Influence of walk-off, dispersion, and diffraction on the coherence of parametric fluorescence

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We consider the basic problem of spontaneous parametric generation from quantum noise fluctuations in the presence of a continuous plane-wave pump. We show, both numerically and analytically, that the walk-off between the down-converted fields is the key ingredient that leads to the generation of coherent fields in the parametric process. Along these lines, our theory reveals that, in the absence of walk-off, diffraction and chromatic dispersion in usual quadratically nonlinear materials only lead to incoherent erratic dynamics. Moreover, a two-dimensional study shows that, when the walk-off is exclusively temporal (or spatial), the parametric process is not able to yield the generation of spatially (or temporally) coherent fields. This study sheds light on the problem of coherence in parametric fluorescence and, in particular, allows us to explain various recent experimental observations.

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I. INTRODUCTION

The phenomenon of parametric generation from quantum noise is a fundamental physical process that has been widely studied since the advent of nonlinear optics in the 1960s. It refers to a parametric amplification process where energy is transferred from a pump laser beam into two fields of larger wavelength usually called the signal and the idler. The quantum noise amplification that is observed in the absence of any signal and idler seeds in quadratic optical crystals is called parametric fluorescence.

In the literature, parametric fluorescence was first extensively studied in the low-conversion-efficiency regime (i.e., linear regime), in particular for the characterization of its angular intensity distribution [1,2]. More recently, the growing interest in traveling-wave optical parametric generators as sources of high-power tunable femtosecond pulses motivated the study of parametric fluorescence in the nonlinear regime of pump depletion [3–6]. For instance, generation through parametric fluorescence of two-dimensional spatial solitary waves [7] and optical vortices [8] was recently reported. These experiments revealed, in particular, that in the parametric fluorescence process a transition occurs between two regimes characterized, respectively, by the incoherence and coherence of the generated signal and idler fields. In the transition toward the coherent regime the initial incoherence of the quantum noise is apparently smoothed down by the parametric process, to lead to regular spatiotemporal field distributions.

These experimental observations naturally raise the following question: What is the mechanism responsible for the incoherence-to-coherence transition in parametric fluorescence? The problem is not trivial, because the existence of this transition seems to depend critically on the experimental conditions, as revealed in Refs. [7,9], that report on the generation of spatially incoherent down-converted fields in the spontaneous parametric process.

An attempt to answer this question was proposed by Di Trapani *et al.* in Ref. [7]. In that work the authors interpreted the spontaneous formation of coherent solitary waves from quantum noise in terms of ''spatial mode locking,'' relying upon the analogy of this process and the temporal mode-locking in laser resonators. Following this interpretation, the generation of coherent down-converted fields would be closely related to the self-trapping mechanism characteristic of the solitary-wave formation process. Accordingly, as soon as self-trapping does not occurs, the generated signal and idler fields remain incoherent even in the strong pump depletion regime [7]. However, this physical interpretation in terms of ''spatial mode locking'' does not seem to provide a complete picture of the problem, since down-converted fields have also been generated with a high degree of coherence far from the solitary wave regime, as shown by studies of coherent pattern formation in parametric down-conversion [5,6].

In the present paper we address the problem from a different point of view. We consider the basic problem of parametric fluorescence from a continuous plane-wave pump. By means of numerical simulations and a very simple mathematical analysis of the equations that rule optical three-wave mixing in the classical limit, we show that the walk-off between the signal and idler waves is the key ingredient responsible for the onset of coherence in parametric down-conversion. The term ''walk-off'' refers here to either the spatial beam divergence due to the crystal birefringence or the group velocity difference that affects short pulses in dispersive crystals. The mathematical description of the role of the signal-idler walk-off is based on a linear analysis of the three-wave mixing equations. It reveals that coherent wave generation takes place in the linear regime of parametric fluorescence (i.e., with arbitrarily small pump depletion). Conversely, the natural chromatic dispersion (or diffraction in the spatial domain) encountered in usual materials with quadratic nonlinearities, only leads to an incoherent evolution of the optical fields. Moreover, a two-dimensional study reveals that when the nature of the walk-off is exclusively temporal (or spatial), the parametric process is not able to yield the generation of spatially (temporally) coherent fields. To motivate an experimental confirmation of our theory, we present a numerical simulation of a realistic experiment in which the transition from incoherent to coherent fluorescence could be observed and studied.

The paper is organized as follows. In Sec. II, we present a typical three-wave interaction model that describes the parametric process. The relevant role of the walk-off on the coherence of parametric fluorescence is discussed through numerical simulations in Sec. III, and through a linear stability analysis in Sec. IV. In Sec. V we discuss the practical relevance of the walk-off-induced coherence mechanism in the context of some recent experimental results. The comparison between the roles of walk-off and of dispersion (or diffraction) in the coherence of the generated fields is presented in Sec. VI. This analytical study is then confirmed by numerical simulations in a realistic experimental situation in Sec. VII. In order to complete our analysis, in Sec. VIII we consider the two-dimensional geometry of the parametric process, and study the influence of the walk-off and diffraction on both the spatial and temporal coherences of the down-converted fields. Finally, Sec. IX gives a summary of our results, and briefly discusses the related issues and links with other problems in physics.

II. GOVERNING EQUATIONS

Our starting points are the usual equations governing the spatiotemporal evolution of optical fields in quadratic nonlinear media in one space dimension. The evolution of the slowly varying envelopes A_i of these fields, of frequency ω_i and wave number k_i, obeys the coupled partial differential equations

$$\frac{1}{v_1}\frac{\partial A_1}{\partial t}+\frac{\partial A_1}{\partial z}+i\beta_1\frac{\partial^2 A_1}{\partial z^2}=\sigma_1 A_3 A_2^*+\sqrt{\epsilon_1}\xi_1(z,t), \tag{1a}$$

$$\frac{1}{v_2}\frac{\partial A_2}{\partial t}+\frac{\partial A_2}{\partial z}+i\beta_2\frac{\partial^2 A_2}{\partial z^2}=\sigma_2 A_3 A_1^*+\sqrt{\epsilon_2}\xi_2(z,t), \tag{1b}$$

$$\frac{1}{v_3}\frac{\partial A_3}{\partial t}+\frac{\partial A_3}{\partial z}+i\beta_3\frac{\partial^2 A_3}{\partial z^2}=-\sigma_3 A_2 A_1+\sqrt{\epsilon_3}\xi_3(z,t), \tag{1c}$$

with $\omega_3=\omega_2+\omega_1$. For definiteness we call A_1, A_2, and A_3 the signal, idler, and pump waves, respectively. $\sigma_i=dk_i/n_i^2$, v_i and $\beta_i=v_i^2k_i''/2$ $[k_i''=(\partial^2k/\partial\omega^2)_i]$ are the coupling constants, the velocities, and the dispersion coefficients of the crystal at frequency ω_i, respectively, d being the effective nonlinear susceptibility. From now on we will assume for simplicity and without loss of generality that $\sigma=\sigma_1=\sigma_2=\sigma_3/2$ and $k''=k_i''$ ($i=1,2$, and 3). The complex stochastic variables $\xi_i(z,t)$ are Gaussian, with a zero mean $\langle\xi_i(z,t)\rangle=0$ and correlation $\langle\xi_i(z,t)\xi_j^*(z',t')\rangle=\delta_{i,j}\delta(t-t')\delta(z-z')$, and ϵ_i represents the noise intensity of the field A_i. This description of noise by means of classical-looking Langevin equations was introduced in Ref. [10] to study quantum fluctuations in optical parametric oscillators below threshold. It was also used to show the formation of noise-sustained convective structures in both cubic [11] and quadratic [12] nonlinear optical cavities.

