Pulse evolution in nonlinear optical fibers with sliding-frequency filters

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The effect of fiber loss, amplification, and sliding-frequency filters on the evolution of optical pulses in nonlinear optical fibers is considered, this evolution being governed by a perturbed nonlinear Schrödinger (NLS) equation. Approximate ordinary differential equations (ODE's) governing the pulse evolution are obtained using conservation and moment equations for the perturbed NLS equation together with a trial function incorporating a solitonlike pulse with independently varying amplitude and width. In addition, the trial function incorporates the interaction between the pulse and the dispersive radiation shed as the pulse evolves. This interaction must be included in order to obtain approximate ODE's whose solutions are in good agreement with full numerical solutions of the governing perturbed NLS equation. The solutions of the approximate ODE's are compared with full numerical solutions of the perturbed NLS equation and very good agreement is found.

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I. INTRODUCTION

Sliding-frequency filters (SFF's) are used to suppress the effect of the Gordon-Haus (GH) jitter due to erbium-doped fiber amplifiers in optical communication systems [1,2]. These all-optical amplifiers are used to counteract the loss inherent in fiber optic cables. However, while the erbium-doped fiber amplifiers amplify the signal, they also amplify the noise in the system. In a soliton based communication system, this noise amplification causes a shift in the soliton parameters, most importantly, in its amplitude and frequency. As the soliton frequency is coupled to its velocity, this then causes random fluctuations in the soliton velocity and thus in the arrival time of the soliton. This negative effect of amplification is known as GH jitter [3].

To reduce the effect of noise, optical filters are used [4]. A fixed-frequency filter can reduce GH jitter by creating an attractive value of soliton frequency, and thus velocity. Random noise will therefore not drive the soliton too far from its preferred velocity, reducing fluctuations in the soliton arrival time. However, a fixed-frequency filter is unable to reduce radiation within the filter's passband. To reduce this noise, a sliding-frequency filter is employed [1]. A SFF allows the central frequency of the filter to change along the length of the fiber. As the filter passband changes, the nonlinear soliton readjusts to this new frequency, while the nearly linear radiation underneath does not. In this way radiation around and under the soliton is filtered out, reducing GH jitter.

The basic equation governing pulse propagation in a nonlinear optical fiber is the nonlinear Schrödinger (NLS) equation

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}+|u|^2u=0 \qquad (1)$$

[5]. While an exact inverse scattering solution of the NLS equation exists [6], there is no exact solution of the perturbed NLS equation that results when the effect of SFF's is added. Therefore approximate and/or numerical methods must be used to study pulse evolution governed by this perturbed equation. To counteract the pulse damping due to the filtering and the inherent fiber losses, periodically spaced optical amplifiers are used, which in the limit of the amplifier spacing much smaller than the dispersion length can be modeled by a continuous system of amplifiers [2]. In an experimental and numerical study, Mamyshev and Mollenauer [7] showed that stable soliton propagation was possible with amplification for a range of filter sliding rates and strengths. In particular it was found that there are upper and lower bounds on the soliton energy for which stable propagation is possible. For energies below the lower bound, the pulse decays into dispersive radiation due to excessive filtering and, for energies above the upper bound, a second soliton is formed, as may be expected from inverse scattering theory [8]. Kodama and Wabnitz [9] used a multiple scale analysis based on a slowly varying NLS soliton to derive ordinary differential equations governing the propagation of a soliton in the presence of SFF's and amplification. It was shown that these equations possessed two fixed points, one of which was stable and the other unstable. It was further shown that the stable fixed point existed for energies above a certain threshold, in agreement with Mamyshev and Mollenauer [7]. However, the multiple scale analysis did not predict the upper energy bound. Burtsev and Kaup [10] used perturbed inverse scattering theory to derive the same approximate equations governing the soliton as Kodama and Wabnitz [9]. As a perturbed soliton propagates, it sheds dispersive radiation and Burtsev and Kaup [10] deduced that it was this radiation that gave rise to the second soliton. By extending their perturbation analysis to higher order, Burtsev and Kaup [10] obtained estimates on this radiation which enabled them to find an approximation to the upper energy bound. This upper bound was found to be in good agreement with that found from numerical results in [7]. Soliton propagation in the presence of SFF's was also studied by Malomed and Tasgal [11] using

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the same multiple scale method as Kodama and Wabnitz [9], but for ultrashort pulses for which the amplifiers must be taken as discrete.

