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Estimating Lyapunov exponents in biomedical time series

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Among nonlinear dynamical invariants, determination of the largest Lyapunov exponent is well suited to positive identification of chaos in observed time series. When analyzing the dynamics of biomedical series, such as an electro-encephalogram (EEG), model-based methods should be used. Moreover, in the absence of any well founded theoretical model, and because of unexplained variability in the data, candidate models must provide for a stochastic component. Here we use nonlinear autoregressive stochastic modeling to estimate the dominant Lyapunov exponent in an EEG series and compute confidence intervals from surrogate data. The results are found to differ from those of approaches which aim at deleting noise prior to analysis.

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I. INTRODUCTION

Originally, chaos and dynamical invariants (e.g., dimension parameters and Lyapunov exponents) were defined in the context of purely deterministic systems. This point of view will be adequate for the analysis of data from well-controlled physical or chemical experiments. However, when considering biomedical series, the analyst should introduce a noise component in the model because of unexplained variability in the data and unavoidable model misspecification. Indeed, controversies about the chaotic nature of biomedical series, such as epidemics or electro-encephalogram (EEG) recordings might be, at least in part, due to inadequate accounting for noise (e.g., Ref. [1]).

Results from EEG analysis offer a particularly clear illustration of the controversies arising from nonlinear analysis of noisy biomedical series: whether the irregular pattern of routine scalp EEG recordings is best explained as arising from deterministic nonlinear chaotic dynamics or from stochastic fluctuations is still a matter of debate [2]. Some authors have provided evidence for low dimensional chaos [3–5]. However, this conclusion has been challenged, because of the poor reliability of parameter estimates from experimental series, insofar as positive identification of chaos is concerned [6–8]. Presently, modeling the EEG as a nonlinear stochastic system with additive noise is often preferred [9–13].

The above issues should obviously be solved before one attempts to compare nonlinear invariants between groups of subjects. For instance, it has been reported that correlation dimension is lower in schizophrenic versus normal subjects [14]. While the conclusion is that there are differences in the EEGs of the two groups, the interpretation of the difference is unclear [15,16]. In effect, estimation [15,17–19] and interpretation [6,2,7] of dimension parameters in noisy series is far from straightforward because the attractor is blurred by noise.

Estimation of the largest Lyapunov exponent is computationally more demanding [20], but estimates of this parameter are more readily interpreted with respect to the presence of chaos, as positive Lyapunov exponents are the hallmark of chaos [21]. Some authors have expressed doubts as to whether Lyapunov exponents could be defined for a random series (see, e.g., Ref. [20]). However, the multiplicative ergodic theorem of Oseledeč [22] provides a clear answer in the affirmative, as was explicated by Arnold [23,24] within the formalism of random dynamical systems. Because these results seem to have been overlooked, most authors have treated noise as a nuisance phenomenon, to be eliminated prior to computing Lyapunov exponents. Several approaches such as filtering, computation along noise-free trajectories, or along averaged trajectories have been considered in order to ''eliminate'' noise [25]. In the presence of dynamic (system) noise, however, Lyapunov exponents should be computed along the actual (noisy) sample path. They may be interpreted as quantifying the rate of divergence of initially nearby trajectories, subject to the constraint of identical sequences of random shocks [16].

Here we investigate EEG dynamics by computing Lyapunov exponents of the random dynamical system along the actual sample path and show that conclusions derived from these parameters differ markedly from those derived with the noise deleted. Furthermore, although Lyapunov exponents are statistical dynamical invariants which are independent of the observed motion along trajectories, values estimated from finite amounts of data will be random quantities. Thus, to draw any reliable conclusion from the estimation of Lyapunov exponents as to the presence of chaos, one must quantify the variability of the estimates. This is best done in the form of confidence intervals, which can be estimated via bootstrapping, an approach akin to hypothesis testing using surrogate data [26].

II. LYAPUNOV EXPONENTS IN NOISY TIME SERIES

Assume that scalar observations $x_1, \ldots, x_n, \ldots$ are made at regular time intervals on some biomedical process, for which, motivated by Takens' theorem [27], we postulate

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the following autoregressive model:

$$x_n = f(x_{n-1}, \ldots, x_{n-p}) + e_n, \quad n > p, \tag{1}$$

where f is a possibly nonlinear autoregressive function, e_n is a white noise component with zero mean and finite variance, and p is the order of the autoregression. Note that p is also the embedding dimension of the process being observed.