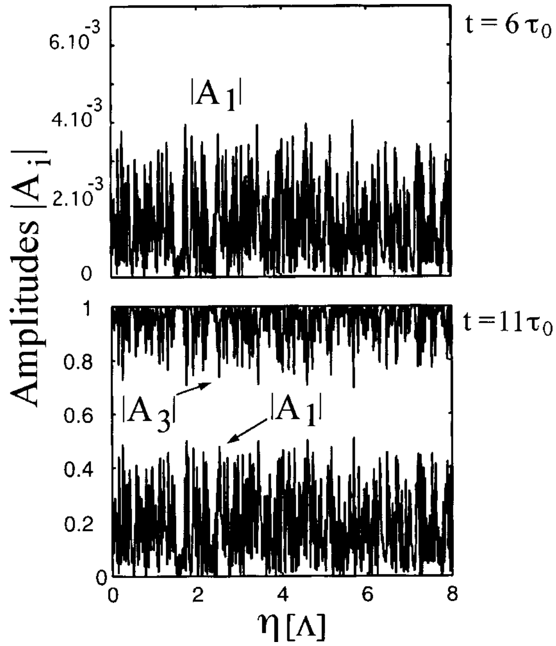

FIG. 1. Evolution of the interacting envelopes without walk-off in their own reference frame (degenerate case) in the linear ($t=6\tau_0$) and nonlinear ($t=11\tau_0$) regimes: the parametric amplification is incoherent (amplitudes are given in units of e_0, and η in units of $\Lambda=v_3\tau_0=0.022$ mm).

Note that Eqs. (1) also hold for a description of purely transverse spatial evolution governed by diffraction and spatial walk-off. Indeed, the substitutions $(1/v_i)(\partial/\partial t)\rightarrow\chi_i\partial/\partial y$ where χ_i represents the spatial walk-off, and $\beta_i(\partial^2/\partial z^2)\rightarrow-\kappa_i(\partial^2/\partial y^2)$, where $\kappa_i=1/2k_i$ is the diffraction parameter, transform Eqs. (1) into the well-known equation of transverse effects in quadratic crystals [13].

III. ROLE OF WALK-OFF

To obtain a basic insight into the role of walk-off in parametric fluorescence, it is interesting to consider first the artificial situation of a degenerate down-conversion ($\omega_1=\omega_2$, $A_1=A_2$) in an ideal dispersionless quadratic crystal, so that there is no walk-off between the signal and pump waves ($v_{1,2}=v_3$) and $\beta_i=0$ in Eqs. (1). In this situation, through Eqs. (1) we numerically simulated the basic propagation problem of a continuous pump with a zero initial signal field amplitude apart from the presence of quantum noise fluctuations (vacuum field). A typical result is illustrated in Fig. 1, that shows the evolution of the pump and signal envelopes in the reference frame traveling at their common group velocity. In this example the injected pump intensity is $I=100$ MW/cm^2 for $\sigma=2.8\times10^{-5}$ V^{-1}. As an initial condition in $t=0$, we take a continuous wave envelope for the pump $A_3(z,t=0)=e_0$ and a small amplitude noisy signal field $A_1(z,t=0)=\sqrt{\epsilon_1}\xi_1(z,t=0)$. To evaluate the noise in-

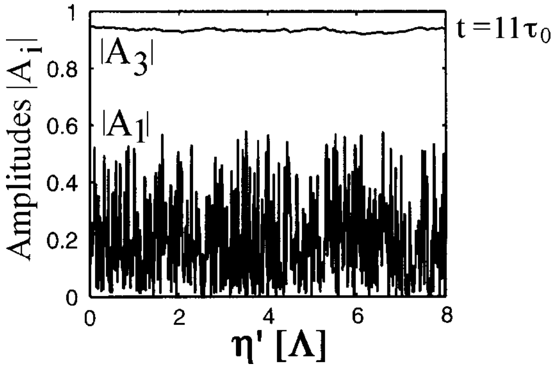

FIG. 2. Same as in Fig. 1, but in the presence of a walk-off between the pump and the down-converted degenerate signal wave ($\eta' = z - v_1 t$).

tensity ϵ_1, we consider that an amplification factor of 10 orders of magnitude is necessary to obtain a signal intensity comparable to that of the pump [7], which gives a value of the noise intensity of $\epsilon_1 = 10^{-10} e_0^2$. It is worth noting that the value of the noise intensity ϵ_1 does not affect our numerical results. Indeed, we obtained the same qualitative results for amplification factors ranging from 10^8 to 10^{14}. As revealed by the structure of Eqs. (1), this problem of parametric amplification is characterized by the time constant $\tau_0 = 1/(\sqrt{v_1 v_2}\sigma e_0)$ that determines the temporal growth rate of the signal field amplitude in the crystal. For this reason we will use τ_0 as the time unit in the presentation of our results. With the parameters considered above, we obtain $\tau_0 = 12$ ps.

As can be seen in Fig. 1, the signal field keeps its initial incoherence (determined by the quantum noise) even in the nonlinear regime of pump depletion ($t > 10\tau_0$). In other words, the initial incoherent fluctuations affect the long term evolution of the parametric process. This result can be easily interpreted through a geometrical approach of the parametric process proposed in Refs. [14,15]. First we note that, in the absence of walk-off and dispersion, Eqs. (1) can be viewed as a continuous set of ordinary differential equations for the variable $\eta = z - v_{1,3} t$. This means that the field evolution in a particular point η is independent of the field evolution in the neighboring point $\eta + d\eta$, so that there is no means to smooth down the initial noisy signal envelope. Following the geometrical approach, due to the Hamiltonian nature of the problem, the dynamical evolution is fully described by closed orbit trajectories in a two-dimensional phase space representation [15,16]. Consequently, the field amplitude at any particular point η sits on a specific orbit determined by the random initial condition. It thus appears evident that the signal field keeps its initial incoherence, and that the pump field loses its initial coherence.

Following this very simple reasoning, one would think that the introduction of the walk-off between the pump and signal waves is sufficient to increase the signal coherence during the amplification process. Indeed, in the presence of walk-off the system of equations (1) is no longer equivalent to a set of ordinary differential equations, and the field amplitudes in η and $\eta + d\eta$ are therefore coupled so as to increase coherence. However, our numerical simulations reveal that the situation is more complex than this simple picture. Figure 2 shows a typical result obtained when introducing a walk-off between the pump and signal waves for the same parameters as in Fig. 1, but with the velocities $v_3 = 1.28 \times 10^8$ ms^{-1} and $v_1 = 1.32 \times 10^8$ ms^{-1} corresponding to a realistic walk-off parameter of $1/v_3 - 1/v_1 = 0.24$ ps/mm. As in Fig. 1, the signal is amplified, but remains incoherent up to the nonlinear regime. However, the presence of the walk-off drastically changes the evolution of the pump, that no longer loses its coherence since, as is apparent in Fig. 2, its envelope now remains smooth for arbitrarily long interaction times.

