Defect statistics in the two-dimensional complex Ginzburg-Landau model

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The statistical correlations between defects in the two-dimensional complex Ginzburg-Landau model are studied in the defect-coarsening regime. In particular the defect-velocity probability distribution is determined and has the same high velocity tail found for the purely dissipative time-dependent Ginzburg-Landau (TDGL) model. The spiral arms of the defects lead to a very different behavior for the order parameter correlation function in the scaling regime compared to the results for the TDGL model.

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I. INTRODUCTION

We study here the statistical properties of a collection of point defects generated during the evolution of the two-dimensional complex Ginzburg-Landau equation (CGLE) [1–3]. We will be interested in that portion of the parameter space where the CGLE, driven by random initial conditions, has a regime of defect coarsening where the density of defects falls off with a power law in time. Our interest here is in the statistical properties of these defects and ultimately properties of the associated order parameter driven by the dynamics of the defects. Initially we will focus on the velocity distribution of the defects and the spatial correlations between defects.

The approach developed here is based on the use of a set of topological invariants applicable to a large set of systems that generate defects as a part of an ordering process. In particular one is led to a clean expression for the velocity of the defect cores in terms of derivatives of the order parameter field evaluated at the core position. This approach not only allows one to investigate equations of motion obeyed by individual defects, but opens up the possibility of treating the statistical properties of an ensemble of interacting defects. We have, from previous work [4] in the area of phase-ordering kinetics [5], the analog of the Maxwell velocity distribution for a collection of phase-ordering defects.

In the defect-coarsening regime for the CGLE, the defect density $\overline{n}(t)$ scales as $L^{-2}(t)$ where $L(t)$ is a characteristic length that grows with t, and t is the time of the evolution of the system starting with random initial conditions. In these circumstances, as shown in detail below, the defect-velocity probability distribution is given, as in the purely dissipative time-dependent Ginzburg-Landau (TDGL) case, by

$$P(\mathbf{V})=\Gamma\left(\frac{3}{2}\right)\left(\frac{1}{\pi\overline{v^2}}\right)(1+\mathbf{V}^2/\overline{v^2})^{-2}, \tag{1}$$

where the characteristic velocity $\overline{v}\approx L^{-1}$ is given explicitly below. Similarly, the defect-defect equal-time correlation function has the same form (see below) as found for the TDGL case. In the case of the correlations between defect densities at different times, we find some rather weak deviations from the results in the TDGL case.

These results for the statistical properties of the defects, inspire one to look at the order parameter correlations, using ideas that have been successful for treating the TDGL case. In this case we find results quite different from the TDGL case. This is due to both the spiral arms and the precessional motion characteristic of defects in the CGLE. The spiral arms render order parameter correlations shorter in range, compared to the TDGL case, and the order parameter correlation function shows the behavior $\approx\overline{n^3}(t)W(r/L(t))$. The precessional effects are predicted to be prominent in the two-time order parameter correlation function.

II. BACKGROUND

The complex Ginzburg-Landau equation can be written [1] in the form

$$\partial_t\psi=b\nabla^2\psi+(1-u|\psi|^2)\psi, \tag{2}$$

where ψ is a complex field and b and u are complex parameters. For the appropriate set of parameters (choice of b and u) we find on quenching from an initially disordered state that the CGLE generates a set of coarsening point defects. The characteristic distance between the defects increases with time due to the annihilation process between defects and antidefects [6].

For b and u real Eq. (2) reduces to the dissipative TDGL equation which is the most widely studied model for phase ordering [5]. If we set $b=u=i\eta$ in Eq. (2) and take η large, we find that after a simple gauge transformation Eq. (2) reduces to

$$-i\partial_t\psi=\nabla^2\psi+(1-|\psi|^2)\psi. \tag{3}$$

This equation, the nonlinear Schrödinger equation (NLSE) [7], gives a highly idealized description of the low temperature properties of a neutral superfluid. Unlike the TDGL system, the NLSE supports several conserved quantities. In particular the quantity $\int d^dr|\psi(\mathbf{r},t)|^2$ does not change with time. This model supports the same defects as the TDGL model, but the dynamics of the defects are quite different. Two oppositely charged vortices in the TDGL model move along the line connecting them toward annihilation. In the NLSE the same two vortices move at right angles to the line connecting them.

III. DEFECT CHARGED DENSITY AND VELOCITY FIELDS

The approach developed here allows for a direct connection between a set of field equations, like the complex Ginzburg-Landau equation satisfied by an order parameter field, and the equations of motion of the cores of a set of defects. It has only recently been understood, as discussed below, that these expressions for the defect velocity reduce to the same form as found in pattern forming studies using very different arguments.

The approach developed here is motivated by addressing the question: What is the probability of finding a defect a distance r from an antidefect? In work on phase-ordering kinetics [8] we developed methods that are convenient for handling such questions. A motivating factor was the realization that in treating statistical properties of defects one does not want to work with formal structures which require an explicit treatment in terms of the defect positions. This leads to problems of specification of initial conditions. Instead we looked for a way of implicitly finding the positions of the defects using the order parameter field ψ itself.

Let us consider the case of two dimensions where we have point defects. The case of line defects can also be treated [9,10] using these ideas but will be discussed elsewhere. The basic idea is that the positions of defects are located by the zeros [11–13,8] of the order parameter field ψ. Suppose, instead of the positions $\mathbf{r}_i(t)$ we want to write our description in terms of the zeros of $\psi(\mathbf{r},t)$. It is not difficult to see that the defect charged density has the two representations

$$\rho(\mathbf{r},t)=\delta(\boldsymbol{\psi}(\mathbf{r},t))\mathcal{D}(\mathbf{r},t)=\sum_{i=1}^{N} q_i\delta(\mathbf{r}-\mathbf{r}_i(t)) \tag{4}$$

where $q_i=\mathcal{D}(\mathbf{r}_i)/|\mathcal{D}(\mathbf{r}_i)|=\pm 1$, and $\mathcal{D}(\mathbf{r})$ is the Jacobian associated with the change of variables from the set of defect positions to the field ψ:

$$\mathcal{D}=\frac{i}{2!}\epsilon_{\mu_1\mu_2}\nabla_{\mu_1}\psi\nabla_{\mu_2}\psi^*, \tag{5}$$

where we sum over the μ_i, $\epsilon_{\mu_1\mu_2}$ is the two-dimensional antisymmetric tensor, and summation over repeated indices is implied. For later reference, the unsigned defect density is given by $n(\mathbf{r},t)=|\rho(\mathbf{r},t)|$.

For systems where only unit charges are present, ρ is the topological charge density. Notice that q_i is well defined even for systems like classical fluids where the circulation associated with a defect is not quantized.

