Markov models of non-Gaussian exponentially correlated processes and their applications

S. Primak*
Department of Electrical and Computer Engineering, The University of Western Ontario, London, Ontario, Canada N6A 5B9

V. Lyandres†
Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, Beer-Sheva, 84105, Israel

V. Kontorovich‡
Department of Electrical Engineering (Communications), Research Center for Advanced Study of Polytechnic Institute of Mexico, Av IPN No. 2508, Col. Zacatenco, Caixa Postale 07000, Mexico, Distrito Federal, Mexico
(Received 8 December 2000; published 21 May 2001)

We consider three different methods of generating non-Gaussian Markov processes with given probability density functions and exponential correlation functions. All models are based on stochastic differential equations. A number of analytically treatable examples are considered. The results obtained can be used in different areas such as telecommunications and neurobiology.

DOI: 10.1103/PhysRevE.63.061103 PACS number(s): 05.40.−a, 02.50.Ga, 02.50.−r, 05.10.Cc

I. INTRODUCTION

Modeling of signals and interference in the fields of communications, radar, sonar, and speech processing is usually based on the assumptions that the processes may be considered as stationary (or at least locally stationary) and that experimental estimations of their simplest statistical characteristics the autocovariance function (ACF) and marginal probability density function (PDF), are available. While the generation of stationary random processes (sequences) with a specified PDF or ACF does not present any major difficulty, the solution of the joint problem requires much more effort. This was considered in [1] where some sequential combinations of linear filtering and zero-memory nonlinear transformations of white Gaussian noise (WGN) were used. A different approach is based on treatment of a process with the prescribed characteristics as a stationary solution of the appropriate system of stochastic differential equations (SDE's) with the WGN on the right-hand side [2,3]. Such an interpretation seems attractive as it takes advantage of Markov processes theory and appears to be efficient in the modeling of correlated non-Gaussian processes.

Markov chains with exponential correlation function are effective models for video conference traffic, as used, for example, in [4,5]. While it is mentioned there that any discrete distribution can be represented, and the continuous time limit of the Markov chain is considered, the continuum limit was not considered. In particular, as we will show in this paper, it was not clarified whether the Markov chain is approaching a continuous Markov process or rather one with jumps. It will be shown here that this limit is a process with zero drift and jumps and is thus a good model for impulsive noise [6]. Continuous exponentially correlated processes were extensively studied in [1] and have been used to generate continuous non-Gaussian processes with more complicated correlation structure. Other possible applications can be found in the literature and include radar [7–9], biology [10], statistical electromagnetics [11], etc. However, there is little work reported on generation of processes that are mixed, i.e., have a continuous (nonzero drift) and a jump part. Such processes can be used as models for bursty internet traffic, Middleton class *B* noise, intersymbol interference combined with additive noise [12], speech [13], and stochastic ratchets [14,15]. In this paper we provide a unifying approach for modeling non-Gaussian Markov random process with exponential correlation functions.

In the general case a nonlinear system can be driven by a mixture of white Gaussian noise $\xi(t)$ and a Poisson flow of δ pulses $\eta(t)$,

$$\frac{dx}{dt}=f(x)+g(x)\xi(t)+\eta(t). \tag{1}$$

The statistical properties of the solution $x(t)$ can be completely described by its transitional probability density $\pi(x,t;x_0,t_0)$, which must obey the differential Chapman-Kolmogorov equation [16]

$$\begin{aligned}\frac{\partial}{\partial t}\pi(x|x_0;\tau)=&-\frac{\partial}{\partial x}[K_1(x)\pi(x;x_0,t_0)]\\&+\frac{1}{2}\frac{\partial^2}{\partial x^2}[K_2(x)\pi(x,t;x_0,t_0)]\\&+\lambda\int_{-\infty}^{\infty}[W(x|z,t)\pi(z,t;x_0,t_0)\\&-W(z|x,t)\pi(z,t;x_0,t_0)]dz,\end{aligned} \tag{2}$$

where the drift $K_1(x)$, diffusion $K_2(x)$, and probability of jumps $W(x|z,t)$ can be obtained from the corresponding parameters of Eq. (1) as

$$K_1(x)=f(x), \tag{3}$$

*Electronic address: primak@engga.uwo.ca
†Electronic address: lyandres@newton.bgu.ac.il
‡Electronic address: valeri@enigma.red.cinvestav.mx

$$K_2(x)=g^2(x), \tag{4}$$

$$\lim_{t-t_0\to 0}\pi(z,t;x_0,t_0)=\lambda W(x|z,t), \tag{5}$$

$$\int_{-\infty}^{\infty}W(x|z,t)dx=1. \tag{6}$$

(in the Ito form of stochastic integrals). In the stationary case, if it exists, Eq. (2) becomes

$$\lambda\int_{-\infty}^{\infty}[W(x|z,t)\pi(z|x_0;\tau)-W(z|x,t)\pi(x|x_0;\tau)]dz$$
$$=\frac{\partial}{\partial x}[K_1(x)\pi(x|x_0;\tau)]-\frac{1}{2}\frac{\partial^2}{\partial x^2}[K_2(x)\pi(x|x_0;\tau)]. \tag{7}$$

Different particular cases of the SDE (1) have been considered in a number of publications [2,3,17–24]. The main goal here is to show how different Markov processes with the same non-Gaussian probability density and exponential correlation function can be obtained. A method of numerical simulation of such processes is also considered and some examples are given.