All the previous analytical work on pulse propagation in the presence of SFF's is based on a slowly varying soliton. However, the choice of a slowly varying NLS soliton as the approximate solution of the perturbed NLS equation has two drawbacks. The first is that the amplitude and width variations of the pulse are linked, as the amplitude and width of a NLS soliton are inversely proportional. It will be shown in the present work that decoupling the amplitude and width results in better agreement with full numerical solutions. The second is that the approximate solution of [9,11] takes no account of the dispersive radiation shed as the pulse evolves down the fiber. While the second order perturbed inverse scattering work of [10] does find equations for this shed radiation, it does not include the damping effect of this radiation on the evolving pulse and the equations for the soliton amplitude and velocity are the same as those of [9,11].

Anderson [12] developed an approximate Lagrangian method that allows independent amplitude and width oscillations of a pulse. This method is based on using a ''chirped'' NLS soliton with varying amplitude, width, and velocity as a trial function in an averaged Lagrangian. However, while the amplitude and width oscillations are now independent under this method, Anderson's method also does not take account of the dispersive radiation shed as the pulse evolves. In many situations the inclusion of the shed radiation is vital for obtaining good agreement with numerical solutions [13–15].

An approximate method that does include the dispersive radiation shed as a pulse evolves was developed by Kath and Smyth [13] for the NLS equation (1). Their approximate method was again based on an averaged Lagrangian. However, the trial function used was different from that of Anderson [12]. As well as a varying solitonlike pulse with independent amplitude and width, their trial function included a ''shelf'' term, which accounted for the dispersive radiation in the vicinity of the evolving pulse. Furthermore, the solitonlike pulse was not chirped. Kath and Smyth showed that the dispersive radiation shed by the evolving pulse is governed by a linearized NLS equation. By analyzing solutions of this linearized NLS equation, the mass lost from the pulse to shed dispersive radiation was calculated. Kath and Smyth [13] thus added terms to the variational equations derived from the averaged Lagrangian that included the effect of this shed radiation. It was found that the solutions of the modified variational equations were in excellent agreement with full numerical solutions of the NLS equation (1).

The method of Kath and Smyth [13] has been extended to study pulse propagation in other optical systems. It has been used to study pulse propagation and switching in nonlinear twin-core fibers [14] and pulse propagation in nonuniform fibers [15]. In both of these extensions, it was found that the inclusion of the shed radiation was necessary in order to obtain good agreement with numerical solutions, and that the agreement thus obtained was better than that obtained using the chirp method of Anderson [12].

In the present work, the shelf method of Kath and Smyth [13] will be extended to study the evolution of pulses under the action of SFF's, fiber loss, and amplification. As the governing perturbed NLS equation is not conservative, an averaged Lagrangian cannot be used to obtain approximate equations governing pulse evolution, as in [13]. Equations for mass, momentum, and energy and their moments are therefore used to obtain the approximate equations. In the case of a conservative system, this is equivalent to using an averaged Lagrangian, due to Nöther's theorem. Using a similar analysis to that of Kath and Smyth [13], the effect of the dispersive radiation shed as the pulse evolves is added to the approximate equations. Solutions of these approximate equations are then compared with full numerical solutions of the governing perturbed NLS equation and excellent agreement is found. It is found that the approximate equations of the present work give solutions in better agreement with numerical solutions than do those of [9–11]. The main reason for this is that the amplitude and width oscillations of the pulse are now independent. The present approximate equations also give the same lower energy bound for stable pulse propagation as [9,10]. By using mass and energy conservation, an approximation to the upper energy bound due to the formation of a second soliton is also found. This bound is found to be in good agreement with the numerical results of [7].