The dynamical implications of this approach are simple. If indeed topological charge is conserved then we would expect the charge density to obey a continuity equation. It was shown in Ref. [4] that ρ satisfies a continuity equation of the form

$$\partial_t\rho=-\boldsymbol{\nabla}\cdot(\rho\mathbf{v}), \tag{6}$$

where the defect velocity field $\mathbf{v}$ is given explicitly by

$$\mathcal{D}v_\alpha=-\frac{i}{2}\sum_\beta \epsilon_{\alpha\beta}(\dot{\psi}\nabla_\beta\psi^*-\dot{\psi}^*\nabla_\beta\psi). \tag{7}$$

Here $\mathcal{D}$ is defined by Eq. (5) and we must remember that $\mathbf{v}$ is multiplied by the defect-core-locating δ function in ρ in Eq. (6). Equation (7) gives one an explicit expression for the defect-velocity field expressed in terms of derivatives of the order parameter. This expression for the defect velocity seems to be very general. Notice that we have not specified the form of the equation of motion for the order parameter, only that the order parameter be complex and $d=2$. For the CGLE, our expression for the defect velocity reduces to

$$\mathcal{D}v_\alpha=-\frac{i}{2}\sum_\beta \epsilon_{\alpha\beta}(b\nabla^2\psi\nabla_\beta\psi^*-b^*\nabla^2\psi^*\nabla_\beta\psi). \tag{8}$$

Does this expression for the velocity agree with our expectations for known cases? Let us assume that we have a defect of charge m at the origin of our two dimensional system and write the order parameter in the form $\psi=Re^{i\theta}$ ($R=r^{|m|}e^w$ and $\theta=m\phi+\theta_B$), where again r and ϕ are the cylindrical coordinates relative to the core at the origin. It is then a straightforward bit of calculus to show that the velocity given by Eq. (8) reduces to

$$v_\alpha=2b''\left(\nabla_\alpha\theta_B+\frac{m}{|m|}\sum_\beta \epsilon_{\alpha\beta}\nabla_\beta w\right) -2b'\left(\nabla_\alpha w-\frac{m}{|m|}\sum_\beta \epsilon_{\alpha\beta}\nabla_\beta\theta_B\right). \tag{9}$$

If we ignore the contributions due to the variation in the amplitude w, Eq. (9) reduces [14] to $v_\alpha=2b''\nabla_\alpha\theta_B+b'(m/|m|)\Sigma_\beta\epsilon_{\alpha\beta}\nabla_\beta\theta_B$. The first term is the only contribution in the NLSE case and states that a vortex moves with the local superfluid velocity [15]. The second term is the Peach-Koehler [16] term first found in this context by Kawasaki [17]. These are the results from the phase-field approach and lead, for example, to the same type of interaction between two vortices as found in fluids. The velocity of a single isolated vortex is zero. For a set of two isolated vortices one has the expected behavior for the TDGL and NLSE cases.

For our purposes here the more important point is to consider the work of Törnkvist and Schröder [18]. Using methods of differential geometry, they looked at the derivation of the form of the velocity of a defect in the case of the CGLE. They comment, ''The evolution of a system with (spiral) vortices may be described in terms of the defects, or filaments, along with values of the fields'' R and θ ''at positions away from the defects of filaments. Such a separation into collective coordinates and field variables is nontrivial, and the present work comprises the first exact treatment of this kind for a dissipative system.'' The final equation they obtain, in our notation here and for two-dimensional systems, is precisely given by Eq. (9). Thus the velocity given by Eq. (9) reproduces the most sophisticated results obtained using other methods.

IV. AUXILIARY FIELD METHOD

A. Overview

How can we use these expressions for ρ and $\mathbf{v}(\psi)$ to compute the measurable statistical properties of an evolving CGLE system? We will use a generalization of an approximate method which has led to good results for the TDGL case. The basic idea is to assume that there is mapping from the order parameter field onto an auxiliary field m that shares the same zeros as ψ in space. In particular, we require $\rho[\psi]=\rho[m]$ and $\mathbf{v}[\psi]=\mathbf{v}[m]$, where again we use the result that the velocity is multiplied by a defect-zero-finding δ function. These requirements are not very constraining since they only require that ψ be proportional to m for small m with corrections that are cubic in m. It has been convenient to think of $\mathbf{m}(\mathbf{x})$ as a two-vector whose magnitude gives the distance from $\mathbf{x}$ to a defect core. Thus, as discussed in more detail below, near the core we can take $\mathrm{Re}\,\psi=m_x$ and $\mathrm{Im}\,\psi=m_y$.

The main assumption [9] in the theory is that the field m is Gaussian and the variance in m is determined by requiring that the defect charge density continuity equation be satisfied on average:

$$\frac{\partial G_{\rho\rho}(12)}{\partial t_1}=-\boldsymbol{\nabla}_1\cdot\langle\rho(1)\mathbf{v}(1)\rho(2)\rangle\equiv G_{J\rho}(12), \tag{10}$$

where

$$G_{\rho\rho}(12)=\langle\rho(1)\rho(2)\rangle \tag{11}$$

and $\rho(1)=\rho(t_1,\mathbf{x}_1)$. With these assumptions and assumptions about the initial conditions, one can work out all of the statistical properties of the defects including $G_{\rho\rho}(12)$ and the defect-velocity probability distribution function defined by

$$\bar{n}P(\mathbf{V})\equiv\langle|\rho|\,\delta(\mathbf{V}-\mathbf{v}[\psi])\rangle. \tag{12}$$

Corrections to this Gaussian approximation can be investigated using methods of the type developed in Refs.[19,20].

The procedure then is to first compute $G_{\rho\rho}(12)$ and $G_{J\rho}(12)$ assuming that m is a Gaussian field. This will give $G_{\rho\rho}(12)$ and $G_{J\rho}(12)$ as functions of the auxiliary field correlation function

$$C_{\alpha\beta}(12)=\langle m_\alpha(1)m_\beta(2)\rangle, \tag{13}$$

where α and β take on the values x and y. Inserting these results for $G_{\rho\rho}(12)$ and $G_{J\rho}(12)$ back into Eq. (10) gives an equation for $C(12)$. It will turn out that this equation for $C(12)$ can be solved analytically. This result can then be fed back into the result for $G_{\rho\rho}(12)$ to obtain an explicit expression for the defect density correlation function. As part of this calculation we obtain the average defect density $\bar{n}=\langle|\rho|\rangle$. Finally we can carry out the average over the Gaussian variable m to obtain $P(\mathbf{V})$ as a function of $C(12)$, and in turn obtain an explicit expression for $P(\mathbf{V})$.