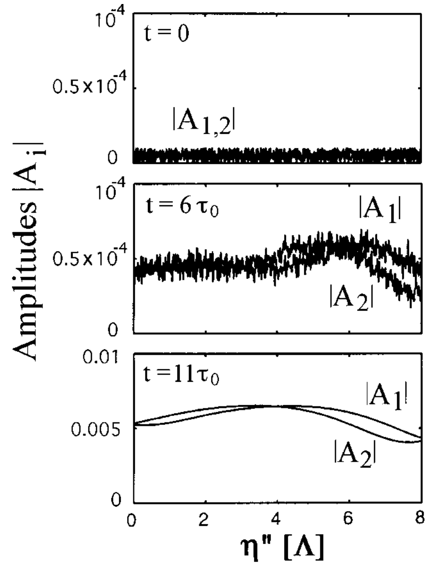

FIG. 3. Amplitude evolution in the linear regime of the nondegenerate parametric process: the walk-off between the signal and the idler waves leads to their coherent amplification in the linear regime [amplitudes are given in units of e_0; $\tau_0 = 12$ ps, $\Lambda = v_3 \tau_0 = 0.022$ mm, and $\eta'' = z - (v_1 + v_2)t/2$].

Let us now consider the situation where there is a walk-off between the down-converted fields, namely, between the signal and the idler waves ($v_1 \neq v_2$). This configuration typically arises in the nondegenerate configuration of the parametric interaction ($\omega_1 \neq \omega_2$, $A_1 \neq A_2$). In this case, the amplification scenario is of a fundamentally different nature. A typical result is represented in Fig. 3, that illustrates the evolution of the down-converted amplitudes $A_{1,2}$ in the reference frame of their average group velocity $(v_1 + v_2)/2$. The parameters of the simulation are the same as in Fig. 1, except that we introduce a temporal walk-off with the velocities $v_1 = 1.32 \times 10^8$ ms^{-1} and $v_2 = 1.28 \times 10^8$ ms^{-1} for the signal and idler waves, respectively. The incoherent initial noise is rapidly smoothed down during propagation, which leads to a coherent parametric amplification process ($t > 9\tau_0$).

Note that these numerical results are not dependent on the statistical properties of the stochastic functions $\xi_i(z,t)$. In particular, the same qualitative results were also obtained in the absence of the stochastic functions $\xi_i(z,t)$ [$\epsilon_i = 0$ in Eqs. (1)] and starting the numerical simulations from a weak random complex noise for the signal and idler fields.

IV. STABILITY ANALYSIS

The remarkable aspect of the scenario presented above is that the amplification process becomes coherent at the very beginning of the parametric interaction. In other words, the transition occurs within the linear regime of the parametric interaction, far from the nonlinear regime of pump depletion. This is visible in Fig. 3 from the values of the signal and idler amplitudes. This aspect of the problem suggests that a simple linear analysis of Eqs. (1) is sufficient to explain the role of walk-off in the coherence of the parametric generation process.

Indeed, linearizing Eqs. (1) with respect to the signal and idler amplitudes leads to a linear problem that may be treated through the classical theory of instabilities in wave propagation [17]. Since our purpose here is to see how the walk-off affects the nature of the parametric interaction, we neglect dispersion for clarity [i.e., we consider $\beta_i = 0$ in Eqs. (1)]. It is convenient to carry out the stability analysis in the reference frame of the average group velocity $(v_1+v_2)/2$ of the down-converted fields. This reference frame is defined by the variables $[\zeta = z-(v_1+v_2)t/2, \tau = t]$. Thus, assuming an undepleted pump wave in the whole interaction domain, we can linearize Eqs. (1) near the trivial solution ($A_3 = e_0$, $A_1 = A_2 = 0$, for $\epsilon_{1,2,3} = 0$) to obtain the following equation for the evolution of the signal and idler envelopes,

$$\frac{\partial^2 A_{1,2}}{\partial \tau^2} - \delta^2 \frac{\partial^2 A_{1,2}}{\partial \zeta^2} = v_1 v_2 \sigma^2 e_0^2 A_{1,2}, \tag{2}$$

where $\delta = (v_2 - v_1)/2$. The general solution of this equation reads

$$A_{1,2}(\zeta,\tau) = \int_{-\infty}^{+\infty} \tilde{A}_{1,2}(q) \exp[\gamma_w(q)\tau + iq\zeta] dq, \tag{3}$$

where γ_w and q obey the following dispersion relation:

$$\gamma_w(q) = \sqrt{v_1 v_2 \sigma^2 e_o^2 - \delta^2 q^2}. \tag{4}$$

The amplitudes $\tilde{A}_{1,2}(q)$ in Eq. (3) represent the initial signal and idler perturbations in the wave number space.

We can analyze Eq. (3) by keeping only the unstable modes, i.e., $\mathrm{Re}[\gamma_w(q)] > 0$, since only these modes contribute significantly to the integral over a certain time τ. It is easy to see from the dispersion relation [Eq. (4)] that the function $\mathrm{Re}[\gamma_w(q)]$ exhibits a maximum at $q = 0$. This means that the mode $q = 0$ that corresponds to a homogenous perturbation is preferentially amplified in the parametric process. Equation (4) shows that the associated spectral gain curve $\mathrm{Re}[\gamma_w(q)]$ becomes narrower as the walk-off increases. We can therefore conclude that the selection of the homogeneous mode is more efficient when the walk-off is larger. Conversely, when there is no walk-off, i.e., $\delta = 0$, the spectral gain curve $\mathrm{Re}[\gamma_w(q)]$ becomes flat and the homogeneous mode is no longer favored. In this case all the modes that are present in the initial condition $\tilde{A}_{1,2}(q)$ are amplified in the same way, and no regular pattern is selected in the system. This brief analysis explains in very simple terms the essential role of walk-off in the coherence of parametric fluorescence. The walk-off simply appears as having a filtering action that favors the formation of homogeneous patterns.