To define Lyapunov exponents we write model (1) in p-dimensional space

$$X_n = F(X_{n-1}) + E_n, \quad n > p, \tag{2}$$

where $X_n = (x_n, \ldots, x_{n-p+1})^T$, $E_n = (e_n, 0, \ldots, 0)^T$, and the map F is defined in obvious fashion.

In the terminology of Arnold [23], model (2) specifies a one-sided discrete time random dynamical system on $\mathbb{R}^p$ generated by the random mapping

$$\psi(\omega)X = F(X) + E_1, \tag{3}$$

where $\omega = (E_1, E_2, \ldots)$.

Furthermore, the Jacobian matrix $J = \partial F / \partial X$ generates a linear cocycle

$$B_n(\omega, X_0) = J(X_0) J(X_1) \ldots J(X_{n-1}) \tag{4}$$

on $\mathbb{R}^p$ [24]. Oseledeč's multiplicative ergodic theorem [22] then ensures that, under general ergodicity and integrability conditions, the Lyapunov exponent

$$\lim_{n\to\infty} \frac{1}{n} \log \| B_n(\omega, X_0) u \|, \tag{5}$$

where $\|.\|$ is Euclidean norm in p-space and $u \in \mathbb{R}^p$, exists and is independent of ω and X_0. Moreover, if the coordinates of u are randomly generated from the uniform distribution on $[0,1]$, the above limit is independent of u and equals the dominant Lyapunov exponent λ.

For purpose of comparison, the Lyapunov exponent

$$\tilde{\lambda} = \lim_{n\to\infty} \frac{1}{n} \log \| A_n(X_0) u \|, \tag{6}$$

for the system with noise deleted was also considered. Here, $A_n(X_0) = J(X_0) J[F(X_0)] \ldots J[F^{n-1}(X_0)]$.

III. STATISTICAL ESTIMATION

For estimating Lyapunov exponents in noisy systems, direct methods that track small orbit differences [20] are improper because, as far as the trajectories being compared will not correspond to the same sequence of random shocks, the divergence between them might simply be due to the noise component. We thus resort to model based methods.

Because in the biomedical setting, one is generally unable to specify a reliable parametric model for f, the autoregressive function was estimated by nonparametric regression, using the multivariate adaptive splines of Friedman [28]. Contrary to classical spline estimation, the knots are adaptively selected from the data, resulting in a parsimonious flexible estimation method. The procedure operates as follows. For fixed p, an approximation to f is built in the form

$$\hat{f}_{p,D}(X) = \sum_{j=0}^{D} a_j \Phi_j(X), \tag{7}$$

where $X \in \mathbb{R}^p$, the a_j's are real coefficients, and the Φ_j's are $D+1$ basis functions defined as tensor products of univariate truncated linear splines. More specifically,

$$\Phi_0(X) = 1,$$

$$\Phi_j(X) = \prod_{k=1}^{K_j} [s_{j,k}(x_{\nu(j,k)} - t_{j,k})]_+, \quad 1 \leqslant j \leqslant D,$$

where $s_{j,k} = \pm 1$, the $x_{\nu(j,k)}$ are (distinct) components of X, and the $t_{j,k}$ are the corresponding knot locations. The K_j's are upper bounds specified by the user.

The basis functions Φ_j and the coefficients a_j are computed stepwise by least-squares. The initial basis function is Φ_0. At each step, a least-squares optimization is performed to select a parent basis function $\Phi_\star$ (already in the model), a component variable $x_\star$ (which is not among the arguments of $\Phi_\star$), and a knot location $t_\star$. Two daughter basis functions $\Phi_\star(X)(x_\star - t_\star)_+$ and $\Phi_\star(X)(t_\star - x_\star)_+$ are then added to the model.

Optimal values $\hat{p}$ and $\hat{D}$ for p and D, are selected by minimization of a penalized least-squares criterion [29].