B. Expressing $G_{\rho\rho}$ in terms of C

The defect density in the defect-defect correlation function defined by Eq. (11) can be written explicitly in terms of the Gaussian auxiliary field $\mathbf{m}$ in the form

$$\rho(1)=\frac{1}{2}\epsilon_{\mu_1\mu_2}\epsilon_{\nu_1\nu_2}\nabla_{\mu_1}m_{\nu_1}(1)\nabla_{\mu_2}m_{\nu_2}(1)\,\delta(\mathbf{m}(1)) \tag{14}$$

and we sum over all the indices ν and μ. In the isotropic case, worked out previously, the evaluation of $G_{\rho\rho}$ for the n-vector model for the general case of $n=d$ was facilitated by the decomposition of the average for $G_{\rho\rho}$ into a product of averages corresponding to each component. This decomposition is not possible here because the complex coefficients in the CGLE couple the components of the order parameter as the system evolves. We need a more general approach. This more general approach involves using the general identity valid for Gaussian fields:

$$\langle m_\nu(1)F[\mathbf{m}]\rangle=\sum_{\nu'}\int dt_2d^2x_2C_{\nu\nu'}(12)\left\langle\frac{\delta}{\delta m_{\nu'}(2)}F[\mathbf{m}]\right\rangle. \tag{15}$$

Using this result for all of the fields in $G_{\rho\rho}$ acted upon by a gradient in Eq. (11), one can bring all the gradients outside the average. This generates many terms that are products of the matrix C and averages proportional to the quantities

$$\mathcal{G}(12)=\langle\delta(\mathbf{m}(1))\delta(\mathbf{m}(2))\rangle \tag{16}$$

and

$$\mathcal{G}_{\nu_1\nu_2}(12)=\left\langle\frac{\partial}{\partial m_{\nu_1}(1)}\delta(\mathbf{m}(1))\frac{\partial}{\partial m_{\nu_2}(2)}\delta(\mathbf{m}(2))\right\rangle, \tag{17}$$

and similar higher order derivatives of the δ functions which do not contribute to the final result.

A key assumption in the evaluation of $G_{\rho\rho}$ is that the system is isotropic in space and we can write

$$C_{\nu\nu'}(12)=C_{\nu\nu'}(r,t_1t_2), \tag{18}$$

$$\nabla_\mu^{(1)}C_{\nu\nu'}(12)=C'_{\nu\nu'}(12)\hat{r}_\mu, \tag{19}$$

and

$$\nabla_\mu^{(1)}\nabla_{\mu'}^{(2)}C_{\nu\nu'}(12)=-[C^L_{\nu\nu'}(12)-C^T_{\nu\nu'}(12)]\hat{r}_\mu\hat{r}_{\mu'}-C^T_{\nu\nu'}(12)\,\delta_{\mu\mu'}, \tag{20}$$

where $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2$, and

$$C^L_{\nu\nu'}(12)\equiv C''_{\nu\nu'}(12), \tag{21}$$

$$C^T_{\nu\nu'}(12)\equiv\frac{1}{r}C'_{\nu\nu'}(12), \tag{22}$$

and the primes indicate derivatives with respect to r. Using these results one can then carry out the sums over the spatial coordinate labels, the μ's in $G_{\rho\rho}$, to obtain

$$G_{\rho\rho}(12)=G^{(1)}_{\rho\rho}(12)+G^{(2)}_{\rho\rho}(12), \tag{23}$$

where

$$G^{(1)}_{\rho\rho}(12)=\mathcal{G}(12)\epsilon_{\nu_1\nu_2}\epsilon_{\nu'_1\nu'_2}C^{L}_{\nu_1\nu'_1}(12)C^{T}_{\nu_2\nu'_2}(12) \tag{24}$$

and

$$G^{(2)}_{\rho\rho}(12)=-\epsilon_{\nu_1\nu_2}\epsilon_{\nu'_1\nu'_2}\mathcal{G}_{\sigma_2\sigma_1}(12)\times C'_{\nu_1\sigma_1}(12)C'_{\sigma_2\nu'_2}(12)C^{T}_{\nu_2\nu'_1}(12). \tag{25}$$

It is easy to evaluate, using the results from the Appendix, the remaining averages over the auxiliary field:

$$\mathcal{G}_{\sigma_2\sigma_1}(12)=D^2C_{\sigma_2\sigma_1}(12)\mathcal{G}(12), \tag{26}$$

$$\mathcal{G}(12)=\frac{D^2}{(2\pi)^2}, \tag{27}$$

where D is defined by Eq. (A10). Expressing $C_{\nu\nu'}(12)$ in terms of $C_0(12)$ and $\Delta(12)$, as given by Eq. (A2), and doing the sums over the ν's, we find after some rearrangement the result for the defect density correlation function:

$$G_{\rho\rho}(12)=\frac{1}{(2\pi)^2}\frac{1}{r}\frac{d}{dr}(Q\gamma_T^2), \tag{28}$$

where

$$\gamma_T^2=(1-f_T^2)^{-1}, \tag{29}$$

$$f_T=\sqrt{f_0^2+\Delta_0^2}, \tag{30}$$

$$f_0=\frac{C_0}{\sqrt{S_0(1)S_0(2)}}, \tag{31}$$

$$\Delta_0=\frac{\Delta}{\sqrt{S_0(1)S_0(2)}}, \tag{32}$$

where $S_0(i)=C_0(ii)$ and

$$Q=(f'_0)^2+(\Delta'_0)^2. \tag{33}$$

We still need to determine the auxiliary field correlation functions C_0 and Δ. It is easy to see that the result given by Eq. (28), in the isotropic limit where $\Delta=0$, reduces to the result first reported by Halperin [13]

$$G_{\rho\rho}(12)=\frac{1}{r}\frac{d}{dr}(h^2), \tag{34}$$

where

$$h=\frac{\gamma_T f'_0}{2\pi}. \tag{35}$$

C. Satisfying conservation of topological charge

The calculation of the current contribution of $G_{J\rho}$ on the right-hand side of Eq. (10) is much the same as for $G_{\rho\rho}$ except for terms that involve the on-site correlation function

$$S^{(2)}(1)=\frac{1}{2}\langle[\nabla\mathbf{m}(1)]^2\rangle=-[\nabla^2C_0(r,t_1t_1)]_{r=0}. \tag{36}$$