This conclusion is confirmed by a rigorous mathematical treatment applied to Eq. (3). Indeed, the dispersion relation γ_w of the real variable q [Eq. (4)] can be analytically continued in the complex q plane, and one can simply notice that the complex dispersion relation $\gamma_w(q)$ exhibits a saddle point of the first order in $q = 0$, i.e., $[\partial \gamma_w / \partial q]_{q=0} = 0$, and $[\partial^2 \gamma_w / \partial q^2]_{q=0} \neq 0$. We remark that, due to the presence of the square root in the expression (4) of γ, the integrand exhibits two branch cuts that can always be chosen so that they do not cross the real axis of integration. In this way, the integral in Eq. (3) can be calculated through the steepest descent method, which yields a solution for long times τ [18],

$$A_{1,2}(\zeta,\tau) \propto \frac{1}{\sqrt{-[\partial^2 \gamma_w / \partial q^2]_{q=0} \tau}} \tilde{A}_{1,2}(q=0) \exp(\tau/\tau_0),$$

where $\tau_0 = 1/\sqrt{v_1 v_2}\sigma e_0$. This clearly shows that, independently of the initial condition, the signal and idler fields evolve toward a homogeneous pattern (ζ independent) in the amplification process, provided that they exhibit a mutual walk-off, in accordance with the numerical simulation of Fig. 3. In the absence of walk-off, the dispersion relation becomes q independent, so that the saddle point no longer exists and the steepest descent method can no longer be applied. In this case, since all the modes of the initial quantum noise are amplified with the same gain, there is no transition toward coherent fields, as discussed in our numerical analysis (see Figs. 1 and 2). Note finally that the above analysis predicts that a single walk-off between the pump and the down-converted signal and idler fields does not lead to a coherent parametric process, as shown in Fig. 2. Indeed, the walk-off that exists with respect to the pump does not enter the linearized problem [Eq. (2)], and is therefore irrelevant as regards the coherence of the generated fields, in agreement with our numerical simulations (see Fig. 2).

Let us note that this analysis corroborates the early results obtained in the pioneering works on parametric fluorescence (Refs. [2]). Although the role of walk-off on parametric fluorescence was not discussed explicitly in these works, it can be found mathematically that the walk-off affects the spectral bandwidth of spontaneous parametric emission. By considering the characteristic function of the parametric amplification $f(\Delta k) = \sin(\Delta k L/2)/(\Delta k L/2)$, where Δk is the phase mismatch and L the crystal length, the authors expanded the phase matching Δk in frequency; it was then straightforward to find that the walk-off narrows the bandwidth of the parametric amplifier. Here we find this result from a completely different formalism, that will allow us to compare the roles of the walk-off and dispersion (or diffraction) in a very simple way, even when the parametric process takes place in two dimensions (see Secs. VI–VIII).

V. DISCUSSION

Before proceeding with our study it is interesting to discuss the practical relevance of the mechanism of walk-off-induced coherence studied above through the analysis of some recent experimental results reported in the literature. Our conclusions are consistent with the results of a recent study that reports on a phenomenon of incoherent parametric amplification [7]. It is thus interesting to consider the specific experimental conditions in that previous study in order to show whether its outcome can be explained in the framework of our theory or not. First let us note that the experiment was carried out with a nonlinear crystal used in the type-I temperature tuned noncritical phase-matching condition, i.e., in the absence of spatial walk-off. The authors observed a process of coherent parametric amplification only with sufficiently narrow pump beams. Under these conditions, they observed the formation of coherent spatial solitary waves in which diffraction is balanced by quadratic nonlinearity and the so-called ''spatial mode locking'' mechanism occurs. This situation cannot be described in our theory because of the predominant role of diffraction [i.e., the role of the second derivatives in our model equations (1) that will be discussed in Sec. VI]. Conversely, as the beam size of the injected pump increases (and then tends to the plane wave configuration considered in our theory), the process of parametric amplification becomes incoherent. Note that our theory also agrees with an other experiment performed with a LBO (lithium triborate) crystal in the type-I noncritical phase-matching configuration, which also led to the generation of incoherent nondegenerate signal and idler fields [9].

Let us remark that as these experiments were realized in two dimensions, their interpretation thus requires a two-dimensional analysis. However, since these experiments were realized in the absence of spatial walk-off, it is clear that the extension of our analysis to two dimensions is straightforward and does not alter the validity of our conclusions as regards the coherence of the generated fields. Nevertheless, the study of the two-dimensional problem in the presence of walk-off is not trivial, and will be discussed in detail in Sec. VIII.

VI. ROLE OF DISPERSION OR DIFFRACTION

Although our interpretation of the role of the walk-off on the coherence of the parametric process is corroborated by various experiments, it is interesting to analyze the role of the dispersion effect (or either diffraction in the spatial case) that has been neglected in our theory. The motivation for this analysis is that, since walk-off and dispersion are two processes that share the same origin, one would be tempted to conclude that dispersion leads to the coherence of the parametric fluorescence exactly as walk-off does. In order to show that this conclusion is not valid, here we study the influence of dispersion on the parametric fluorescence process in the framework of the linear stability analysis considered above (Sec. IV). Our purpose is to show how the dispersion affects the parametric instability. We then first assume the absence of walk-off between the down-converted fields for the sake of clarity. In other words, we consider a degenerate configuration of the parametric interaction where the signal and idler fields propagate with the same group velocity v_1. It is convenient to carry out the stability analysis in the reference frame of a signal group velocity that is defined by the variables $[\zeta=z-v_1 t, \tau=t]$. Note that since we consider a degenerate configuration ($v_1=v_2$), the variable ζ is the same as that introduced in Sec. IV. Assuming an undepleted pump wave in the whole interaction domain, we can derive the evolution of the signal envelope by linearizing Eqs. (1):

$$\partial_\tau A_1=\sigma v_1 e_0 A_1^* - i\beta\partial_{\zeta\zeta}A_1 .$$

The solution of this equation is given by the general expansion (3); however, in this case, the corresponding dispersion relation $\gamma_d(q)$ depends on the dispersion parameter β, instead on the walk-off δ:

$$\gamma_d(q)=\sqrt{v_1^2\sigma^2 e_o^2-\beta^2 v_1^2 q^4}. \tag{5}$$

Following the same procedure as that outlined in Sec. IV, we can look at the long term evolution of the parametric instability by studying the unstable modes q of the above dispersion relation, i.e., the modes satisfying $\mathrm{Re}[\gamma_d(q)]>0$. At first sight, it seems that the dispersion relation that accounts for dispersion [Eq. (5)] exhibits the same properties as that obtained with the walk-off alone [Eq. (4)]. Indeed, since $\gamma_d(q)$ exhibits a maximum at $q=0$, one may expect that the homogenous mode will be selected by the system in its long term evolution, exactly as in the case of the walk-off effect. However, let us point out that, quite importantly, the dispersion relation $\gamma_d(q)$ displays a flatter peak at $q=0$ as compared to that of $\gamma_w(q)$ [note that $\gamma_w^2(q)$ varies as q^2, while $\gamma_d^2(q)$ as q^4]. This difference can be simply interpreted by stating that the selection of the homogenous mode $q=0$ is less efficient when the process is ruled by dispersion rather than walk-off.