II. APPROXIMATE EQUATIONS

Light propagating in a monomode, polarization-preserving, nonlinear optical fiber operating in the anomalous group-velocity dispersion regime is described by the NLS equation [5]. When the effects of fiber loss and a SFF are added, the governing equation is a perturbed NLS equation, which is

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}+|u|^2u=-i\sigma u+i\gamma\left(\frac{\partial}{\partial t}+i\Omega z\right)^2 u \tag{2}$$

in nondimensional form [2]. Here u is the complex-valued envelope of the pulse, z is the normalized spatial variable along the length of the fiber, and t is the normalized time (in a frame moving with the linear group velocity). The first term on the right represents uniform fiber loss, where σ is the loss parameter [5]. The second term represents a SFF with filter strength γ and frequency sliding rate Ω [2]. When the amplifier spacing is much smaller than the dispersion length scale, the term $-i\sigma u$ on the right hand side of Eq. (2) represents the excess of gain over loss for $\sigma<0$ [2,5].

The approximate method of Kath and Smyth [13] is now extended to obtain an approximate solution of Eq. (2) that describes pulse propagation under the influence of a SFF and loss. In the work of Kath and Smyth [13], a trial function was substituted into an averaged Lagrangian for the NLS equation. Variations were then taken with respect to the pulse parameters and ordinary differential equations (ODE's) for these parameters were thus obtained. However, a Lagrangian only exists for conservative systems. With the loss terms on the right hand side of the perturbed NLS equation

(2), this equation does not describe a conservative system. However for nonconservative systems, approximate ODE's governing pulse evolution can be obtained by substituting the trial function of [13] into conservation and moment equations. In the present context, a conservation equation does not mean that the quantity is conserved. What is meant are equations for mass, momentum, and energy modified by loss terms due to the SFF and fiber loss. The ODE's obtained from these conservation and moment equations are exactly those of [13] when the loss terms on the right hand side of the perturbed NLS equation (2) are set equal to zero. It is this conservation and moment equation approach that is used in the present work.

The perturbed NLS equation (2) has three conservation equations, commonly referred to as mass, momentum, and energy [16], although in the context of optical fibers they do not physically correspond to these quantities. In this regard, the three quantities

$$\rho=|u|^2, \tag{3a}$$

$$J=\frac{i}{2}(uu_t^*-u^*u_t), \tag{3b}$$

$$E=|u_t|^2-|u|^4, \tag{3c}$$

are defined [16], which are referred to as mass density, momentum density, and energy density, respectively. Here * denotes the complex conjugate. From the perturbed NLS equation (2), the conservation equations

$$\frac{d}{dz}\int_{-\infty}^{\infty}\rho\,dt=-\int_{-\infty}^{\infty}[(2\sigma+2\gamma\Omega^2z^2)\rho+4\gamma\Omega zJ+2\gamma(E+\rho^2)]\,dt, \tag{4a}$$

$$\frac{d}{dz}\int_{-\infty}^{\infty}J\,dt=-\int_{-\infty}^{\infty}[(2\sigma+2\gamma\Omega^2z^2)J+4\gamma\Omega z(E+\rho^2)+i\gamma(u_tu_{tt}^*-u_t^*u_{tt})]\,dt, \tag{4b}$$

$$\frac{d}{dz}\int_{-\infty}^{\infty}E\,dt=\int_{-\infty}^{\infty}\{(2\sigma+2\gamma\Omega^2z^2)(\rho^2-E)+2\gamma[4|u_t|^2|u|^2+(u^*)^2u_t^2+u^2(u_t^*)^2-|u_{tt}|^2]+2i\gamma\Omega z(u_t^*u_{tt}-u_tu_{tt}^*+2|u|^2uu_t^*-2|u|^2u^*u_t)\}dt \tag{4c}$$

can be directly derived. The perturbed NLS equation (2) also possesses moment equations. The moment equation used in the present work is the moment of momentum equation

$$\frac{d}{dz}\int_{-\infty}^{\infty}tJ\,dt=\int_{-\infty}^{\infty}\left[E+\frac{1}{2}\rho^2-(2\sigma+2\gamma\Omega^2z^2)tJ+i\gamma t(u_t^*u_{tt}-u_tu_{tt}^*)-4\gamma\Omega zt(E+\rho^2)\right]dt, \tag{5}$$

which again can be directly derived from the perturbed NLS equation (2).