$G_{J\rho}$ is also proportional to the factor $[1/(2\pi)^2](1/r)d/dr$ and, after performing an integration over r, we obtain the averaged conservation law, given by Eq. (10), which can be rewritten as

$$\frac{\partial}{\partial t_1}(Q\gamma_T^2)=2b'M+2b''N, \tag{37}$$

where

$$M=\gamma_T^4Q(\omega_0(1)+f_0\nabla^2f_0+\Delta_0\nabla^2\Delta_0)+\gamma_T^2(f'_0\nabla^2f'_0+\Delta'_0\nabla^2\Delta'_0) \tag{38}$$

and

$$N=\gamma_T^4Q(f_0\nabla^2\Delta_0-\Delta_0\nabla^2f_0)+\gamma_T^2(f'_0\nabla^2\Delta'_0+\Delta'_0\nabla^2f'_0), \tag{39}$$

and we have introduced the time-dependent quantity

$$\omega_0(1)=\frac{S^{(2)}(1)}{S_0(1)}=-[\nabla^2f_0(r,t_1t_1)]_{r=0}. \tag{40}$$

Equation (37) looks very complicated but simplifies if we replace f_0 and Δ_0 with

$$f_0=f_T\cos\Omega \tag{41}$$

and

$$\Delta_0=f_T\sin\Omega. \tag{42}$$

Then Eq. (37) can be rewritten as

$$\gamma_Tf'_T[\gamma_T(2\dot{f}_T-R)]'+\gamma_T^4f_T(\Omega')^2(2\dot{f}_T-R)+\gamma_T^2\Omega'[2f_T^2\dot{\Omega}'-f'_TS+f_TS']=0, \tag{43}$$

where

$$R=2b'\omega_0(1)f_T+2b'A+2b''B, \tag{44}$$

$$S=-2b'B+2b''A, \tag{45}$$

and

$$A=\nabla^2f_T-f_T(\nabla\Omega)^2, \tag{46}$$

$$B=2\nabla f_T\cdot\nabla\Omega+f_T\nabla^2\Omega. \tag{47}$$

A solution to Eq. (43) is given by

$$2\dot{f}_T = R \tag{48}$$

and

$$2f_T^2\dot{\Omega}' = f_T'S - f_T S'. \tag{49}$$

This last equation can be reduced to

$$2f_T\dot{\Omega} = -S. \tag{50}$$

The set of coupled equations given by Eqs. (48) and (50) are equivalent to the equations for f_0 and Δ_0 given by

$$\dot{f}_0 = b'(\omega_0(1) + \nabla^2)f_0 + b''\nabla^2\Delta_0 \tag{51}$$

and

$$\dot{\Delta}_0 = b'(\omega_0(1) + \nabla^2)\Delta_0 - b''\nabla^2 f_0. \tag{52}$$

This is the set of equations that must be solved self-consistently to obtain the unknown quantities f_0, Δ_0, and $\omega(1)$.

D. Auxiliary field correlation function

Equations (51) and (52) are reduced to a set of differential equations in time if we Fourier transform in space and put in the time labels explicitly:

$$\frac{\partial}{\partial t_1} f_0(\mathbf{q},t_1t_2) = \alpha(q,t_1)f_0(\mathbf{q},t_1t_2) - \beta_q\Delta_0(\mathbf{q},t_1t_2), \tag{53}$$

$$\frac{\partial}{\partial t_1} \Delta_0(\mathbf{q},t_1t_2) = \alpha(q,t_1)\Delta_0(\mathbf{q},t_1t_2) + \beta_q f_0(\mathbf{q},t_1t_2), \tag{54}$$

where

$$\alpha(q,t_1) = b'(\omega_0(t_1) - q^2) \tag{55}$$

and

$$\beta_q = b''q^2. \tag{56}$$

Equations (53) and (54) need to be solved together with the symmetry condition

$$f(\mathbf{q},t_1,t_2) = f_0(\mathbf{q},t_1,t_2) + i\Delta_0(\mathbf{q},t_1,t_2) = f^*(-\mathbf{q},t_2,t_1) \tag{57}$$

and the initial condition

$$f(\mathbf{q},t_0,t_0) = 2\pi\ell^2 e^{-(q\ell)^2/2} \equiv g(q). \tag{58}$$

This particular choice of initial conditions, corresponding to an initial correlation length ℓ, is very convenient since all integrals can be carried out analytically for all times. Finally we must remember the normalization that follows from the definition of $f(12)$ given by Eqs. (31) and (32):

$$f(11) = \int \frac{d^2q}{(2\pi)^2} f(\mathbf{q},t_1,t_1) = 1. \tag{59}$$

It is not difficult to construct the appropriate solution given by

$$f(\mathbf{q},t_1,t_2) = R(t_1,t_0)R(t_2,t_0)g(q) \times e^{-b'q^2(t_1+t_2-2t_0)}e^{i\beta_q(t_1-t_2)}, \tag{60}$$

where

$$R(t_1,t_0) = \exp\left(b'\int_{t_0}^{t_1} d\tau\,\omega_0(\tau)\right). \tag{61}$$

It is straightforward to take the inverse Fourier transform of Eq. (60) with the result

$$f(12) = R(t_1,t_0)R(t_2,t_0)\left(\frac{\ell^2}{\tilde{L}^2}\right)e^{-(r/\tilde{L})^2/2}, \tag{62}$$

where

$$\tilde{L}^2 = \ell^2 + 4b'T - 2ib''(t_1 - t_2) \tag{63}$$

and

$$T = \frac{t_1 + t_2}{2}. \tag{64}$$

We must stop here and satisfy the constraint given by Eq. (59). We have from Eq. (62)

$$1 = R^2(t_1,t_0)\left(\frac{\ell^2}{L^2}\right), \tag{65}$$

where

$$L^2(t_1) = \tilde{L}^2(t_1,t_1) = \ell^2 + 4b't_1. \tag{66}$$

Equation (65) serves as an equation for $\omega_0(t_1)$ which can be easily solved to give

$$\omega_0(t_1) = \frac{2}{L^2(t_1)} = \frac{2}{\ell^2 + 4b't_1}. \tag{67}$$

Using Eq. (65) to express $R(t_1,t_0)$ in terms of $L(t_1)$, we find

$$f(12) = \Phi(t_1,t_2)\frac{1}{1-i\omega}e^{-(r/\tilde{L})^2/2}, \tag{68}$$

where

$$\Phi(t_1,t_2) = \frac{L(t_1)L(t_2)}{L^2(T)} \tag{69}$$

and

$$\omega = \frac{2b''(t_1 - t_2)}{L^2(T)}. \tag{70}$$

This last definition implies

$$\tilde{L}^2 = L^2(T)(1 - i\omega) \tag{71}$$

and

$$f(12) = \Phi(t_1, t_2)\frac{e^{-iz}}{1 - i\omega}e^{-y^2/2}, \tag{72}$$

where

$$y^2 = \frac{x^2}{1 + \omega^2}, \tag{73}$$

$$x = r/L(T), \tag{74}$$

and

$$z = \frac{1}{2}\omega y^2. \tag{75}$$

There are a number of comments relevant to this result for $f(12)$ given by Eq. (68). First note that there is consistency between the definition of $\omega_0(t)$ given by Eq. (40) and the solution for f that leads to Eq. (67). For equal times $t_1 = t_2 = t$, we have