II. EXPONENTIALLY CORRELATED MARKOV CHAIN

In this section we consider the discrete time scheme that generates a non-Gaussian Markov chain with an exponential correlation function and a given arbitrary PDF. Following [25], let us assume that the stationary distribution of the Markov chain y_n with N states

$$\gamma_1<\gamma_2<\cdots<\gamma_N \tag{8}$$

is described by the following probabilities of an individual state:

$$q_k=\mathrm{Prob}\{y_n=\gamma_k\}. \tag{9}$$

Any Markov chain can be completely described by its transitional probability matrix $\mathbf{T}=[T_{k,l}]$ where

$$T_{k,l}=\mathrm{Prob}\{y_m=\gamma_k|y_{m-1}=\gamma_l\},\quad k,l=1,2,\ldots,N. \tag{10}$$

which is the probability of the event $y_m=\gamma_k$ when $y_{m-1}=\gamma_l$ and satisfies the conditions

$$T_{k,l}\geqslant 0,\quad \sum_{k=1}^{N}T_{k,l}=1,\quad k=1,2,\ldots,N. \tag{11}$$

It is well known that the stationary probabilities q_k are obtained as the eigenvectors of the transition probability matrix $\mathbf{T}$ corresponding to the eigenvalue $\lambda=1$:

$$\sum_{i=1}^{N}T_{k,i}q_i=q_k,\quad k=1,2,\ldots,N. \tag{12}$$

The same matrix defines the power spectrum in the stationary case. However, we consider the inverse problem, having defined only the stationary probabilities of the states. For the case where the correlation function is exponential the solution was obtained in [25]. We follow this procedure here to obtain the chain approximation with infinite number of states as the limit of a finite state Markov chain.

To achieve the first goal, we define the following matrix $\mathbf{Q}$ in terms of the probabilities of the states:

$$\mathbf{Q}=\underbrace{\begin{bmatrix} q_1 & q_1 & \cdots & q_1\\ q_2 & q_2 & \cdots & q_2\\ \cdots & \cdots & \cdots & \cdots\\ q_N & q_N & \cdots & q_N\end{bmatrix}}_{N\ \text{times}}. \tag{13}$$

It is easy to check that

$$\mathbf{Q}^2=\mathbf{Q}, \tag{14}$$

and [25]

$$\det[\mathbf{Q}-\lambda\mathbf{I}]=(1-\lambda)(-\lambda)^{N-1}. \tag{15}$$

In terms of the matrix $\mathbf{Q}$, the transition matrix $\mathbf{T}$ can be defined as [25]

$$\mathbf{T}=\mathbf{Q}+d(\mathbf{I}-\mathbf{Q}), \tag{16}$$

where $0\leqslant d<1$ will define the correlation properties (described below) and $\mathbf{I}$ is the identity matrix. At the same time $\mathbf{T}$ satisfies the condition (12).

For any integer m one can obtain the following expression from Eqs. (16) and (14):

$$\mathbf{T}^m=\mathbf{Q}+d^m(\mathbf{I}-\mathbf{Q}). \tag{17}$$

Since d is a positive number less than 1, one has

$$\lim_{m\to\infty}\mathbf{T}^m=\mathbf{Q}, \tag{18}$$

which means that the Markov chain described by Eq. (16) becomes ergodic [26] and has a stationary probability given by q_k.

The next step is to consider the correlation function R_m of the Markov chain y_m. The average value and the average of the squared value can be obtained in terms of the stationary probability q_k as

$$\langle y_m\rangle=\sum_{i=1}^{N}\gamma_k q_k, \tag{19}$$

$$\langle y_m^2\rangle=\sum_{i=1}^{N}\gamma_k^2 q_k, \tag{20}$$

To calculate the correlation function, one has to consider the two-dimensional probability, which may be obtained from Eq. (17) as

$$\mathbf{Q}^{(m)}(k,l)=\mathrm{Prob}\{y_m=\gamma_k,y_0=\gamma_l\}=\{q_k+d^m(\delta_{k,l}-q_k)\}q_l, \tag{21}$$

where $m>0$ and $\mathbf{Q}^{(m)}(k,l)$ stands for the m-step transitional probability. It is easy to show that Eq. (21) fits the consistency relation

$$\sum_{k=1}^{N} \mathbf{Q}^{(m)}(k,l)=\sum_{k=1}^{N} \mathbf{Q}^{(m)}(l,k)=q_l. \tag{22}$$

The correlation function R_m is an even function of m defined as

$$R_m=R_{-m}=\langle y_m y_0\rangle-\langle y_m\rangle^2. \tag{23}$$

Substitution of Eqs. (19)–(22) into Eq. (23) produces

$$\begin{aligned}
R_m&=R_{-m}=\langle y_m y_0\rangle-\langle y_m\rangle^2\\
&=\sum_{k,l=1}^{N} \gamma_k\gamma_l \,\mathrm{Prob}\{y_m=\gamma_k, y_0=\gamma_l\}-\langle y_m\rangle^2\\
&=\sum_{k,l=1}^{N} \gamma_k\gamma_l\{q_k+d^m(\delta_{k,l}-q_k)\}q_l-\langle y_m\rangle^2\\
&=\sum_{k,l=1}^{N} \gamma_k\gamma_l q_k q_l+d^m\sum_{k,l=1}^{N} \gamma_k\gamma_l(\delta_{k,l}-q_k)q_l-\langle y_m\rangle^2\\
&=\langle y_m\rangle^2+d^{|m|}(\langle y_m^2\rangle-\langle y_m\rangle^2)-\langle y_m\rangle^2\\
&=d^{|m|}(\langle y_m^2\rangle-\langle y_m\rangle^2),
\end{aligned} \tag{24}$$

which is an exponential function with correlation length defined as

$$N_{\mathrm{corr}}=(-1)/\ln d. \tag{25}$$

Formula (17) can be extended to the finite state continuous time Markov chain $y_m(t)$ as in [25]:

$$\mathbf{T}(t)=\mathbf{Q}+\exp(-\mu t)(\mathbf{I}-\mathbf{Q}). \tag{26}$$

The expression for the correlation function can be given in this case as

$$R_{yy}(\tau)=\exp(-\mu\tau)(\langle y_m^2\rangle-\langle y_m\rangle^2). \tag{27}$$