It should be mentioned that the phase of the soliton is not determined by the conservation or moment equations as these equations are all independent of phase. An equation for the phase for the NLS equation (1) can be derived from an extension of Nöther's theorem [17] based on scale invariance of the NLS equation [13]. However, the equations for the other parameters (amplitude, width, and velocity) will be found to be independent of the phase and so the phase equation is not dealt with in the present work.

The key to the method of Kath and Smyth is the choice of trial function to use in the conservation and moment equations (4) and (5). Based on numerical solutions, previous experimental work by other authors, and perturbed inverse scattering theory for the NLS equation, a solution of the form

$$u=\left(\eta\,\mathrm{sech}\frac{t}{w}+ig\right)\exp[i\theta] \tag{6}$$

was sought. The same trial function, extended to allow for a variable pulse velocity, will be used in the present work. Hence a solution of the form

$$u=\left(\eta\,\mathrm{sech}\frac{t-y}{w}+ig\right)\exp[i\theta+iV(t-y)] \tag{7}$$

will be sought for the perturbed NLS equation (2). Here the amplitude η, width w, velocity V, mean position y, phase θ, and g are functions of z. The first term in sech is a varying solitonlike pulse. The second term in g accounts for the low frequency radiation in the vicinity of the pulse [13]. From numerical solutions of the NLS equation and perturbed inverse scattering theory, it was found by Kath and Smyth [13] that the radiation in the vicinity of the pulse is independent of t. The reason for this is that the group velocity for the linearized NLS equation is $c_g=-2k$, where k is the wave number, so that low frequency radiation stays in the vicinity of the pulse. High frequency radiation rapidly propagates away from the pulse, leaving a flat shelf of radiation on which the pulse remains. The radiation cannot continue to be flat away from the pulse, otherwise it would contain infinite mass. It is therefore assumed that the radiation is flat in the region $-\ell/2<t<\ell/2$ about the pulse. The form of the radiation outside this region is dealt with in the next section. Furthermore, numerical solutions show that the radiation is of small amplitude, so that $|g|\ll\eta$.

Substituting the trial solution (7) into the conservation laws and moment equations (4) and (5) yields

$$(2\eta^2w+\ell g^2)\frac{dV}{dz}=-\frac{8}{3}\gamma(\Omega z+V)\frac{\eta^2}{w}, \tag{8a}$$

$$\frac{d}{dz}\left(\frac{\eta^2}{w}-2\eta^4w\right)=2(\sigma+\gamma\Omega^2z^2+\gamma V^2+2\gamma\Omega zV)\times\left(4\eta^4w-\frac{\eta^2}{w}\right)+\gamma\left(\frac{24\eta^4w^2-14\eta^2}{5w^3}\right), \tag{8b}$$

$$\frac{d}{dz}(\pi\eta g w)=\frac{2}{3}\left(\frac{\eta^2}{w}-\eta^4 w\right)-2(\sigma+\gamma\Omega^2 z^2+3\gamma V^2+4V\gamma\Omega z)\pi\eta g w, \tag{8c}$$

$$\frac{d}{dz}(2\eta^2 w+\ell g^2)=-2(\sigma+\gamma\Omega^2 z^2+\gamma V^2+2\gamma\Omega z V)(2\eta^2 w+\ell g^2)-\frac{4}{3}\gamma\frac{\eta^2}{w}. \tag{8d}$$

The last equation is

$$\frac{dy}{dz}=V, \tag{9}$$

which links the soliton center position to the velocity. After some manipulation, the conservation and moment equations (8a)–(8d) become

$$(2\eta^2 w+\ell g^2)\frac{dV}{dz}=-\frac{8}{3}\gamma(\Omega z+V)\frac{\eta^2}{w}, \tag{10a}$$

$$\frac{d}{dz}\left(\frac{\eta^2}{w}-2\eta^4 w\right)=2(\sigma+\gamma\Omega^2 z^2+\gamma V^2+2\gamma\Omega z V)\times\left(4\eta^4 w-\frac{\eta^2}{w}\right)+\gamma\left(\frac{24\eta^4 w^2-14\eta^2}{5w^3}\right), \tag{10b}$$

$$\frac{d}{dz}(\eta w)=-(\sigma+\gamma\Omega^2 z^2+\gamma V^2+2\gamma\Omega z V)\eta w-\frac{\ell g}{2\pi}\frac{1-2(\eta w)^2}{w^2}+\frac{\gamma}{5}\frac{\eta}{w}, \tag{10c}$$

$$\frac{dg}{dz}=\frac{2}{3\pi}\frac{\eta}{w^2}[1-(\eta w)^2]-(\sigma+\gamma\Omega^2 z^2+5\gamma V^2+6\gamma\Omega z V)g-\frac{\gamma}{5}\frac{g}{w^2}. \tag{10d}$$