$$f(r,t) = e^{-x^2/2}, \tag{76}$$

which is of the same form as in the purely dissipative case [20,21] with a characteristic length $L \approx \sqrt{b' t}$. If we look at the on-site $r = 0$ autocorrelation function,

$$f(0,t_1,t_2) = \Phi(t_1, t_2)\frac{1 + i\omega}{1 + \omega^2}, \tag{77}$$

we can write for $t_1, t_2 \gg t_0$,

$$\Phi(t_1, t_2) = \left(\frac{\sqrt{t_1 t_2}}{T}\right)^{\lambda_0}. \tag{78}$$

For $t_1 \gg t_2$, ω approaches a constant and the nonequilibrium exponent λ_0 for Φ also governs $f(0,t_1,t_2)$ and is given by $\lambda_0 = 1$, which is the same [20] as for the TDGL case for $n = d = 2$.

The main result here is that for nonequal times the auxiliary field correlation function shows an oscillatory behavior. One of our chief goals below is discuss the possibility of observing this phenomenon. We note here that $f(12)$ does obey a form of scaling for $t_1, t_2 \gg t_0$:

$$f(12) = f(x,\tau) = \Phi(\tau)\frac{e^{-iz(x,\tau)}}{1 - i\omega(\tau)}e^{-(1/2)x^2/[1 + \omega^2(\tau)]}, \tag{79}$$

where $\tau = t_1/t_2$,

$$\Phi(\tau) = \frac{2\sqrt{\tau}}{1 + \tau}, \tag{80}$$

$$\omega(\tau) = \frac{b''}{b'}\left(\frac{\tau - 1}{\tau + 1}\right), \tag{81}$$

and

$$z(x,\tau) = \frac{1}{2}\frac{\omega(\tau)x^2}{1 + \omega^2(\tau)}. \tag{82}$$

Rather than discussing this result for $f(12)$ in more detail, it is prudent to remember that $f(12)$ is not itself directly observable. Thus let us turn back to observables and their dependence on $f(12)$. We delay discussing the details of the oscillations in $f(12)$ until after discussing how this feeds back into the determination of observables.

V. DEFECT-DEFECT CORRELATION FUNCTION

A. General result

Given the explicit solution for $f(12)$, Eq. (79), we can return to the evaluation of the density correlation function $G_{\rho\rho}(12)$ given by Eq. (28). The input we need for its determination is $\gamma_T^{-2} = 1 - F^2$, where

$$F^2 = |f|^2 = \frac{\Phi^2}{1 + \omega^2}e^{-x^2/(1 + \omega^2)} \tag{83}$$

and

$$Q = (f_0')^2 + (\Delta_0')^2 = \frac{x^2 F^2}{L^2(1 + \omega^2)}. \tag{84}$$

Inserting these results for γ_T and Q back into Eq. (28) gives

$$G_{\rho\rho}(12) = \frac{\Phi^2(\tau)}{2\pi^2 L^4(T)[1 + \omega^2(\tau)]}g\left(\frac{x^2}{1 + \omega^2(\tau)}\right), \tag{85}$$

where

$$g(s) = \frac{e^s(1 - s) - \Phi^2(\tau)}{[e^s - \Phi^2(\tau)]^2}. \tag{86}$$

In analyzing $G_{\rho\rho}(12)$ we must be careful to distinguish the equal-time case from the unequal-time case.

B. The equal-time case

If $t_1 = t_2 = t$ and $\tau = 1$, the density correlation function can be written as

$$G_{\rho\rho}(r,t) = \frac{1}{2\pi^2 L^4(t)}g(x), \tag{87}$$

where

$$g(x)=\frac{e^{x^2}(1-x^2)-1}{(e^{x^2}-1)^2}. \quad (88)$$

This is the same result found in the purely dissipative case. It is known [13] that the conservation of topological charge for equal times requires one to include in the defect-defect correlation function the correlation of a defect with itself:

$$\tilde{G}_{\rho\rho}(r,t)=\delta(\mathbf{r})\overline{n(t)}+G_{\rho\rho}(r,t), \quad (89)$$

where $\overline{n(t)}$ is the average defect density. Then conservation of topological charge is given by

$$\int d^2r\,\tilde{G}_{\rho\rho}(r,t)=0. \quad (90)$$

Inserting Eq. (89) into Eq. (90) gives

$$\overline{n(t)}=-\int d^2r\,G_{\rho\rho}(r,t). \quad (91)$$

However, using the form given by Eq. (28) we can do the integral in Eq. (91) and obtain for the average defect density

$$\overline{n(t)}=\lim_{r\to 0}\frac{Q\gamma_T^2}{2\pi}=\frac{1}{2\pi L^2(t)}. \quad (92)$$

This is the expected result if scaling holds. One can also find $\overline{n(t)}$ by direct computation and obtain

$$\overline{n(t)}=\frac{\omega_0(t)}{4\pi}, \quad (93)$$

where ω_0 is defined by Eq. (40) and given in this approximation by Eq. (67). We see that the two determinations of $\overline{n(t)}$ agree.

C. The unequal-time case

For the case $\tau\neq 1$, we have that the conservation of topological charge holds directly for $G_{\rho\rho}(12)$ since

$$\int d^2r\,G_{\rho\rho}(12)=-\lim_{r\to 0}\frac{Q\gamma_T^2}{2\pi}=0. \quad (94)$$

The final step follows since $Q\approx r^2$ for small r and γ_T^2 is regular in this limit. If we set $r=0$ in $G_{\rho\rho}(12)$ given by Eq. (85) we obtain

$$G_{\rho\rho}(0,t_1,t_2)=\frac{1}{2\pi^2L^4(T)}\frac{1}{[1+\omega^2(\tau)]}\frac{4\tau}{(1-\tau)^2}. \quad (95)$$

We see that this quantity blows up at $\tau\to 1$ signaling the existence of the δ function at $r=0$ obtained for equal times. Thus we see that the limits $r\to 0$ and $\tau\to 1$ do not commute.