Before turning to the continuous time infinite state Markov chain (a Markov process with a continuum of states) let us point out an important property of the exponentially correlated Markov chain. It follows from the definition of the matrices $\mathbf{T}$ and $\mathbf{T}(t)$ as in Eqs. (17) and (26) that the transition probability density does not depend on the current state. If N tends to infinity, $N\to\infty$, then the Markov continuous time chain tends to a Markov process, which can be a nondiffusion one. In this case Eq. (26) can be written as

$$\pi(x|x_0;\tau)=\exp(-\mu\tau)\,\delta(x-x_0)+[1-\exp(-\mu\tau)]p_s(x). \tag{28}$$

Here $p_s(x)$ is the stationary distribution of the limit process.

It is important to validate that the expression indeed defines a proper PDF of the Markov process. Positivity is obvious, since both summands are positive numbers; thus

$$\pi(x|x_0;\tau)\geqslant 0. \tag{29}$$

When τ approaches infinity the transitional PDF must correspond to the stationary PDF of the process since the values of the process far away from the observation points are independent of this observation:

$$\begin{aligned}
\lim_{\tau\to\infty}\pi(x|x_0;\tau)&=\lim_{\tau\to\infty}\{\exp(-\lambda\tau)\,\delta(x-x_0)\\
&\quad+[1-\exp(-\lambda\tau)]p_s(x)\}\\
&=p_s(x).
\end{aligned} \tag{30}$$

At the same time the limit of the PDF when τ approaches zero must be the delta function $\delta(x-x_0)$, since the process cannot assume two different values at the same moment. Taking the limit of Eq. (28) one finds that this condition is indeed satisfied:

$$\begin{aligned}
\lim_{\tau\to 0}\pi(x|x_0;\tau)&=\lim_{\tau\to 0}\{\exp(-\lambda\tau)\,\delta(x-x_0)\\
&\quad+[1-\exp(-\lambda\tau)]p_s(x)\}\\
&=\delta(x-x_0).
\end{aligned} \tag{31}$$

Finally, in order to represent a Markov process, the PDF $\pi(x|x_0;\tau)$ must obey the Smoluchovski equation [1]

$$\pi(x|x_0;\tau)=\int_{-\infty}^{\infty}\pi(x|x_1;\tau_2)\,\pi(x_1|x_0;\tau_1)\,dx_1 \tag{32}$$

with $\tau=\tau_1+\tau_2$. It is easy to check that this is the case for the PDF given by Eq. (28). Indeed,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\pi(x|x_1;\tau_2)\,\pi(x_1|x_0;\tau_1)\,dx_1\\
&=\int_{-\infty}^{\infty}\{\exp(-\lambda\tau_2)\,\delta(x-x_1)+[1-\exp(-\lambda\tau_2)]p_s(x)\}\\
&\quad\times\{\exp(-\lambda\tau_1)\,\delta(x_1-x_0)\\
&\quad+[1-\exp(-\lambda\tau_1)]p_s(x_1)\}dx_1\\
&=\exp[-\lambda(\tau_1+\tau_2)]\,\delta(x-x_0)\\
&\quad+\{1-\exp[-\lambda(\tau_1+\tau_2)]\}p_s(x)\\
&=\pi(x|x_0;\tau_1+\tau_2).
\end{aligned} \tag{33}$$

Thus, in fact, the PDF (28) defines a Markov process. The correlation function of this process is, indeed, exponential:

$$\begin{aligned}
B_{xx}(\tau)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x x_1\,\pi(x|x_1;\tau)p_s(x_1)\,dx\,dx_1\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x x_1[\exp(-\lambda\tau)\,\delta(x_1-x)]p_s(x_1)\,dx\,dx_1\\
&\quad+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x x_1\{[1-\exp(-\lambda\tau)]p_s(x)\}\\
&\quad\times p_s(x_1)\,dx\,dx_1\\
&=\exp(-\lambda\tau)[\sigma_x^2+m_x^2]+[1-\exp(-\lambda\tau)]m_x^2
\end{aligned}$$

$$=\sigma_x^2\exp(-\lambda\tau)+m_x^2. \quad (34)$$

The next step is to understand if the Markov process defined by the PDF (28) represents a diffusion Markov process or if it is a process with jumps. In order to do this one must calculate the following limit, which describe the nondiffusion part of any general Markov process [Eq. (3.4.1) in 1]:

$$W(x|x_0;\tau)=\lim_{\substack{\tau\to 0\\ |x-x_0|>\varepsilon}}\frac{\pi(x|x_0;\tau)}{\tau}=\lim_{\substack{\tau\to 0\\ |x-x_0|>\varepsilon}}\frac{[1-\exp(-\lambda\tau)]p_s(x)}{\tau}=\lambda p_s(x). \quad (35)$$

The last equation implies that the Markov process defined by the PDF (30) is a process with jumps since $W(x|x_0;\tau)\neq 0$. However, it is important to note that the probability of jumps $W(x|x_0;\tau)$ does not depend on the current state x_0. This property is inherited from the fact that the prelimit Markov chain $\{y(t)\}$ has the same property.