In deriving these equations, terms of $O(g^2)$ and higher have been dropped, except for the quadratic term in g proportional to ℓ. This term was not dropped as it is important in mass conservation [13].

For the case of the NLS equation (1), Kath and Smyth [13] found the length ℓ of the shelf by matching the frequency of oscillation of the solution of the NLS approximate equations near their fixed point $\eta=\hat{\kappa}$ to the steady NLS soliton oscillation frequency $\frac{1}{2}\hat{\kappa}$, obtaining

$$\ell=\frac{3\pi^2}{8\hat{\kappa}}. \tag{11}$$

In turn, the fixed point amplitude $\hat{\kappa}$ was found from the NLS energy conservation equation, which is the energy equation (10b) with $\sigma=\gamma=0$. It was thus found that the fixed point amplitude is

$$\hat{\kappa}=\left(2\eta^4 w-\frac{\eta^2}{w}\right)^{1/3}. \tag{12}$$

However, the only fixed point of the present approximate equations (9)–(10d) in the absence of amplification is $\eta=0$ due to the loss terms. Therefore the method of [13] cannot be used to determine ℓ. If the loss parameters σ and γ are small, then a slowly varying approximation to ℓ is Eq. (11). In a similar vein, since the fixed point of the present approximate equations (9)–(10d) (in the absence of amplification) is $\eta=\hat{\kappa}=0$, the NLS fixed point (12) will be used to determine ℓ via Eq. (11). For σ and γ small, $\hat{\kappa}$ given by Eq. (12) is not constant, but slowly varying in z. This slowly varying approximation gives the local value of amplitude $\hat{\kappa}$ that the pulse would achieve if the fiber loss and filter strength were set to zero instantaneously. Furthermore, with this value of ℓ, setting $\sigma=\gamma=0$ in the present approximate equations (10a)–(10d) reduces the equations to those derived from an averaged Lagrangian by Kath and Smyth [13].

Kodama and Wabnitz [9] and Malomed and Tasgal [11] derived another set of approximate equations for the pulse amplitude and velocity based on another trial function. Their method uses a trial solution in the form of a NLS soliton with variable parameters

$$u=\eta\,\mathrm{sech}[\eta(t-y)]\exp[-i\theta-iV(t-y)]. \tag{13}$$

This trial solution assumes that the amplitude η and width η^{-1} of the pulse are inversely related. The pulse phase is θ and the pulse velocity is V. The parameters are all functions of z. Based on this trial solution, Kodama and Wabnitz [9] used the method of multiple scales and Malomed and Tasgal [11] used the balance-equation technique [18] to derive the following approximate equations for the pulse parameters of Eq. (13):

$$\frac{d\eta}{dz}=-2\eta\left\{\sigma+\gamma\left[\frac{\eta^2}{3}+(V-\Omega z)^2\right]\right\}, \tag{14a}$$

$$\frac{dV}{dz}=-\frac{4}{3}\eta^2\gamma(V-\Omega z). \tag{14b}$$

Notice that setting $\eta=1/w$ and $g=0$ in the present approximate equations (10a) and (10b) gives these equations, as expected.

The approximate equations (9)–(10d) along with Eqs. (11) and (12) for ℓ and $\hat{\kappa}$ are not yet complete as they do not incorporate the effect of the dispersive radiation shed by the evolving pulse. This shed radiation is the subject of the next section.

III. RADIATION LOSS

As the pulse propagates it sheds radiation, so losing mass and momentum (and energy to higher order). The effect of this shed radiation on the evolution of the pulse will now be analyzed in a similar manner to that of Kath and Smyth [13,19].