VI. DEFECT VELOCITY PROBABILITY DISTRIBUTION FUNCTION

The defect-velocity probability distribution function is defined by

$$\overline{n(t)}P(\mathbf{V})\equiv\langle|\rho(\psi)|\,\delta(\mathbf{V}-\mathbf{v}[\psi])\rangle=\langle|\rho(m)|\,\delta(\mathbf{V}-\mathbf{v}[m])\rangle. \quad (96)$$

One of the main results from the last section is that at equal times the auxiliary field probability distribution is isotropic and has the same form as in the purely dissipative case. This means that we obtain the same result here as found in Ref. [4] and given earlier by Eq. (1), where the characteristic velocity $\overline{v(t)}$ is given by

$$\overline{v(t)}=2(b')^2\frac{S^{(4)}(t)}{\omega_0(t)} \quad (97)$$

with

$$S^{(4)}(t)=\nabla^4 f(r,t)|_{r=0}-\omega_0^2(t), \quad (98)$$

and using the explicit results for $f(r,t)$ given by Eq. (76) we obtain

$$\overline{v(t)}=\frac{4(b')^2}{L^2(t)}. \quad (99)$$

The result for $P(\mathbf{V})$ given by Eq. (1) indicates that the probability of finding a defect with a large velocity decreases with time. However, since this distribution falls off only as V^{-4} for large V only the first moment beyond the normalization integral exists. This seems to imply the existence of a source of large velocities. Assuming that the large velocities of defects can be associated with the final collapse of a defect structure (defect-antidefect pair annihilation for point defects), Bray [22] used general scaling arguments to obtain the same large velocity tail given by Eq. (1).

One probe of the defect dynamics is to study the correlation between two defects including the correlation between their velocities. The two-defect velocity probability distribution, $P[\mathbf{V}_1,\mathbf{V}_2,\mathbf{r}]$ gives the probability that two defects separated by a distance r have velocities $\mathbf{V}_1,\mathbf{V}_2$. This quantity was determined in Ref. [23] and since it is an equal-time quantity the results found there hold here. The physical results from the calculation [23] of this quantity for the TDGL model, carried out in detail for $n=d=2$ using the same approximations as indicated above, are relatively simple to state. For a given separation r, the most probable configuration corresponds, as expected, to a state with zero total velocity and a nonzero relative velocity only along the axis connecting the defects: $\mathbf{V}_1=-\mathbf{V}_2\equiv v\hat{x}$. Moreover, there is a definite most probable nonzero value for $v=v_{\max}$ for a given value of r. The most striking feature of these results is that for small r the most probable velocity goes as $v_{\max}=\kappa/r$ and $\kappa=2.19$ in dimensionless units defined in Ref. [23].

VII. ORDER PARAMETER CORRELATION FUNCTION

Thus far we have focused on the statistical properties of the defects in the system and found results for $G_{\rho\rho}$ and $P(\mathbf{V})$ which are very similar to the purely dissipative case. Only rather small differences arise when one looks at unequal times. Unfortunately neither of these quantities probes the full phase dependence of the auxiliary field correlation function which shows interesting oscillations in space at unequal times. We show here that this phase dependence may be probed via the order parameter correlation function evaluated at unequal times. Indeed this quantity, within the approximate treatment given here, is quite different from the purely dissipative case even at equal times.

The order parameter correlation function is defined by

$$C_{\psi}(12)=\langle \psi^{*}(2)\psi(1)\rangle, \tag{100}$$

and our approach toward its evaluation will be to find the relationship between the order parameter and the auxiliary field $\mathbf{m}$. In Sec. IV we required that ψ be proportional to $\mathbf{m}$ for small m near the core of a defect. In evaluating Eq. (100) we need a more general mapping. The procedure we will use here has been successful in the purely dissipative case [24]. Picking up on the point made in Sec. IV, we choose $\mathbf{m}(\mathbf{x})$ to represent the distance from $\mathbf{x}$ to the closest defect. This physical picture can be realized by constructing $\psi(\mathbf{m})$ as a solution to the equation for a single stationary defect:

$$b\nabla_m^2\psi+(1+i\omega_1-u|\psi|^2)\psi=0, \tag{101}$$

where $\mathbf{m}$ serves as the coordinate and ω_1 is the oscillatory frequency. In the purely dissipative case, b real and $\omega_1=0$, one has for field points well away from a defect core

$$\psi(\mathbf{m})=\psi_0 e^{i\phi(\mathbf{m})}, \tag{102}$$

where, for a defect of charge n,

$$\phi(\mathbf{m})=n\tan^{-1}(m_y/m_x). \tag{103}$$

In the purely dissipative case, insertion of Eq. (102) for $\psi(\mathbf{m})$ into Eq. (100) and carrying out the Gaussian average over $\mathbf{m}$ leads to the result [25]

$$C_{\psi}(12)=\psi_0^2 f\int_0^1 dz\,\frac{(1-z^2)^{1/2}}{(1-z^2f^2)^{1/2}} \tag{104}$$

with $f(12)$ given by Eq. (68) with $\omega=0$. This approximate result has been rather extensively tested in the TDGL case [26].

In the CGLE case we have a different and interesting element. There is a range of parameters where one has spiral defects. Thus, unlike the TDGL case, one has spatial structure associated with individual defects beyond the core. In particular Hagan [27] showed that the far-field solution of Eq. (101) is given by

$$\psi(\mathbf{m}(\mathbf{x}))=\psi_0 e^{i[\phi(\mathbf{x})+qm(\mathbf{x})]}, \tag{105}$$

where q is the wave number of the spiral arms asymptotically far from the defect core and $\psi_0=\psi_0(q)$. q depends on the particular parameters of the CGLE as discussed by Hagan. While there are values for which q vanishes, as in the TDGL limit, we will assume that we work in a region of parameter space where $q\neq 0$.

Using this set of mappings the order parameter correlation function is given by

$$\begin{aligned}C_{\psi}(12)&=\psi_0^2\langle e^{-i[\phi(2)+qm(2)]}e^{i[\phi(1)+qm(1)]}\rangle\\&=\psi_0^2\frac{D^2}{(2\pi)^2}\int d^2x(1)d^2x(2)\\&\quad\times e^{i(\phi_1-\phi_2)}e^{iq[x(1)-x(2)]}e^{-A/2},\end{aligned} \tag{106}$$

where $d^2x(i)=x(i)dx(i)d\phi(i)$ for $i=1,2$. The action A and determinant D are given in the Appendix. In particular A is given by Eq. (A13) in terms of polar coordinates and we have, more explicitly,

$$\begin{aligned}C_{\psi}(12)=\psi_0^2\frac{D^2}{(2\pi)^2}&\int_0^{\infty}x(1)dx(1)\\&\times\int_0^{\infty}x(2)dx(2)e^{iq[x(1)-x(2)]}\\&\times\exp\left(-\frac{1}{2}\sum_i x^2(i)W_0(i)\right)J(x(1),x(2)),\end{aligned} \tag{107}$$

where the angular integrations are given by

$$\begin{aligned}J(x(1),x(2))=&\int_0^{2\pi}d\phi(1)\int_0^{2\pi}d\phi(2)\\&\times e^{i[\phi(1)-\phi(2)]}e^{D^2C_Tx(1)x(2)\cos[\phi(1)-\phi(2)-\theta]}\end{aligned} \tag{108}$$

and θ is defined by $\tan\theta=\Delta/C_0$. Shifting the angular integrations we see that the θ dependence factors out:

$$J(x(1),x(2))=2\pi e^{i\theta}\int_0^{2\pi}d\phi\, e^{i\phi}e^{D^2C_Tx(1)x(2)\cos\phi}. \tag{109}$$