It is interesting to determine which SDE generates such a process. In order to accomplish that one has to calculate the drift $K_1(x)$ and the diffusion $K_2(x)$ coefficients, which are defined as [1]

$$\begin{aligned}K_1(x)+O(\varepsilon)&=\lim_{\tau\to 0}\frac{1}{\tau}\int_{|x-z|<\varepsilon}(z-x)\pi(z|x;\tau)dz\\&=\lim_{\tau\to 0}\frac{1}{\tau}\int_{|x-z|<\varepsilon}(z-x)\{\exp(-\lambda\tau)\delta(x-z)\\&\quad+[1-\exp(-\lambda\tau)]p_s(z)\}dz\\&=\lim_{\tau\to 0}\frac{\exp(-\lambda\tau)}{\tau}\int_{|x-z|<\varepsilon}(z-x)\delta(x-z)dz\\&\quad+\lim_{\tau\to 0}\frac{[1-\exp(-\lambda\tau)]}{\tau}\\&\quad\times\int_{|x-z|<\varepsilon}(z-x)p_s(z)dz\\&=0+O(\varepsilon)\end{aligned} \quad (36)$$

and

$$\begin{aligned}K_2(x)+O(\varepsilon)&=\lim_{\tau\to 0}\frac{1}{\tau}\int_{|x-z|<\varepsilon}(z-x)^2\pi(z|x;\tau)dz\\&=\lim_{\tau\to 0}\frac{1}{\tau}\int_{|x-z|<\varepsilon}(z-x)^2\{\exp(-\lambda\tau)\delta(x-z)\\&\quad+[1-\exp(-\lambda\tau)]p_s(z)\}dz\\&=\lim_{\tau\to 0}\frac{\exp(-\lambda\tau)}{\tau}\int_{|x-z|<\varepsilon}(z-x)^2\delta(x-z)dz\\&\quad+\lim_{\tau\to 0}\frac{[1-\exp(-\lambda\tau)]}{\tau}\\&\quad\times\int_{|x-z|<\varepsilon}(z-x)^2p_s(z)dz\\&=0+O(\varepsilon).\end{aligned} \quad (37)$$

The last two equations show that the SDE generating the continuous Markov process with transitional PDF given by Eq. (28) is

$$\frac{dx}{dt}=\eta(t), \quad (38)$$

where $\eta(t)$ is a stream of δ pulses

$$\eta(t)=\lambda\sum_{t_k}A_k\delta(t-k\Delta t) \quad (39)$$

with amplitudes A_k distributed according to the stationary PDF $p_s(x)$ and time between two sequential arrivals Δt. In order to obtain the continuous time chain Δt must approach zero.

III. EXPONENTIALLY CORRELATED DIFFUSION PROCESS

In order to make this paper self-explanatory, some basic equations obtained earlier in [2] are represented here. These equations allow one to generate an exponentially correlated diffusion Markov random process with an arbitrary probability density function. It is well known that the solution of a SDE (Ito form [20])

$$\dot{x}=f(x)+g(x)\xi(t) \quad (40)$$

is a diffusion Markov random process, whose PDF $p(x,t)$ [and the transition probability density function $\pi(x,t|x_0,t_0)$] obeys the Fokker-Planck equation [20]

$$-\frac{\partial}{\partial t}p(x,t)=\frac{\partial}{\partial x}[K_1(x)p(x,t)]-\frac{1}{2}\frac{\partial^2}{\partial x^2}[K_2(x)p(x,t)]. \quad (41)$$

Here $\xi(t)$ is a WGN of unit variance, and

$$K_1(x)=f(x), \quad (42)$$

$$K_2(x)=g^2(x) \quad (43)$$

are the drift and diffusion of the Markov process $x(t)$. The nonstationary PDF $p(x,t)$ of the process $x(t)$ converges to the stationary PDF $p_s(x)$ when t approaches infinity, i.e.,

$$\lim_{t\to\infty}p(x,t)=p_s(x). \quad (44)$$

There is a simple relation between $K_1(x)$, $K_2(x)$, and $p_s(x)$ [20]:

$$p_s(x)=\frac{C}{K_2(x)}\exp\left[2\int_a^x\frac{K_1(x)}{K_2(x)}dx\right], \quad (45)$$

where the constant C is chosen to normalize the PDF $p_s(x)$.

At the same time, the correlation function $K_x(\tau) = \langle x(t)x(t+\tau)\rangle$ can be considered as the solution of the following ordinary differential equation [20]:

$$\frac{d}{d\tau}K_x(\tau) = \langle x(t)K_1[x(t+\tau)]\rangle \tag{46}$$

with the initial condition

$$K_x(0) = \sigma_x^2 = \langle(x-\langle x\rangle)^2\rangle = \langle(x-m_x)^2\rangle. \tag{47}$$

Here $\langle\cdot\rangle$ stands for the statistical average over the realizations [20]. If one chooses

$$K_1(x) = -\alpha(x-m_x), \tag{48}$$

then Eq. (46) has a solution of the form

$$K_{xx}(\tau) = \sigma_x^2 \exp(-\alpha|\tau|). \tag{49}$$

After substituting Eq. (48) into Eq. (45) and solving for $K_2(x)$ one can obtain that

$$K_2(x) = -\frac{2\alpha}{p_x(x)}\int_{-\infty}^{x}(x-m_x)p_s(x)dx. \tag{50}$$

It is proven in [2] that $K_2(x) \geqslant 0$ for any $p_s(x)$; thus the last equation is a meaningful one for any stationary PDF. The drift $K_1(x)$ and the diffusion $K_2(x)$ now define the SDE

$$\dot{x} = -\alpha(x-m_x) + \left(-\frac{2\alpha}{p_s(x)}\int_{-\infty}^{x}(x-m_x)p_s(x)dx\,\xi(t)\right)^{1/2}, \tag{51}$$

whose solution has the given stationary PDF $p_s(x)$ and exponential correlation function (49). In turn, the SDE (51) can be numerically simulated, using a technique suggested in [18], providing one with a convenient tool for generating non-Gaussian exponentially correlated random processes.