As the shed radiation has small amplitude, the nonlinear term in the perturbed NLS equation (2) is negligible away from the pulse. Therefore, the equation governing the shed radiation is

$$iu_z+\frac{1}{2}u_{tt}=-i\sigma u+i\gamma u_{tt}-2\Omega z\gamma u_t-i\gamma\Omega^2z^2u. \quad (15)$$

The substitution

$$u=U(z,t)\exp\left(-\sigma z-\frac{1}{3}\gamma\Omega^2z^3\right) \quad (16)$$

transforms the linearized equation (15) to

$$iU_z+\left(\frac{1}{2}-i\gamma\right)U_{tt}=-2\Omega z\gamma U_t. \quad (17)$$

The conservation of mass equation for this transformed equation (17) is

$$\frac{\partial|U|^2}{\partial z}=\frac{i}{2}\frac{\partial}{\partial t}(U^*U_t-UU_t^*)+\gamma\frac{\partial}{\partial t}(U^*U_t+UU_t^*)-2\gamma|U_t|^2-2i\Omega z\gamma(UU_t^*-U^*U_t). \quad (18)$$

Integrating this mass equation from the edge of the shelf $t=y+\ell/2$ to $t=\infty$, and noticing that the last two terms in Eq. (18) are already included in the original mass conservation equation (4), yields an expression for the mass radiated to the right away from the pulse as

$$\frac{d}{dz}\int_{y+\ell/2}^{\infty}|U|^2\,dt=-V|U|^2_{t=y+\ell/2}+\mathrm{Im}(U^*U_t)|_{t=y+\ell/2}-2\gamma\,\mathrm{Re}(U^*U_t)|_{t=y+\ell/2}. \quad (19)$$

Furthermore, for small γ, the velocity V may be taken to be a constant to first order, as in [15]. If this approximation were not made, then the radiation would be determined by a moving boundary problem whose boundary $t=y+\ell/2$ is unknown and determined by the approximate equations of Sec. II.

Kath and Smyth [13] used Laplace transforms to solve the linearized NLS equation and thus determined an expression for the mass radiated by the pulse. However, the linearized equation (17) has nonconstant coefficients and so Laplace transforms are not really useful for its analysis. In the preceding section, it was assumed that γ is small. For small γ, the nonconstant coefficient in the linearized equation (17) is slowly varying, and so may be taken to be constant on the fast scale z. Laplace transforms may then be used to solve the linearized equation (17). Denoting the Laplace transform of $U(z,t)$ by $\overline{U}(s,t)$, it is found that

$$\left(\frac{1}{2}-i\gamma\right)\overline{U}_{tt}+2\Omega z\gamma\overline{U}_t+is\overline{U}=0. \quad (20)$$

Keeping only leading order terms in γ, we obtain the following expression for U_t:

$$U_t=-2\Omega z\gamma U-\sqrt{2}(e^{-i\pi/4}+\gamma e^{i\pi/4})\left(\frac{d}{dz}\int_0^z\frac{U}{\sqrt{\pi(z-\tau)}}\,d\tau\right) \quad (21)$$

on solving Eq. (20) for $\overline{U}$ and inverting the transform. Substituting this expression into the mass conservation relation (19) and ignoring quadratic terms in γ, we obtain

$$\frac{d}{dz}\int_{y+\ell/2}^{\infty}|U|^2\,dt=-V|U|^2_{t=y+\ell/2}+(1+\gamma)U^*\frac{d}{dz}\int_0^z\frac{U}{\sqrt{\pi(z-\tau)}}\,d\tau \quad (22)$$

for the mass radiated to the right of the pulse (i.e., $t>y+\ell/2$). An expression for the mass radiated to the left of the pulse (i.e., $t<y-\ell/2$) may be obtained in a similar manner, the only difference to the mass expression (22) being that the sign of the $V|U|^2$ term is reversed. Then inverting the transformation (16) and substituting the right and left mass loss expressions into the mass equation (4) for the pulse results in the modified mass equation for the pulse

$$\frac{d}{dz}\int_{-\infty}^{\infty}\rho\,dt=-\int_{-\infty}^{\infty}[(2\sigma+2\gamma\Omega^2z^2)\rho+4\gamma\Omega zJ+2\gamma(E+\rho^2)]\,dt-2(1+\gamma)r\times\exp\left(-\sigma z-\frac{1}{3}\gamma\Omega^2z^3\right)\times\frac{d}{dz}\int_0^z\frac{r\exp\left(\sigma\tau+\frac{1}{3}\gamma\Omega^2\tau^3\right)}{\sqrt{\pi(z-\tau)}}\,d\tau. \quad (23)$$