If we change integration variables from $x(i)$ to

$$y_i=\sqrt{W_0(i)}x(i), \tag{110}$$

we can rewrite Eq. (107) in the form

$$\begin{aligned}C_{\psi}(12)=\psi_0^2\frac{f}{f_T}\gamma_T^{-2}&\int_0^{\infty}y_1dy_1\int_0^{\infty}y_2dy_2\\&\times e^{-(y_1^2+y_2^2)/2}e^{i(q_1y_1-q_2y_2)}\\&\times\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{i\phi}e^{f_Ty_1y_2\cos\phi},\end{aligned} \tag{111}$$

where

$$q_i = q\sqrt{S_0(i)}\,\gamma_T^{-1}. \tag{112}$$

Notice that the phase dependence of the auxiliary field correlation function is isolated in the overall factor of f in Eq. (111). The integral over ϕ in Eq. (111) gives a modified Bessel function, but for our purposes we need only the power-series result:

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{i\phi} e^{f_T y_1 y_2 \cos\phi} = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}\left(\frac{f_T y_1 y_2}{2}\right)^{2k+1} \tag{113}$$

and

$$C_\psi(12) = \psi_0^2 \frac{f}{2}\gamma_T^{-2}\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(\frac{f_T}{2}\right)^{2k} J_k(q_1)J_k^*(q_2), \tag{114}$$

where

$$J_k(q_1) = \int_0^\infty y\,dy\, e^{-y^2/2} e^{iq_1 y} y^{2k+1}. \tag{115}$$

In the limit $q \to 0$ Eq. (114) does, after some manipulations, reduce to the result found in the TDGL case for $n=2$. Note that for $q \neq 0$, except for $r = |\mathbf{r}_1 - \mathbf{r}_2|$ very small, q_i, given by Eq. (112), is becoming increasingly large with $\sqrt{S_0(i)} \approx L(t_i)$. This means we need evaluate $J_k(q_i)$ only for large q_i. Evaluation of $J_k(q_i)$ for large q_i is facilitated by writing

$$J_k(q_i, a) = \int_0^\infty y\,dy\, e^{-ay^2} e^{iq_i y} y^{2k+1},$$

where $J_k(q_i) = J_k(q_i, 1/2)$. We have then

$$J_k(q_i, a) = \left(-\frac{\partial}{\partial a}\right)^{k+1} J(q_i, a) \tag{116}$$

and

$$J(q_i, a) = \int_0^\infty dy\, e^{-ay^2} e^{iq_i y} = \sqrt{\frac{\pi}{2a}} e^{-q_i^2/4a} + iJ''(q_i, a). \tag{117}$$

We see that the real part of J is exponentially small for large q_i^2. However, it is easy to see that for large q_i

$$J''(q_i, a) = \frac{1}{q_i} + \frac{2a}{q_i^3} + \frac{3(2a)^2}{q_i^5} + \cdots. \tag{118}$$

This means that the leading nonexponential contribution to the order parameter correlation function comes from $J_0(q_1)$ and is given to leading order by

$$J_0(q_1, a) = \left(-\frac{\partial}{\partial a}\right) i\frac{2a}{q_i^3} + \cdots = -\frac{2i}{q_i^3} + \cdots. \tag{119}$$

Inserting this result back into Eq. (114) we obtain

$$C_\psi(12) = \psi_0^2 \frac{f}{2}\gamma_T^{-2} J_0(q_1) J_0^*(q_2) = \psi_0^2\frac{f}{2}\gamma_T^{-2}\frac{4}{(q_1 q_2)^3} = \frac{\psi_0^2}{q^6}\frac{2f\gamma_T^4}{[S_0(1)S_0(2)]^{3/2}}. \tag{120}$$

The scaled portion of the order parameter correlation function for $x \neq 0$ can be written as

$$W(x,\tau) = \frac{q^6}{\psi_0^2} S_0^{3/2}(t_1) S_0^{3/2}(t_2) C_\psi(12) = 2f\gamma_T^4. \tag{121}$$

For $\tau \neq 1$, the correlation function depends strongly on $\eta = b''/b'$ via ω. This is not true at equal times where $\eta = b''/b'$ does not appear in the scaling function. For $\tau \neq 1$ the oscillations in f are now clear in the order parameter correlation function. Writing out the real and imaginary parts we obtain

$$W' = \frac{2F}{\sqrt{1+\omega^2}}\frac{1}{(1-F^2)^2}[\cos z + \omega \sin z], \tag{122}$$

$$W'' = \frac{2F}{\sqrt{1+\omega^2}}\frac{1}{(1-F^2)^2}[-\sin z + \omega \cos z], \tag{123}$$

where F is given by Eq. (83) and ω and z by Eqs. (81) and (82). We are interested in the oscillations associated with $\eta = b''/b' \neq 0$. These are most clearly manifested in W'' and characterized by the zeros at

$$\omega = \tan z_0. \tag{124}$$

The first zero as a function of scaled distance is given by

$$x_0^2 = 2 + \frac{4}{3}\omega^2 + \cdots \tag{125}$$

for small ω and

$$x_0^2 = \pi\omega - 2 + (1/\omega) \tag{126}$$

for large ω.

VIII. CONCLUSIONS

By using some different ideas about how to characterize defect dynamics, we have shown how one can determine local expressions for the defect density and defect velocity in terms of derivatives of the order parameter fields. These exact results were then used to derive approximate results for the defect-defect density correlation function, defect-velocity probability distribution, and order parameter correlation functions. Within these approximations, which work well for the purely dissipative case, we find that the results for the defect-defect density correlation function and the defect-velocity probability distribution are substantially unchanged from the TDGL case. Thus these results seem robust. The results for the two-time auxiliary field correlation function indicate some interesting oscillations of its phase as a function of scaled distance. Since the defect-defect density correlation function depends only on the amplitude of the auxiliary field correlation function these oscillations are not present. In the last section we have seen that some remnant of these oscillations is present in the order parameter correlation function. However, another different element for the order parameter correlation function is that the spiral arms for the defects render the interactions between different spatial points much shorter range than for the purely dissipative case. Thus for different spatial points at equal times the order parameter correlation function is down by a factor of $\overline{n^3}(t)$ relative to the TDGL case. All these results can be tested via numerical simulation.

