),$$

so the poles of $A(-ik)$ and $A(k)$ occur at Λ_1, Λ_2, and at Λ_1, $-\Lambda_2$, respectively. Similarly, the poles of $A(ik)$ and $A(-k)$ occur at $-\Lambda_1$, $-\Lambda_2$, and $-\Lambda_1$, Λ_2, respectively. Using these facts it follows that

$$q(x,y)=G(x,y)+\sum_{j=1}^{3} R_j(x,y)+P(x,y), \qquad (2.18)$$

where $G(x,y)$ is defined by Eq. (2.15), $P(x,y)$ is the contribution to J_1 and J_2 due to the poles of $\alpha(k)A(-ik)$ and of $A(k)A(-ik)^2$, and the R_j are defined as follows:

$$R_1=i\sum_j \frac{e(\kappa_j,z,\overline{z})A(\kappa_j)A(-i\kappa_j)^2E(\kappa_j)E(i\kappa_j)}{\kappa_j\Delta'(\kappa_j)}\psi(\kappa_j), \qquad (2.19a)$$

$$R_2=-i\sum_j \frac{e(\lambda_j,z,\overline{z})A(-\lambda_j)E(\lambda_j)^2}{\lambda_j\Delta'(\lambda_j)}\psi(\lambda_j) +2\frac{e(\Lambda_2,z,\overline{z})}{\Delta(\Lambda_2)}E(\Lambda_2)^2\psi(\Lambda_2), \qquad (2.19b)$$

$$R_3=-i\sum_j \frac{e(\mu_j,z,\overline{z})A(\mu_j)E(-i\mu_j)E(\mu_j)}{\mu_j\Delta'(\mu_j)}\psi(\mu_j) -2\frac{e(-\Lambda_2,z,\overline{z})}{\Delta(-\Lambda_2)}E(-\Lambda_2)E(i\Lambda_2)\psi(-\Lambda_2), \qquad (2.19c)$$

$\psi(k)$, $\Delta'(k)$ denote

$$\psi(k)=\psi(-ke^{-i\pi/4}), \quad \Delta'(k)=\frac{d\Delta(k)}{dk}, \qquad (2.20)$$

and κ_j, λ_j, μ_j denote the zeros of $\Delta(k)$ in $0<\arg k<\pi/2$, $\pi/4<\arg k<5\pi/4$, $5\pi/4<\arg k<2\pi$, respectively. Multiplying Eq. (2.11c) by $\Delta(k)$ and evaluating the resulting expression at $k_j=\{\kappa_j,\lambda_j,\mu_j\}$, we find

$$\psi(k_j)=-\frac{[1-A(k_j)][F(-ik_j)-A(-ik_j)F(ik_j)]+[1-A(-ik_j)][F(k_j)-A(k_j)F(-k_j)]}{\delta(k_j)}, \quad \delta(k_j)\neq 0, \qquad (2.21)$$

where

$$\delta(k)=A(-ik)E(k)+A(k)E(-ik). \qquad (2.22)$$

Noting that $\alpha(k)=(k+\Lambda_1)(k-\Lambda_2)/2k$, $\alpha(ik)=-(k-\Lambda_1)(k-\Lambda_2)/2ik$, and evaluating Eq. (2.8) at $k=\Lambda_2$, we find $\psi(\Lambda_2)$. Similarly, evaluating Eq. (2.9) at $k=-\Lambda_2$ we find $\psi(-\Lambda_2)$:

$$\psi(\Lambda_2)=E(-\Lambda_2)[F(\Lambda_2)+F(i\Lambda_2)],$$
$$\psi(-\Lambda_2)=E(-i\Lambda_2)[F(\Lambda_2)+F(i\Lambda_2)]. \qquad (2.23)$$

The term $P(x,y)$ arises from $\alpha(k)A(-ik)$ in J_1, and $A(k)A(-ik)^2/\Delta(k)$ $[\Delta(k)\neq 0]$ in J_2, each of which has a simple pole at $k=\Lambda_1$. Evaluation of the pertaining residues yields

$$P(x,y)=2e(\Lambda_1,z,\overline{z})[\alpha(\Lambda_1)\varphi(i\Lambda_1)+E(\Lambda_1)\psi(\Lambda_1)].$$

Evaluating Eq. (2.8) at $k=\Lambda_1$, we find

$$\alpha(\Lambda_1)\varphi(i\Lambda_1)+E(\Lambda_1)\psi(\Lambda_1)=F(\Lambda_1)+F(i\Lambda_1).$$

Thus,

$$P(x,y)=2e(\Lambda_1,z,\overline{z})[F(\Lambda_1)+F(i\Lambda_1)]. \qquad (2.24)$$

In summary, assume that $\delta(k_j)\neq 0$, where k_j is a zero of $\Delta(k)$, and $\delta(k)$, $\Delta(k)$ are defined by Eqs. (2.12), (2.22), respectively. Then $q(x,y)$ is given by Eq. (2.18), where $G(x,y)$ is defined by Eq. (2.15), $P(x,y)$ is defined by Eq. (2.24), and $R_j(x,y)$, $j=1,2,3$, are defined by Eqs. (2.19), with $\psi(k_j)$, $\psi(\Lambda_2)$, $\psi(-\Lambda_2)$ defined by Eqs. (2.21), (2.23).