It is impossible to obtain the exact equation for the transitional probability function, except for a number of cases considered in [27]. However, for a small transitional time τ an approximate formula can be obtained. Indeed, since the solution of the SDE is a diffusion Markov process, it can be approximated by a Gaussian random process with the same local drift and local diffusion as in [1],

$$\pi(x|x_0;\tau) = \frac{1}{\sqrt{2\pi K_2(x)\tau}}\exp\left[-\frac{[x-x_0-K_1(x)\tau]^2}{2K_2(x)}\right], \tag{52}$$

or, taking Eqs. (50)–(51) into account,

$$\pi(x|x_0;\tau) = \frac{\sqrt{p_s(x)}}{[4\pi\alpha(\int_{-\infty}^{x}(m_x-x)p_s(x)dx)\tau]^{1/2}} \times\exp\left[-\frac{[x-x_0+\alpha(x-m_x)\tau]^2 p_s(x)}{4\alpha(\int_{-\infty}^{x}(m_x-x)p_s(x)dx)\tau}\right]. \tag{53}$$

This transitional probability can be used to numerically simulate the random process using the chain method, as described in Sec. V.

IV. MIXED PROCESS WITH EXPONENTIAL CORRELATION

Another possibility of a non-Gaussian Markov process with exponential correlation was considered in [19]. In this case, the generating SDE was chosen in the form of a linear system excited by a train of δ functions, similar to Eq. (39),

$$\frac{dx}{dt} = -\alpha x + \eta(t). \tag{54}$$

However, it was found in [19] that a relatively small class of non-Gaussian processes can be represented in this form. It is shown in this paper that using a slightly different approach one can widen the class of processes represented. Without loss of generality one may consider the case of zero mean since a constant value can easily be added to the zero-mean random process to account for it. In this case the desired amplitude distribution and the intensity of $\eta(t)$ can be adjusted to obtain the desired properties. In the case of Eq. (52), the differential Chapman-Kolmogorov equation (7) becomes

$$\lambda\int_{-\infty}^{\infty}[W(x|z,t)\pi(z|x_0;\tau) - W(z|x,t)\pi(x|x_0;\tau)]dz = -\alpha\frac{\partial}{\partial x}\{[x\pi(x|x_0;\tau)]\} \tag{55}$$

since the diffusion coefficient is zero and the drift is a linear term. Multiplying both parts by $p_s(x_0)$ and integrating over x_0 one can obtain that

$$\lambda\int_{-\infty}^{\infty}W(x|z,t)p_s(z)dz = -\alpha\frac{\partial}{\partial x}\{[xp_s(x)]\} + \lambda p_s(x) = (\lambda-\alpha)p_s(x) - \alpha x\frac{\partial}{\partial x}p_s(x) \tag{56}$$

since

$$\int_{-\infty}^{\infty}\pi(x|x_0;\tau)p_s(x_0)dx_0 = p_s(x) \tag{57}$$

and, according to Eq. (5),

$$\int_{-\infty}^{\infty}W(z|x,t)\pi(x|x_0;\tau)dz = \pi(x|x_0;\tau). \tag{58}$$

Since both $W(x|z)$ and $p_s(x)$ are non-negative functions,

$$\int_{-\infty}^{\infty}W(x|z,t)p_s(z)dz = \left(1-\frac{\alpha}{\lambda}\right)p_s(x) - \frac{\alpha}{\lambda}x\frac{\partial}{\partial x}p_s(x) \geqslant 0. \tag{59}$$

This gives us the weakest test of what kind of distributions can be implemented using this technique. It also gives a lower bound on the intensity of the jumps λ needed for a stationary distribution to exist (recall that the constant α is defined by the required correlation interval $\tau_{\mathrm{corr}}=1/\alpha$ and cannot be chosen arbitrarily [2,19]):

$$x\frac{(\partial/\partial x)p_s(x)}{p_s(x)}\leqslant\frac{\lambda}{\alpha}-1. \tag{60}$$

Let us assume in the following that the condition (60) is indeed satisfied. Detailed investigation of this matter is the subject of an upcoming publication.

Since one has to choose an unknown function $W(x|z)$ of two variables having just one equation (59), it is possible that this choice is not unique. Indeed, two possibilities are considered below. Following the idea of Sec. II one can assume that the jump probability does not depend on the current state, i.e., $W(x|z)=W(x)$. As an alternative, a more common kernel depending on the difference between the current and future states can be chosen, i.e., $W(x|z)=W(x-z)$. Both cases are investigated here.

A. PDF $W(x|z)$ does not depend on the current state

In this case

$$W(x|z)=W(x) \tag{61}$$

and Eq. (56) becomes

$$\begin{aligned}\lambda\int_{-\infty}^{\infty}&W(x|z)p_s(z)dz\\&=\lambda W(x)\int_{-\infty}^{\infty}p_s(z)dz\\&=\lambda W(x)\\&=(\lambda-\alpha)p_s(x)-\alpha x\frac{\partial}{\partial x}p_s(x)\end{aligned} \tag{62}$$

and has the unique solution

$$W(x)=\left(1-\frac{\alpha}{\lambda}\right)p_s(x)-\frac{\alpha}{\lambda}x\frac{\partial}{\partial x}p_s(x). \tag{63}$$

Since it was assumed that Eq. (60) is satisfied, the function $W(x)$ is a positive function. The only additional condition would be its normalization to 1, i.e.,

$$\begin{aligned}\int_{-\infty}^{\infty}W(x)dx&=\int_{-\infty}^{\infty}\left[\left(1-\frac{\alpha}{\lambda}\right)p_s(x)-\frac{\alpha}{\lambda}x\frac{\partial}{\partial x}p_s(x)\right]dx\\&=\left(1-\frac{\alpha}{\lambda}\right)-\frac{\alpha}{\lambda}\int_{-\infty}^{\infty}x\frac{\partial}{\partial x}p_s(x)dx\\&=1,\end{aligned} \tag{64}$$

or, equivalently,

$$\int_{-\infty}^{\infty}x\frac{\partial}{\partial x}p_s(x)dx=-1. \tag{65}$$

Using integration by parts, the last integral can be transformed to

$$\begin{aligned}\int_{-\infty}^{\infty}x\frac{\partial}{\partial x}p_s(x)dx&=\int_{-\infty}^{\infty}x\,dp_s(x)\\&=xp_s(x)|_a^b-\int_{-\infty}^{\infty}p_s(x)dx\\&=xp_s(x)|_a^b-1.\end{aligned} \tag{66}$$