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APPENDIX

In the purely dissipative case all correlations are isotropic:

$$C_{\nu\nu'}(ij)=\langle m_\nu(i)m_{\nu'}(j)\rangle=\delta_{\nu\nu'}C_0(ij), \tag{A1}$$

where $(i,j)=(1,2)$. In the complex case, over time, the real and imaginary components of the order parameter are mixed, and this requires that we treat the more general correlation function for the auxiliary field

$$C_{\nu\nu'}(ij)=\delta_{\nu\nu'}C_0(ij)+\epsilon_{\nu\nu'}\epsilon_{ij}\Delta(12), \tag{A2}$$

which satisfies the required symmetry for classical fields

$$C_{\nu\nu'}(ij)=C_{\nu'\nu}(ji) \tag{A3}$$

if $C_0(ij)=C_0(ji)$. Thus the variance of the Gaussian field $\mathbf{m}$ is determined by the two independent functions $C_0(12)$ and $\Delta(12)$.

We will be concerned with various two-point averages over $\mathbf{m}$ of the general form

$$C_{AB}(12)=\langle A(\mathbf{m}(1))B(\mathbf{m}(2))\rangle$$
$$=\int d^2x(1)d^2x(2)A(\mathbf{x}(1))B(\mathbf{x}(2))\Phi(\mathbf{x}(1),(\mathbf{x}(2)), \tag{A4}$$

where the two-point probability distribution is given by

$$\Phi(\mathbf{x}(1),(\mathbf{x}(2))=\langle\delta(\mathbf{x}(1)-\mathbf{m}(1))\delta(\mathbf{x}(2)-\mathbf{m}(2))\rangle$$
$$=\int\frac{d^2k(1)}{(2\pi)^2}\frac{d^2k(2)}{(2\pi)^2}\exp\left(i\sum_j \mathbf{k}(j)\cdot\mathbf{x}(j)\right)$$
$$\times\exp\left(-\frac{1}{2}\sum_{\nu\nu'}\sum_{ij}k_\nu(i)k_{\nu'}(j)C_{\nu\nu'}(ij)\right)$$
$$=\frac{1}{(2\pi)^2}\frac{1}{(\det C)^{1/2}}e^{-A/2}, \tag{A5}$$

where the argument of the exponential is given by

$$A=\sum_{\nu\nu'}\sum_{ij}x_\nu(i)x_{\nu'}(j)W_{\nu\nu'}(ij) \tag{A6}$$

and the matrix W is the inverse of C defined by

$$\sum_{\gamma k}W_{\nu\gamma}(ik)C_{\gamma\nu'}(kj)=\delta_{\nu\nu'}\delta_{ij}. \tag{A7}$$

W is given explicitly by

$$W_{\nu\nu'}(ij)=\delta_{\nu\nu'}D^{-2}\left[\delta_{ij}\left(\frac{S_0(1)S_0(2)}{S_0(i)}+C_0\right)-C_0\right]$$
$$-\epsilon_{\nu\nu'}\epsilon_{ij}D^{-2}\Delta, \tag{A8}$$

where

$$S_0(i)=C_0(ii), \tag{A9}$$

$$D^{-2}=S_0(1)S_0(2)-C_T^2, \tag{A10}$$

$$C_T^2=C_0^2+\Delta^2, \tag{A11}$$

and finally

$$\det C=D^{-4}. \tag{A12}$$

If we express $\mathbf{x}(i)=x_i(\cos\phi_i,\sin\phi_i)$, the argument of the exponential in the distribution takes on the simple form

$$A=\sum_i x_i^2W_0(i)-2D^{-2}C_Tx_1x_2\cos(\phi_1-\phi_2-\theta), \tag{A13}$$

where

$$C_0=C_T\cos\theta, \tag{A14}$$

$$\Delta=C_T\sin\theta, \tag{A15}$$

and

$$W_0(i)=D^{-2}\frac{S_0(1)S_0(2)}{S_0(i)}=S_0^{-1}(i)\gamma_T^{-2}. \tag{A16}$$

[1] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993). This review gives an extensive discussion of the role of the CGLE in pattern forming systems.
[2] L.M. Pismen, *Vortices in Nonlinear Fields* (Oxford University Press, London, 1999).
[3] I. Aronson and L. Kramer, report (unpublished).
[4] G.F. Mazenko, Phys. Rev. Lett. **78**, 401 (1997).
[5] A.J. Bray, Adv. Phys. **43**, 357 (1994).
[6] While there is clear evidence for such a regime in, for example, G. Huber, P. Alstrøm, and T. Bohr, Phys. Rev. Lett. **69**, 2380 (1992), there are reasons to believe that this coarsening may not go to completion [I. Aronson (private communication)].
[7] E.P. Gross, Nuovo Cimento **20**, 454 (1961); L.P. Pitaevskii, Zh. Éksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].
[8] F. Liu and G.F. Mazenko, Phys. Rev. B **46**, 5963 (1992).
[9] G.F. Mazenko and R. Wickham, Phys. Rev. E **57**, 2539 (1998).
[10] G.F. Mazenko, Phys. Rev. E **59**, 1574 (1999).
[11] S.O. Rice, in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).
[12] B.I. Halperin and M. Lax, Phys. Rev. **148**, 722 (1966).
[13] B.I. Halperin, in *Physics of Defects*, edited by R. Balian *et al.* (North-Holland, New York, 1981).
[14] It turns out, as discussed carefully in Ref. [18], that one can replace θ_B and w by θ and $\ln R$ in Eq. (9).
[15] A.L. Fetter, Phys. Rev. **151**, 100 (1966).
[16] M. Peach and J.S. Koehler, Phys. Rev. **80**, 436 (1950).
[17] K. Kawasaki, Prog. Theor. Phys. Suppl. **79**, 161 (1984).
[18] O. Törnkvist and E. Schröder, Phys. Rev. Lett. **78**, 1908 (1997).
[19] G.F. Mazenko, Phys. Rev. E **58**, 1543 (1998).
[20] G.F. Mazenko, Phys. Rev. E **61**, 1088 (2000).
[21] T. Ohta, D. Jasnow, and K. Kawasaki, Phys. Rev. Lett. **49**, 1223 (1983).
[22] A. Bray, Phys. Rev. E **55**, 5297 (1997).
[23] G.F. Mazenko, Phys. Rev. E **56**, 2757 (1997).
[24] G.F. Mazenko, Phys. Rev. B **42**, 4487 (1990).
[25] F. Liu and G.F. Mazenko, Phys. Rev. B **45**, 6989 (1992).
[26] A.J. Bray and K. Humayun, J. Phys. A **25**, 2191 (1992).
[27] P.S. Hagan, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **42**, 762 (1982).