Let us consider this aspect quantitatively by a rigorous mathematical treatment of Eq. (3). The main difference between the two complex dispersion relations $\gamma_d(q)$ and $\gamma_w(q)$ is that $\gamma_d(q)$ exhibits a saddle point of fourth order (i.e., $[\partial^i\gamma_w/\partial q^i]_{q=0}=0 (i=0,1,2,3)$ and $[\partial^4\gamma_w/\partial q^4]_{q=0}\neq 0$) while it is of second order for $\gamma_w(q)$ (i.e., $[\partial^2\gamma_w/\partial q^2]_{q=0}\neq 0$). This means that the evolution of the down-converted fields in the presence of the walk-off takes the form

$$A_{1,2}(\zeta,\tau)\propto\int_{-\infty}^{+\infty}\exp[-|\tfrac{1}{2}\gamma_w^{(2)}(0)|q^2\tau]\exp(iq\zeta)dq,$$

while in the presence of the dispersion it takes the form

$$A_1(\zeta,\tau)\propto\int_{-\infty}^{+\infty}\exp[-|\tfrac{1}{24}\gamma_d^{(4)}(0)|q^4\tau]\exp(iq\zeta)dq.$$

These expressions allow us to introduce two distinct characteristic times $\tau_w=2/[|\gamma_w^{(2)}(0)|q^2]$ and $\tau_d=24/[|\gamma_d^{(4)}(0)|q^4]$, where $\gamma^{(i)}(0)=[\partial^i\gamma/\partial q^i]_{q=0}$. They represent the times required to obtain the emergence of a coherent parametric process in the presence of walk-off and dispersion, respectively. In order to compare the influence of these two effects, it is more convenient to introduce the characteristic lengths $l_{w,d} = v_1 \tau_{w,d}$ associated with the corresponding characteristic times $\tau_{w,d}$. After some simple algebra, one finds $|\gamma_w^{(2)}(0)| = \delta^2/(\sqrt{v_1 v_2}\sigma e_0)$ and $|\gamma_d^{(4)}(0)| = 12\beta^2 v_1/(\sigma e_0)$ which leads to the following relation between the characteristic lengths of the walk-off and the dispersion effects:

$$l_d = \frac{\delta^4}{2\sigma e_0 \beta^2 v_1^3 v_2} l_w^2.$$

This expression allows us to compare the relative strength of the two basic mechanisms that are liable to yield a coherent parametric amplification process. Let us compare the characteristic lengths l_d and l_w in the context of a realistic experimental situation. As an example, we consider the same parameters as those specified in Fig. 3, and a typical dispersion coefficient of $k'' = 0.5$ ps^2/m. According to the numerical simulation of Fig. 3, we may estimate the characteristic walk-off length by $l_w = 9 v_1 \tau_0 = 1.4$ cm, since we observed the onset of coherence after nine characteristic times τ_0. With these parameters, we obtain $l_d \simeq 17 \times 10^3 l_w^2$, showing that a crystal of approximatively 240 m would be necessary to observe the mechanism of dispersion-induced coherence during parametric amplification. Even if this experiment was feasible, we may reasonably expect that the nonlinear stage of the parametric amplification (i.e., pump depletion) would occur prior to the emergence of the coherent dynamics. In other words, the observation of the mechanism of dispersion-induced coherence appears to be impossible with any realistic quadratic nonlinear materials.

Clearly, the situation remains unchanged when one considers the role of diffraction in the spatial domain, instead of dispersion in the temporal domain. Indeed, the parameter that represents diffraction, i.e., $\kappa_i = 1/2k_i$ (Sec. II), is about one order of magnitude greater than β. In these conditions, one may expect that a crystal length of several meters is necessary in order to observe the mechanism of diffraction-induced coherence in parametric generation, which is well beyond the possibilities of available technology. This conclusion is in agreement with the experiment reported in Ref. [7]. It was shown that diffraction yields a coherent parametric amplification only for very narrow injected pump beams, and thus far from the plane wave configuration considered in our analysis. Conversely, as the beam size of the pump increases, the amplification process becomes incoherent in agreement with our theory.

VII. COMPARISON BETWEEN DISPERSION AND WALK-OFF

According to the above analysis, the walk-off between the down-converted optical fields seems to be a unique mechanism able to yield a coherent parametric process in its linear regime of undepleted pump. Although the various experiments discussed in Sec. V seem to corroborate this conclusion, it would be of great interest to observe, in a single experiment and in a straightforward way, the predicted walk-off-induced coherent transition in parametric amplification. Here we discuss the experimental conditions required for this observation. In our discussion we take into account the role of dispersion (or diffraction) that has been neglected in the numerical simulations presented above. We compare the role of the dispersion to the role of walk-off by comparing the degenerate and nondegenerate configurations of the parametric interaction.

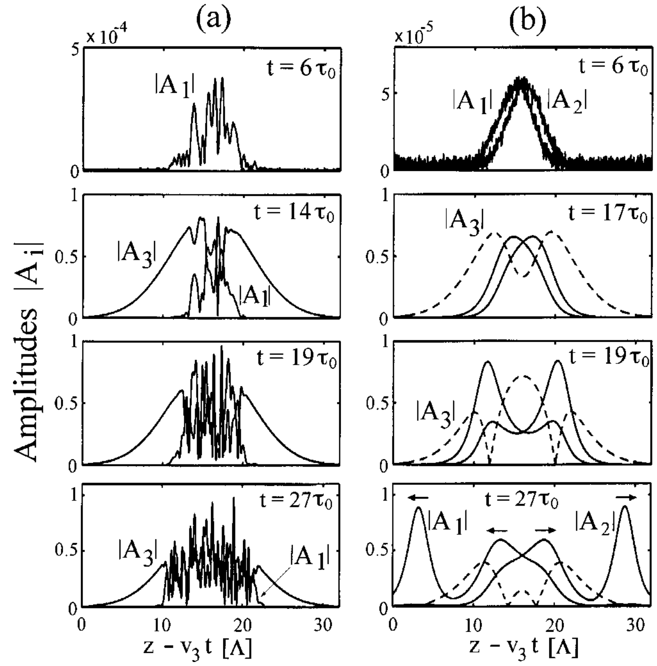

FIG. 4. Evolution of the three interacting envelopes in the reference frame of the pump. (a) In the presence of dispersion. (b) In the presence of both dispersion and walk-off, the signal and idler envelopes are represented by a solid line, and the the pump by a dashed line (amplitudes are given in units of e_0, $\tau_0 = 12$ ps, $\Lambda = v_3 \tau_0 = 0.022$ mm).

Due to the short interaction length available in typical nonlinear crystals, it is more realistic to consider the amplification process from an intense pump pulse instead of a continuous wave pump. A typical example of nonlinear wave dynamics in this condition is shown in Fig. 4, that illustrates the evolution of interacting fields in the reference frame of the pump pulse. Here we considered a Gaussian pump pulse profile. In Fig. 4(a) the role of dispersion is taken into account without walk-off (i.e., the degenerate case), while the effects of both dispersion and walk-off are included in Fig. 4(b) (i.e., the nondegenerate case). In this example the pump pulse width is 60 ps and its peak intensity is $I = 100$ MW/cm^2. It is launched in a crystal of length $L = 4.2$ cm, with an effective nonlinear coefficient $d = 10$ pm/V and a dispersion $k'' = 0.5$ ps^2/m. The velocities $v_3 = 1.30 \times 10^8$ ms^{-1}, $v_1 = 1.32 \times 10^8$ ms^{-1}, and $v_2 = 1.28 \times 10^8$ ms^{-1} are considered, that correspond to the case of the $LiNbO_3$ crystal with the wavelengths $\lambda_3 \simeq 1$ μm, $\lambda_1 \simeq 1.35$ μm, and $\lambda_2 \simeq 3.85$ μm. Experimentally, going from the situation of Fig. 4(a) to the situation of Fig. 4(b) can be done, for instance, through temperature control of a quasiphase-matched interaction [19].