Here a and $b>a$ are the boundaries of the interval $[a,b]$, where $p_s(x)$ differs from zero. Both of them can be infinite. Comparing Eq. (66) to Eq. (65) one can conclude that the function $W(x)$ represents a proper PDF if

$$xp_s(x)|_a^b=0. \tag{67}$$

This condition is satisfied automatically if both boundaries are infinite, since $p_s(x)$ is integrable. If at least one of the boundaries is finite then Eq. (67) constitutes yet another restriction on the class of PDF that can be achieved.

B. PROBABILITY DEPENDING ON THE DIFFERENCE BETWEEN THE CURRENT AND FUTURE STATES

This case was originally considered in [19]. Equation (56) becomes an integral equation of convolution type,

$$\int_{-\infty}^{\infty}W(x-z)p_s(z)dz=\left(1-\frac{\alpha}{\lambda}\right)p_s(x)-\frac{\alpha}{\lambda}x\frac{\partial}{\partial x}p_s(x), \tag{68}$$

and can be solved using the Fourier transform technique. Indeed, let $\Theta_w(j\omega)$ and $\Theta_x(j\omega)$ be characteristic functions, corresponding to the PDF's $W(x)$ and $p_s(x)$, respectively,

$$\Theta_w(j\omega)=\int_{-\infty}^{\infty}W(x)\exp(j\omega x)dx, \tag{69}$$

$$\Theta_x(j\omega)=\int_{-\infty}^{\infty}p_s(x)\exp(j\omega x)dx. \tag{70}$$

Taking the Fourier transform of Eq. (68) and recalling the convolution theorem one can obtain

$$\Theta_w(j\omega)\Theta_x(j\omega)=\Theta_x(j\omega)+\frac{\alpha}{\lambda}\omega\frac{d}{d\omega}\Theta_x(j\omega) \tag{71}$$

and thus

$$\Theta_w(j\omega)=1+\frac{\alpha}{\lambda}\omega\frac{(d/d\omega)\Theta_x(j\omega)}{\Theta_x(j\omega)}. \tag{72}$$

Since the characteristic function of a proper distribution must obey certain conditions [26], not all PDF's of the solution $x(t)$ can be achieved.

V. EXAMPLES

The random processes considered above can be obtained either through generation of the corresponding Markov chain or by direct simulation of the generating SDE. In the first case one obtains an appropriate transition probability matrix $\mathbf{T}=[T_{k,l}]$ first. Equations (13) and (16) give the appropriate transitional matrix for the exponentially correlated Markov chain with zero drift and diffusion. In order to obtain a diffusion process one can approximate the expression for its transitional PDF as given by Eq. (53). The sequence corresponding to the transition matrix $\mathbf{T}$ can be generated by the recurrence equation [25]

$$y_m = F(y_{m-1},\zeta_m), \quad m=0,\pm 1,\pm 2,\ldots, \tag{73}$$

where ζ_m is an independent random number uniformly distributed over the interval [0,1], and $F(y_{m-1},\zeta_m)$ is a discontinuous function of ζ_m defined as

$$F(\gamma_l,\zeta)=\begin{cases}\gamma_1, & 0<\zeta\leqslant T_{1,l}\\ \gamma_2, & T_{1,l}<\zeta\leqslant T_{1,l}+T_{2,l}\\ \gamma_3, & T_{1,l}+T_{2,l}<\zeta\leqslant T_{1,l}+T_{2,l}+T_{3,l}\\ \cdots, & \cdots\\ \gamma_N, & 1-T_{N,l}<\zeta\leqslant 1.\end{cases} \tag{74}$$

Solving Eq. (73) numerically, we obtain a sample of a Markov chain with exponential correlation and given probabilities of the states.

Numerical simulation of the process defined by the SDE (54) can be performed directly using the fact that the external force is a sequence of δ functions, and thus the solution is a sum of the delayed and scaled exponential pulses with decay rate α. Local linearization of the SDE with WGN excitation is used to numerically simulate the solution of Eq. (51). Details of this method can be found in [18]. In all examples we assume that the correlation function has the exponential form

$$K_x(\tau)=\sigma^2\exp[-\alpha|\tau|] \tag{75}$$

or, in the case of discrete time simulations with time step Δt,

$$R_m=\sigma^2 d^m, \quad d=\exp(-\alpha\Delta t). \tag{76}$$

A. Gaussian marginal PDF

In this case the stationary PDF is defined by

$$p_s(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right). \tag{77}$$

The quantization levels γ_i, $1\leqslant i\leqslant N+1$, can be obtained using the entropy method, considered in detail in [28]. The corresponding probabilities q_i can be calculated using the error function [29]. Let us note that, despite the fact that the marginal PDF of this Markov process is a Gaussian one and the correlation function is exponential, the process itself is not Gaussian, i.e., the second order joint probability density is not Gaussian.