III. PHYSICAL PROBLEM

The physical problem corresponds to the homogeneous BCs, i.e., $f=0$. In this case Eq. (2.18) yields $q(x,y)=0$. Thus, we only need to consider the assumption that $\delta(k_n)\neq 0$. If this assumption is violated then the equations $\Delta(k_n)=0$ and $\delta(k_n)=0$ can be rewritten in the form

$$A(ik_n)^2E(-ik_n)^2=1, \qquad (3.1a)$$

$$A(-k_n)^2E(k_n)^2=1. \qquad (3.1b)$$

Equations (3.1) do not have a solution for generic values of γ. Indeed, consider first the limit of infinite back reaction rate, $\gamma L\to\infty$. Inspection of Eqs. (3.1) in this limit yields the asymptotic solution $k_\infty=\pm\Lambda_1, \pm\Lambda_2$. If there exists a steady state other than $q(x,y)=0$, then it would also exist for γL large but finite. We therefore seek solutions of (3.1) of the form $k=k_\infty+\epsilon$. Such solutions do not exist: Using $k_\infty=\Lambda_1$, Eq. (3.1a) yields $\epsilon=(1/4L)(-1+i)$—to first order in ϵ—while Eq. (3.1b) yields the contradictory result $\epsilon=(1/4L)(+1-i)$. The other values of k_∞ lead to similar contradictory results. Thus, the only solution to the physical problem is $q(x,y)=0$, which corresponds to the trivial case of the vacuum, when no particles are left in the system.

Finally, consider the relaxation of the system into the absorbing empty state. Instead of Eq. (1.1), we need to study

$$E_{xx}+E_{yy}+\gamma(-E_x+E_y)=E_\tau, \qquad (3.2)$$

where $\tau=Dt$ is a rescaled time parameter. We turn this into an eigenvalue problem, by writing $E(x,y,t)=1-e^{-\sigma\tau}e^{-\gamma/2(y-x)}q_\sigma(x,y)$. This results in an equation for q_σ identical to Eq. (1.2), valid over the same domain, but with

$4\beta^2=\frac{1}{2}\gamma^2-\sigma$. The BCs for this equation are identical to Eq. (1.3). We have already seen that the problem admits no zero eigenvalue: $q_0(x,y)=0$. The analysis for $\sigma>0$ proceeds along the same lines. Once again, the critical issue is whether there exist solutions of Eqs. (3.1). This time the asymptotic solution for $\gamma L\to\infty$ is $k_\infty=(\gamma/4)[\pm 1\pm i(1-4\sigma/\gamma)^{1/2}]$, $(\gamma/4)[\pm(1-4\sigma/\gamma)^{1/2}\pm i]$. A perturbation analysis shows that solutions exist for finite $\gamma L\ll 1$, provided that $\sigma\sim 2\gamma^2 e^{-\gamma L/2}$. The relaxation time to the empty state is therefore $(D\sigma)^{-1}=(D/2v^2)e^{vL/2D}$.

It is instructive to compare our analysis of $A+A\rightleftharpoons A$ to the mean-field result. The reaction-diffusion equation for the steady state of the process, in a segment demarcated by traps, is

$$D\rho_{xx}+k_1\rho-k_2\rho^2=0, \quad -L/2\leqslant x\leqslant L/2, \tag{3.3}$$

where $\rho(x)$ is the local particle density, k_1 is the rate of the back reaction $A\to A+A$, k_2 is the rate of $A+A\to A$, and the traps impose the BCs $\rho(\pm L/2)=0$. This equation predicts a transition from an empty state ($\rho=0$) to an active state ($\rho>0$), when k_1 exceeds a certain critical value [7,8]. Our exact analysis shows that in the actual system of one-dimensional coalescence the noise destroys the transition and the only existing steady state is the empty state. The nontrivial steady state of the mean-field case is echoed in the exponentially large relaxation time found for large back reaction rates. Although the lack of a transition cannot be established from numerical simulations, especially in view of the long relaxation times for γL large, previous work had suggested that a transition does not take place [9].

While the mean-field approach of Eq. (3.3) fails to describe the $A+A\rightleftharpoons A$ system on the line, it is interesting to speculate about higher dimensions. Imagine the process taking place in an infinite d-dimensional space $(x_1,x_2,\ldots,x_d)$ with absorbing walls at $x_1=\pm L/2$. Is there a dimension above which the mean-field description is valid? If so, what is that dimension? Unfortunately, the method of empty intervals employed in $d=1$ does not generalize for higher d. A related problem, of the propagation of the stable phase of the $A+A\rightleftharpoons A$ process into an unstable empty region, has been studied by numerical simulations [10]. It was found there that the mean-field picture (in this case, the well-known propagation of a Fisher front) applies above the critical dimension $d_c=3$, but the issue is still controversial. It would be interesting to see if $d_c=3$ is also the critical dimension in the present problem.

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