As is clearly visible in Fig. 4(a), even in the presence of dispersion, the parametric process follows an erratic evolution in the absence of walk-off, in agreement with our theory. However, it is worth noting that in the linear regime, the dispersion effect tends to smooth the initial noise fluctuations despite the absence of walk-off [Fig. 4(a), $t=6\tau_0$]. This is no longer the case for the long term evolution, that clearly exhibits erratic dynamics that continuously spread over the whole pump pulse. This scenario, applied to the spatial domain, is similar to that reported in an experiment on spontaneous solitary-wave generation with large input pump beams in which diffraction alone was not able to provide coherence [7].

Conversely, as shown in Fig. 4(b), the walk-off effect induces a coherent parametric amplification whose evolution is not affected qualitatively by dispersion. Note that the walk-off reduces the amplification of the down-converted fields in the linear regime [see Figs. 4(a) and 4(b) at $t=6\tau_0$]. This is a natural consequence of the group-velocity difference that limits the effective duration of the parametric interaction [3,4]. As the down-converted fields grow, they are advected away from each other ($t>17\tau_0$) because of their opposite walk-off directions [note that in the numerical example of Fig. 4(b) we have chosen $v_3=(v_1+v_2)/2$]. In the nonlinear regime, pump depletion around the pulse peak occurs ($t=17\tau_0$), that eventually leads to a π-phase change in the pump envelope ($t=19\tau_0$), that in turn leads to a back-conversion from the down-converted fields to the pump [5]. This back-conversion process, combined with the effect of walk-off, leads to a confinement of the signal and idler pulses that travel away from the pump pulse with their opposite velocities. This process of self-pulse generation repeats until the energy of the pump is exhausted. This numerical example shows that, owing to the walk-off between the interacting waves, one may expect the generation of localized coherent structures from noise fluctuations in a way akin to what has been recently suggested for symbiotic solitary waves in quadratic nonlinear media [20].

VIII. TWO-DIMENSIONAL SPATIOTEMPORAL DYNAMICS

In the previous sections we identified the walk-off as the key ingredient responsible for the onset of coherence in the parametric fluorescence. However, it is worth noting that our analysis was limited to the purely one-dimensional case, and it would be interesting to study the influence of the walk-off in the more complex two-dimensional problem. More precisely, when one considers the two-dimensional parametric process in the presence of temporal or spatial walk-off, the following question naturally raises: Is a pure temporal (or either spatial) walk-off able to yield a coherent spatial and temporal dynamics? This problem is not trivial, since we have two antagonistic effects as regards the coherence of the generated fields: on the one hand, the temporal (or spatial) walk-off favors a coherent behavior, and, on the other hand, diffraction (or dispersion) leads to an incoherent behavior.

Let us study this two-dimensional configuration of the parametric process in the framework of the linear stability analysis considered above (Sec. IV). In the following we will study the particular case where the temporal walk-off competes with diffraction. However, let us emphasize that our results are also relevant for the situation where the spatial walk-off competes with dispersion, or even with diffraction in the case where the parametric process takes place in two transverse dimensions for negligible temporal walk-off. In this view, our purpose here is to see how the diffraction and the temporal walk-off affect the nature of parametric fluorescence. We then neglect the dispersion effect that was revealed to be irrelevant as regards the coherence of the generated fields.

To study the influence of the temporal walk-off, it is more convenient to carry out a stability analysis in the reference frame of the average group velocity $(v_1+v_2)/2$ of the signal and idler fields that is defined (Sec. IV) by the following variables $[\zeta=z-(v_1+v_2)t/2, \tau=t]$. Assuming an undepleted plane wave for the pump field in the whole interaction domain, the down-converted fields evolve according to the linear equations

$$\frac{\partial A_1}{\partial \tau}-\delta\frac{\partial A_1}{\partial \zeta}-i\rho_1\frac{\partial^2 A_1}{\partial y^2}=\sigma v_1 e_0 A_2^*, \tag{6a}$$

$$\frac{\partial A_2}{\partial \tau}+\delta\frac{\partial A_2}{\partial \zeta}-i\rho_2\frac{\partial^2 A_2}{\partial y^2}=\sigma v_2 e_0 A_1^*, \tag{6b}$$

where $\rho_i=v_i\kappa_i$ represents the effective diffraction parameter (Sec. II) and $\delta=(v_2-v_1)/2$ the velocity mismatch of the down-converted fields. The solution to the linear equations (6) reads

$$A_{1,2}(\mathbf{r},\tau)=\int\int_{\Re^2}\hat{A}_{1,2}(q,p)\exp[\gamma(q,p)\tau+i(q\zeta+py)]dq\,dp, \tag{7}$$

where $\mathbf{r}=(\zeta,y)$ and $\hat{A}_{1,2}(q,p)=\int\int_{\Re^2}A_{1,2}(\zeta,y,\tau=0)\exp[-i(q\zeta+py)]dqdp$ is the Fourier transform of the initially fluctuating signal and idler fields. The dispersion relation $\gamma(q,p)$ reads

$$\gamma(q,p)=-i\Delta p^2+\sqrt{1/\tau_0^2-\delta^2 q^2-m^2p^4+2\delta m q p^2},$$

where $\Delta=(\rho_1-\rho_2)/2$ and $m=(\rho_1+\rho_2)/2$ are the difference and average values of the diffraction parameters, respectively. Following the procedure outlined in Sec. IV, it is worth noting that the complex dispersion relation $\gamma(q,p)$ exhibits a saddle point at $q_0=p^2m/\delta$, so that the integral over q in Eq. (7) can be calculated by the steepest method that yields an asymptotic expansion for the generated fields,

$$A_{1,2}(\mathbf{r},\tau)\propto\frac{1}{\delta}\int_{-\infty}^{+\infty}\hat{A}_{1,2}(q_0,p)\exp[\Gamma(p)\tau]\times\exp[i(py+p^2m\zeta/\delta)]dp, \tag{8}$$

where $\Gamma(p)=1/\tau_0-i\Delta p^2$.

At this point, it is interesting to point out some important aspects of the structure of integral (8). First, let us note that, as discussed previously in Sec. IV, the function $\mathrm{Re}[\Gamma(p)]$ is analogous to a spectral gain curve whose typical bandwidth here is inversely proportional to Δ. We may then anticipate that the parameter Δ plays a key role in the coherence of parametric fluorescence.