In contrast, the diffusion process with the same distribution and correlation function is Gaussian [30]. The corresponding SDE of the form (51) is linear:

$$\dot{x}=-\alpha x+2\sigma^2\xi(t) \tag{78}$$

with transitional PDF equal to

$$\pi(x|x_0;\tau)=\frac{1}{\sqrt{2\pi\sigma^2}}\frac{1}{\sqrt{1-\exp(-\alpha\tau)}}\times\exp\left(-\frac{x^2-\exp(-\alpha\tau)xx_0}{2\sigma^2[1-\exp(-2\alpha\tau)]}\right). \tag{79}$$

Condition (60) becomes

$$-\frac{x^2}{\sigma^2}\leqslant\frac{\lambda}{\alpha}-1 \tag{80}$$

and is satisfied once $\lambda\geqslant\alpha$. The corresponding density (63) of jumps then becomes

$$W(x)=\left(1-\frac{\alpha}{\lambda}\right)p_s(x)-\frac{\alpha}{\lambda}x\frac{\partial}{\partial x}p_s(x)=\frac{1-\alpha/\lambda+\alpha x^2/\lambda\sigma^2}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad \lambda>\alpha. \tag{81}$$

It was shown in [19] that it is impossible to obtain a Gaussian process using a SDE of this form. Indeed, in order to achieve a Gaussian process by linear transformation one requires the input to be Gaussian also. However, a Poisson process is not a Gaussian, unless its intensity is infinite.

B. Pearson class of PDF's

In this case the stationary PDF $p_s(x)$ obeys the equation

$$\frac{(d/dx)p_s(x)}{p_s(x)}=\frac{A_1+x+A_0}{B_2x^2+B_1x+B_0}. \tag{82}$$

It was shown in [2,21] that the following SDE (Ito form) allows one to generate all possible Pearson processes with exponential correlation function of this type:

$$\frac{dx}{dt}=\frac{\lambda}{2}(A_1x+A_0)+\sqrt{\lambda(B_2x^2+B_1x+B_0)}\xi(t). \tag{83}$$

In all cases the transition probability density can be expressed in terms of classical orthogonal polynomials [29] or in closed integral form as found in [27]. A few examples are given in Table I.

In order to obtain a convenient expression for the Pearson class of distributions let us rewrite Eq. (63) as

TABLE I. Parameters of the continuous Markov process with exponential correlation.

$p_s(x)$	$\frac{d}{dx}\ln p_s(x)$	$\pi(x,x_0;\tau)$
$\exp(-x)$	-1	$\frac{1}{2\sqrt{\pi\tau}}\exp\left[-\frac{x-x_0}{2}-\frac{\tau}{4}\right]\left\{\exp\left[-\frac{(x-x_0)^2}{4\tau}\right]+\exp\left[-\frac{(x+x_0)^2}{4\tau}\right]\right\}+\frac{1}{\sqrt{\pi}}e^{-x}\int_{(x+x_0-\tau)/2\sqrt{\tau}}^{\infty}e^{-z^2}\,dz$
$\frac{x^\beta\exp(-x)}{\Gamma(\beta+1)}$	$\frac{\beta-x}{x}$	$\frac{1}{1-e^{-\tau}}\left(\frac{x}{x_0\exp(-\tau)}\right)^{\beta/2}\times\exp\left[-\frac{x+x_0e^{-\tau}}{1-e^{-\tau}}\right]I_\beta\left(\frac{2e^{-\tau/2}\sqrt{xx_0}}{1-e^{-\tau}}\right)$
$\frac{\Gamma(\beta+\gamma+2)}{\Gamma(\beta+1)\Gamma(\gamma+1)}\times\frac{(1+x)^\beta(1-x)^\gamma}{2^{\alpha+\gamma+1}}$	$\frac{(\beta-\gamma)-(\beta-\gamma)x}{1-x^2}$	$\frac{(1+x)^\beta(1-x)^\gamma}{2^{\alpha+\gamma+1}}\sum_{n=0}^{\infty}e^{-n(n+\gamma+\beta+1)}\times A_nP_n^{\beta,\gamma}(x_0)P_n^{\beta,\gamma}(x)$, $A_n=\frac{(2n+\alpha+\gamma+1)\Gamma(n+\alpha+\gamma+1)}{\Gamma(n+\alpha+1)\Gamma(n+\gamma+1)n!}$

$$\begin{aligned}W(x)&=\left(1-\frac{\alpha}{\lambda}\right)p_s(x)-\frac{\alpha}{\lambda}x\frac{\partial}{\partial x}p_s(x)\\&=\left[\left(1-\frac{\alpha}{\lambda}\right)-\frac{\alpha}{\lambda}x\frac{(\partial/\partial x)p_s(x)}{p_s(x)}\right]p_s(x)\\&=\left[\left(1-\frac{\alpha}{\lambda}\right)-\frac{\alpha}{\lambda}\frac{A_1x^2+A_0x}{B_2x^2+B_1x+B_0}\right]p_s(x),\quad \lambda>\alpha.\end{aligned}\tag{84}$$

Table II contains the some of the examples given in Table I.

C. Other PDFs

As an example of a non-Pearson distribution one can consider so called Khinchin probability density

$$p_s(x)=\frac{1-\cos x}{\pi x^2}.\tag{85}$$

This distribution is interesting since none of its moments exists. Indeed, the characteristic function corresponding to this distribution is

$$\Theta(\omega)=\begin{cases}0, & |\omega|\leqslant 1\\ 1-|\omega|, & |\omega|<1\end{cases}\tag{86}$$

and is not differentiable at $\omega=0$. The last statement is equivalent to the fact that the PDF (85) does not have moments that converge. Nevertheless, it was shown in [19] that this distribution can be obtained if

TABLE II. Parameters of Markov process with exponential correlation and jumps.