Jacobian matrices J were estimated from $\hat{f}_{\hat{p},\hat{D}}$ by numerical differentiation. The 95% confidence intervals for λ and $\tilde{\lambda}$ were estimated by parametric bootstrap, based on 400 surrogate realizations. The surrogate data were generated from $\hat{f}_{\hat{p},\hat{D}}$ and additive Gaussian noise, with mean zero and variance equal to

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{n} \sum_{i=\hat{p}+1}^{n} [x_i - \hat{f}_{\hat{p},\hat{D}}(x_{i-1}, \ldots x_{i-\hat{p}})]^2. \tag{8}$$

IV. EEG ANALYSIS

We analyzed 18 EEG recordings obtained from 18 asymptomatic female students aged 19–22 years. The EEG data were acquired from nine scalp loci with InstEP software (Canada version 3.1) during 3 min periods, under the eyes closed condition. The sampling rate was 250 Hz. The analysis was performed on stationary 20 s segments that were selected by eye from the central parietal derivation.

Table I presents estimates of λ and 95% bootstrap confidence intervals. The estimates are significantly positive, suggesting the existence of chaos, in 13 out of 18 subjects. Estimates of $\tilde{\lambda}$ and corresponding 95% bootstrap confidence intervals are given in Table II. These estimates, corresponding to the noise free system, are seen to be all nonsignificantly different from zero at the 5% level.

TABLE I. Estimates and 95% bootstrap confidence intervals of Lyapunov exponents (λ) for 18 subjects. Units are s^{-1}.

Subject	λ	Confidence interval
1	6.45	2.48, 11.37
2	−0.78	−0.79, 9.32
3	4.73	1.15, 8.32
4	5.45	0.03, 6.74
5	−0.68	−1.09, 5.48
6	3.45	1.57, 8.42
7	1.83	−3.12, 7.11
8	2.70	0.47, 7.37
9	0.11	−2.43, 4.44
10	0.95	0.19, 4.03
11	0.47	0.02, 2.60
12	1.39	1.35, 9.54
13	6.95	1.98, 10.16
14	1.79	0.16, 7.79
15	2.43	0.11, 6.75
16	5.00	0.95, 8.99
17	4.05	0.83, 8.12
18	3.95	−0.95, 7.00

TABLE II. Estimates and 95% bootstrap confidence intervals of Lyapunov exponents corresponding to the noise free system ($\tilde{\lambda}$) for 18 subjects. Units are s^{-1}.

Subject	$\tilde{\lambda}$	
1	2.80	−4.28, 16.19
2	−6.80	−9.22, 13.86
3	0.49	−2.79, 10.27
4	10.18	−1.90, 13.71
5	−3.08	−3.42, 7.25
6	−0.62	−2.70, 10.01
7	2.08	−8.18, 11.95
8	−1.84	−2.96, 6.54
9	0.39	−6.04, 6.93
10	−0.19	−2.19, 3.95
11	0.35	−1.89, 1.77
12	2.16	−4.11, 11.86
13	7.78	−1.65, 12.27
14	1.14	−3.25, 9.82
15	9.23	−3.97, 10.50
16	0.36	−4.34, 10.26
17	1.45	−2.31, 14.16
18	−3.60	−4.89, 8.83

V. DISCUSSION

Here, we analyzed EEG dynamics using a nonlinear autoregressive stochastic model, which was estimated from the data. The model is comprised of a deterministic autoregressive function and an additive noise component. This structure, however, should not be interpreted in the light of deep (stochastic) projection and (deterministic) dilation theorems established in the context of well understood physicochemical processes [30]. Rather, the noise component is merely a convenient modeling tool, which accounts for unexplained variability in the data and model misspecification. In this sense, stochastic modeling is equivalent to infinite dimensional deterministic modeling [1].

We found that, in biomedical series such as EEG recordings, computing Lyapunov exponents with noise deleted may lead to erroneous conclusions. Moreover, the estimation of confidence intervals was critical to avoid erroneous conclusions about the dynamics of the system. Indeed, the estimation of confidence intervals is analogous to hypothesis tests based on surrogate data advocated by Theiler *et al.* [26]. More specifically, our confidence interval construction may be understood as corresponding to a test of the null hypothesis of nonchaotic dynamics using surrogate data generated by the full model, whereas Theiler *et al.* [26] generate the surrogates from null models.

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