A. Case $\Delta=0$

Let us first consider the simplest case of $\Delta=0$, i.e., $v_1\kappa_1=v_2\kappa_2$. In this situation, we note that, as the ratio m/δ tends to zero (i.e., the diffraction parameter becomes negligible with respect to the walk-off), the second term in the second exponential of Eq. (8) becomes negligible, and the asymptotic evolution of the down-converted fields is simply given by the inverse Fourier transform of $\hat{A}_{1,2}(q_0,p)$ with respect to p,

$$A_{1,2}(\zeta,y,\tau)\propto\frac{1}{\delta}\exp(\tau/\tau_0)\tilde{A}_{1,2}(q_0,y), \tag{9}$$

where $\tilde{A}_{1,2}(q_0,y)$ may be written as

$$\tilde{A}_{1,2}(q_0,y)=\int_{-\infty}^{+\infty}A_{1,2}(\zeta,y,\tau=0)\exp(-iq_0\zeta)d\zeta. \tag{10}$$

It becomes apparent from Eq. (9) that, during the amplification process, the signal and idler fields evolve toward a homogenous pattern along the axis ζ, which simply means that the generated fields are temporally coherent. Conversely, the asymptotic behavior of the down-converted fields along the axis y remains erratic because of the presence in Eq. (9) of a noisy function $\tilde{A}_{1,2}(q_0,y)$ that is closely related to the initial quantum noise $A_{1,2}(\zeta,y,\tau=0)$ through Eq. (10). To summarize, when the parametric process takes place in the presence of a strong temporal walk-off and a perturbative diffraction, the down-converted fields are temporally coherent and spatially incoherent.

This prediction of the linear stability analysis has been checked numerically by solving the complete set of nonlinear two-dimensional equations governing the parametric process. More specifically, we solved Eqs. (1) without dispersion and in the presence of diffraction along the additional axis y. A typical result is shown in Fig. 5, that illustrates the intensity distribution $|A_1|(\zeta,y)$ of the signal field. This intensity distribution has been obtained at time $\tau=13\tau_0$, when we observed a negligible depletion of the pump. As an initial condition, for the signal and idler amplitudes we take a complex random noise, and for the pump beam a super-Gaussian function in both the y and ζ axes with a maximum amplitude e_0. We considered a square grid of 256×256 points, representing 32×32 spatial units of $\Lambda=v_3\tau_0$. In this example, the diffraction effect is perturbative with respect to the temporal walk-off and, as expected, the signal field appears to be coherent along the walk-off axis ζ and incoherent along the transverse axis y. Note that the idler field follows an evolution almost identical to that of the signal field, in agreement with our theory.

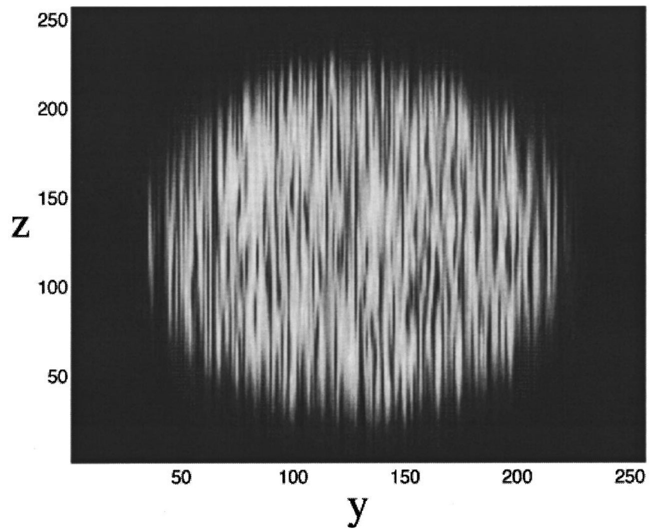

FIG. 5. Intensity distribution of the signal field $|A_1|$ at time $t=13\tau_0$ along the transverse y and longitudinal z axes. Parameters are $\delta=4\times10^7$ m/s, $m=16$ m^2/s, and $\Delta=0$ (the window size is $L=32\Lambda$, $\Lambda=v_3\tau_0=0.022$ mm, and $\tau_0=12$ ps).

These findings, as regards the spatiotemporal coherence of the generated fields in the presence of a perturbative diffraction and a temporal walk-off are not surprising. Indeed, on the basis of a purely one-dimensional model, we showed in Secs. VI–VII that walk-off leads to a coherent behavior, whereas diffraction only yields an erratic dynamics. However, we shall see in the following that, when diffraction is no longer perturbative, its combination with walk-off effects leads to intriguing dynamics.

In the framework of the stability analysis, the asymptotic evolution of the down-converted fields is given by integral (8), where $m/\delta\neq0$ and where we still assume $\Delta=0$. The integrand is thus to be the product of the noisy function $\hat{A}_{1,2}(q_0,p)$ and the phase term $\exp[i(py+p^2m\zeta/\delta)]$. Although this integral cannot be computed analytically in the general case, it is clear that there is no means for the phase term to smooth down the noisy function $\hat{A}_{1,2}(q_0,p)$. Indeed, all the modes that are present in the fluctuating initial condition $\hat{A}_{1,2}(q_0,p)$ are amplified in a similar way, without any mechanism of wave number selection. This is simply because the gain curve $\Gamma(p)=1/\tau_0$ is flat for $\Delta=0$. One can then reasonably expect that the asymptotic evolution of the down-converted fields remain erratic. Importantly, this erratic evolution results in both spatial and temporal incoherence.

We checked these predictions by numerically solving the complete set of nonlinear equations, and we illustrate an example of typical distribution of the signal field in Fig. 6 at time $\tau=13\tau_0$. The parameters are the same as in Fig. 5, except that we increased the ratio m/δ. As expected, the generated fields are incoherent both temporally and spatially. This study then reveals that, quite surprisingly, diffraction is able to break down the temporal coherence of the generated fields, even when the temporal walk-off is present.

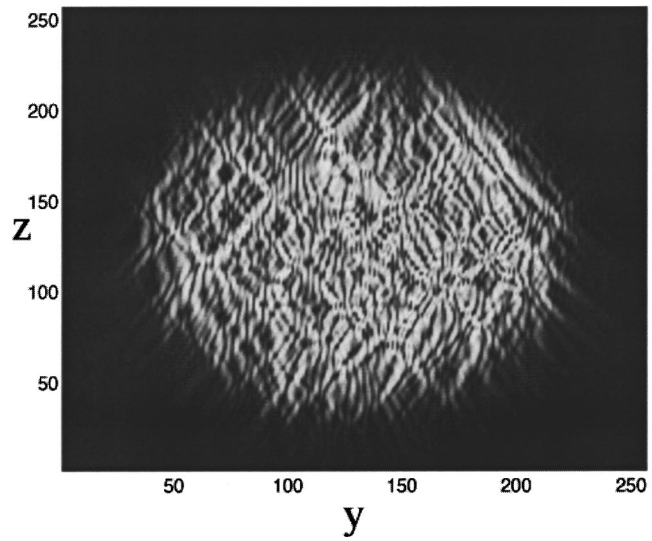

FIG. 6. Intensity distribution of the signal field $|A_1|$ at time $t=13\tau_0$ along the transverse y and longitudinal z axes. Parameters are $\delta=5\times10^6$ m/s and $m=16$ m^2/s, $\Delta=0$ (the window size is $L=32\Lambda$; $\Lambda=v_3\tau_0=0.022$ mm and $\tau_0=12$ ps).