$p_s(x)$	$\frac{d}{dx}\ln p_s(x)$	$W(x)$
$\exp(-x)$	-1	$\left[\left(1-\frac{\alpha}{\lambda}\right)+\frac{\alpha}{\lambda}x\right]\exp(-x)$
$\frac{x^\beta\exp(-x)}{\Gamma(\beta+1)}$	$\frac{\beta-x}{x}$	$\left[\left(1-\frac{\alpha}{\lambda}\right)-\frac{\alpha}{\lambda}(\beta-x)\right]\frac{x^\beta\exp(-x)}{\Gamma(\beta+1)}$
$\frac{\Gamma(\beta+\gamma+2)}{\Gamma(\beta+1)\Gamma(\gamma+1)}\times\frac{(1+x)^\beta(1-x)^\gamma}{2^{\alpha+\gamma+1}}$	$\frac{(\beta-\gamma)-(\beta+\gamma)x}{1-x^2}$	$\left[\left(1-\frac{\alpha}{\lambda}\right)-\frac{\alpha}{\lambda}\frac{(\beta-\gamma)x-(\beta+\gamma)x^2}{1-x^2}\right]\times\frac{\Gamma(\beta+\gamma+2)}{\Gamma(\beta+1)\Gamma(\gamma+1)}\frac{(1+x)^\beta(1-x)^\gamma}{2^{\alpha+\gamma+1}}$

$$W(x|y)=\frac{1}{\pi\exp(-\gamma)}\left[\alpha^{-\gamma}A^{\gamma-1}C_{\gamma}\left(\frac{x-y}{\alpha}\right)+\frac{\nu\exp(\gamma)-\nu\cos[(x-y)/\alpha]+(x-y)\sin[x-y/\alpha]}{\nu^{2}+(x-y)^{2}}\right], \tag{87}$$

where

$$C_{\gamma}(w)=\int_{w}^{\infty}\frac{\cos(z)}{z^{-\gamma}}dz \tag{88}$$

and

$$\gamma=\frac{\alpha}{\lambda}. \tag{89}$$

VI. CONCLUSIONS

In this paper we have considered three different methods of generating non-Gaussian processes with a given marginal PDF $p_s(x)$ and exponential correlation function. The difference between the three types of process lies in the proportion between diffusion, drift, and jumpiness of these processes. At the same time we have shown that all three types of process can be considered in the same framework of Markov processes generated by a stochastic differential equation. The results obtained in this paper can be used in a wide area of applications such as communication systems modeling (fading, error flows), biomedical applications (neuron activities), stochastic control, etc.

[1] S. Primak, Phys. Rev. E **61**, 100 (2000).

[2] V. Kontorovich and V. Lyandres, IEEE Trans. Signal Process. **42**, 1229 (1995).

[3] S. Primak and V. Lyandres, IEEE Trans. Signal Process. **46**, 1229 (1998).

[4] S. Xu, Z. Huang, and Y. Yao, IEEE Trans. Circuits Syst. Video Technol. **10**, 63 (2000).

[5] S. Xu and Z. Huang, IEEE Trans. Circuits Syst. Video Technol. **8**, 138 (1998).

[6] D. Middleton, *An Introduction to Statistical Communication Theory* (McGraw-Hill, New York, 1960).

[7] E. Conte and M. Longo, IEE Proc. F, Commun. Radar Signal Process. **134**, 191 (1987).

[8] A. Farina and P. Lombardo, Electron. Lett. **30**, 520 (1994).

[9] M. Rangaswamy, D. Weiner, and A. Ozturk, IEEE Trans. Aerosp. Electron. Syst. **29**, 111 (1993).

[10] A. Fulinski, Z. Grzywna, I. Mellor, Z. Siwy, and P. N. R. Usherwood, Phys. Rev. E **58**, 919 (1998).

[11] S. Primak and J. LoVetri, CRC, Technical Report, Ottawa, Canada (unpublished).

[12] J. Lee, R. Tafazolli, and B. Evans, Electron. Lett. **33**, 259 (1997).

[13] R. Rosenfeld, Proc. IEEE **88**, 1270 (2000).

[14] J. Luczka, T. Czernik, and P. Hanggi, Phys. Rev. E **56**, 3968 (1997).

[15] M. Popescu, C. Arizmendi, A. Salas-Brito, and F. Family, Phys. Rev. Lett. **85**, 3321 (2000).

[16] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, Berlin, 1983).

[17] V. Kontorovich and V. Lyandres, IEEE Trans. Signal Process. **43**, 2372 (1995).

[18] S. Primak, V. Lyandres, O. Kaufman, and M. Kliger, Signal Process. **72**, 61 (1999).

[19] G. Kotler, V. Kontorovich, V. Lyandres, and S. Primak, Signal Process. **75**, 79 (1999).

[20] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vol. 1.

[21] A. Haddad, IEEE Trans. Inf. Theory **16**, 529 (1970).

[22] T. Felbinger, S. Schiller, and J. Mlynek, Phys. Rev. Lett. **80**, 492 (1998).

[23] A. C. Marley, M. J. Higgins, and S. Bhattacharya, Phys. Rev. Lett. **74**, 3029 (1995).

[24] P. Rao, D. Johnson, and D. Becker, IEEE Trans. Signal Process. **40**, 845 (1992).

[25] J. Nakayama, IEICE Trans. Fundam. Electron. Commun. Comput. Sci. **E77-A**, 917 (1994).

[26] S. Karlin, *A First Course in Stochastic Processes* (Academic, New York, 1968).

[27] E. Wong, Proc. Symp. Appl. Math. **16**, 264 (1964).

[28] N. S. Jayant and P. Noll, *Digital Coding of Waveforms* (Prentice-Hall, Englewood Cliffs, NJ, 1984).

[29] *Handbook of Mathematical Functions*, edited by M. Abramoviz and I. A. Stegun (Dover, New York, 1971).

[30] C. W. Helstrom, *Markov Processes and Application in Communication Theory* (McGraw-Hill, New York, 1968), pp. 26–86.