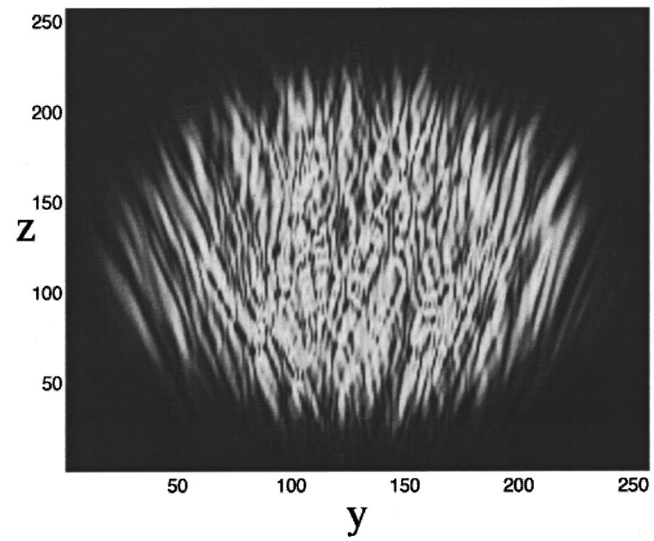

FIG. 8. Intensity distribution of the signal field $|A_1|$ at time $t=13\tau_0$ along the transverse y and longitudinal z axes. Parameters are $\delta=5\times10^6$ m/s and $m=16$ m^2/s, $\Delta=12$ m^2/s (the window size is $L=32\Lambda$; $\Lambda=v_3\tau_0=0.022$ mm and $\tau_0=12$ ps).

B. Case $\Delta\neq0$

At this point we may summarize our results as follows: apart from the particular case where the diffraction effect is negligible with respect to the walk-off, we have shown that, provided $\Delta=0$, the two-dimensional parametric process generally leads to the emergence of both spatially and temporally incoherent fields. This result is due to the simple fact that, when $\Delta=0$, the effective bandwidth of the gain curve $\mathrm{Re}[\Gamma(p)]$ is flat, and there is no means for the emergence of a spatially coherent field. Let us now consider the general case $\Delta\neq0$ in the framework of the linear stability analysis. It is worth noting that for $\Delta\neq0$ the complex dispersion relation $\Gamma(p)=1/\tau-i\Delta p^2$ has a saddle point in $p=0$, and integral (8) may be calculated by the steepest descent method. Nevertheless, note that the real axis $\mathrm{Re}(p)$ is not the right contour of integration because it is not the contour of steepest descent. Using the Cauchy theorem, we can equivalently evaluate integral (8) along any contour C in the complex plane p connecting the extrema of integration, provided the integrand has no singularities in the area bounded by the original and the new contour. The integrand of Eq. (8) being analytic everywhere, we can calculate the corresponding integral on a contour that goes through the saddle point as depicted in Fig. 7. Along this specific contour, $\mathrm{Re}[\Gamma(p)]$ has the steepest ascent for $\mathrm{Re}(p)<0$ and the steepest descent for $\mathrm{Re}(p)>0$, while $\mathrm{Im}[\Gamma(p)]$ is constant, as it must be in order to apply the steepest descent method [18]. Accordingly, the result of the integral reads

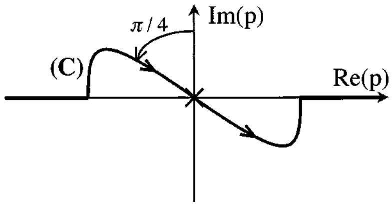

FIG. 7. Contour C of integration in the complex plane p.

$$A_{1,2}(\zeta,y,\tau)\propto\frac{1}{\delta}\exp(\tau/\tau_0) \tag{11}$$

and one may expect the generation of a homogenous pattern along both the walk-off axis ζ and the transverse axis y, which simply means that the generated fields would be both spatially and temporally coherent.

In order to verify the validity of this result, we numerically solved the full set of nonlinear equations for the same conditions as in Fig. 6, except that we impose a nonvanishing value of the parameter Δ. A typical result is illustrated in Fig. 8, that shows the intensity distribution of the signal component at a time $\tau=13\tau_0$. A comparison between Figs. 8 and 6 shows that the generated signal exhibits some degree of spatial and temporal coherence that may be clearly attributed to the nonvanishing value of Δ. However, it is clear that the intensity distribution is far from being homogenous along the axes ζ and y, as the linear theory predicts through Eq. (11). As a matter of fact, here we are in the same situation as that encountered when studying the role of diffraction in the purely one-dimensional case. We have indeed shown in Sec. VI that dispersion or diffraction is able, *a priori,* to yield a coherent behavior in a way similar to the walk-off effect. However, the interaction length required is in practice too long to allow for an observation of the diffraction-induced coherence process. In the two-dimensional case considered here, the requested interaction length is inversely proportional to Δ, i.e., to the bandwidth of the gain curve $\Gamma(p)$.

The parameter Δ being of the same order of magnitude ast-hat of the effective diffraction parameter ρ_i, it is not surprising that the numerical simulations show an erratic spatiotemporal evolution of the down-converted fields for realistic values of the diffraction parameter. The incoherence of the generated fields in the two-dimensional configuration of the parametric process is then of the same nature as that encountered in the purely one-dimensional case considered in Sec. VI.

IX. CONCLUSION

In conclusion, we considered the fundamental physical problem of spontaneous parametric generation from noise fluctuations in the presence of a plane pump wave. We showed that in the absence of walk-off between the down-converted signal and idler fields, the process of parametric amplification is intrinsically incoherent, even in the nonlinear regime of pump depletion. Apart from the particular case where very short pulses (or narrow beams, for the spatial domain) are involved, dispersion (or diffraction) alone is not able to lead to a coherent parametric process. We showed both analytically, through a simple linear analysis of the model equations, and numerically that it is the walk-off between the generated fields that is at the origin of the coherence of the parametric amplification process. Conversely, the dispersion or diffraction effects are not able in practice to increase the coherence of the down-converted fields. Moreover, a two-dimensional study reveals that when the nature of the walk-off is exclusively temporal (or spatial), the parametric process is not able to yield the generation of both spatially and temporally coherent fields. Our physical interpretation of the theory allowed us to explain the results of a series of recent experimental studies of parametric fluorescence. It would be interesting to extend the proposed theory in order to study rigorously the statistical properties of the parametrically generated fields, which is of particular importance for a quantitative comparison between the theory and experiments. This aspect is presently under investigation.

Although we restricted our analysis to the parametric amplification process in quadratic nonlinear media, the mechanism of walk-off-induced coherence is generic, and can be extended to four-wave mixing in cubic nonlinear media or other physical parametric processes, such as, e.g., the effects of external fields on pattern forming systems [21] or coupled molecular and atomic Bose-Einstein condensates [22]. The experimental confirmation of the mechanism of walk-off-induced coherence would be of great interest, on the one hand, for a fundamental study of spontaneous formation of coherent structures in nonlinear physics [7,8,20] and, on the other hand, for a better knowledge and control of practical traveling-wave optical parametric generators [4].

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