Here $r=|u(y+\ell/2,z)|$ is the height of the shelf at its edge. The second term on the right hand side of Eq. (23) is the mass shed by the pulse in the form of dispersive radiation. When σ and γ are small, the height of the shelf is given by the same expression as in Kath and Smyth [13]. Therefore

$$r^2=\frac{3\hat{\kappa}}{8}(2\eta^2w-2\hat{\kappa}+\ell g^2). \quad (24)$$

When the mass loss term in Eq. (23) is added to the approximate equations (9)–(10d) of the previous section, the equation for g is modified to

$$\frac{dg}{dz}=\frac{2}{3\pi}\frac{\eta}{w^2}[1-(\eta w)^2]-(\sigma+\gamma\Omega^2z^2+5\gamma V^2+6\gamma\Omega zV)g$$

$$-\frac{\gamma}{5}\frac{g}{w^2}-2\alpha(1+\gamma)g, \quad (25)$$

where

$$\alpha=\frac{3\hat{\kappa}}{8}\frac{1}{r}\exp\left(-\sigma z-\frac{1}{3}\gamma\Omega^2z^3\right)$$

$$\times\frac{d}{dz}\int_0^z\frac{r\exp\left(\sigma\tau+\frac{1}{3}\gamma\Omega^2\tau^3\right)}{\sqrt{\pi(z-\tau)}}d\tau. \quad (26)$$

In a similar manner, the momentum lost to shed dispersive radiation can be added to the momentum integral (4b). However, it is found that the same momentum equation (4b) results. The energy lost to shed dispersive radiation is of higher order than the lost mass and momentum, so can be neglected. Hence the full set of equations governing pulse evolution with fiber loss and SFF's, including radiation loss, is

$$(2\eta^2w+\ell g^2)\frac{dV}{dz}=-\frac{8}{3}\gamma(\Omega z+V)\frac{\eta^2}{w}, \quad (27a)$$

$$\frac{d}{dz}\left(\frac{\eta^2}{w}-2\eta^4w\right)$$

$$=2(\sigma+\gamma\Omega^2z^2+\gamma V^2+2\gamma\Omega zV)\left(4\eta^4w-\frac{\eta^2}{w}\right)$$

$$+\gamma\left(\frac{24\eta^4w^2-14\eta^2}{5w^3}\right), \quad (27b)$$

$$\frac{d}{dz}(\eta w)=-(\sigma+\gamma\Omega^2z^2+\gamma V^2+2\gamma\Omega zV)\eta w$$

$$-\frac{\ell g}{2\pi}\frac{1-2(\eta w)^2}{w^2}+\frac{\gamma}{5}\frac{\eta}{w}, \quad (27c)$$

$$\frac{dg}{dz}=\frac{2}{3\pi}\frac{\eta}{w^2}[1-(\eta w)^2]-(\sigma+\gamma\Omega^2z^2+5\gamma V^2+6\gamma\Omega zV)g$$

$$-\frac{\gamma}{5}\frac{g}{w^2}-2\alpha(1+\gamma)g, \quad (27d)$$

$$\frac{dy}{dz}=V, \quad (27e)$$

with

$$\alpha=\frac{3\hat{\kappa}}{8}\frac{1}{r}\exp\left(-\sigma z-\frac{1}{3}\gamma\Omega^2z^3\right)$$

$$\times\frac{d}{dz}\int_0^z\frac{r\exp\left(\sigma\tau+\frac{1}{3}\gamma\Omega^2\tau^3\right)}{\sqrt{\pi(t-\tau)}}d\tau, \quad (28a)$$

$$r^2=\frac{3\hat{\kappa}}{8}(2\eta^2w-2\hat{\kappa}+\ell g^2). \quad (28b)$$

IV. RESULTS

In this section, solutions of the approximate equations (27) of the present work and the equations (14a) and (14b) of [9–11] will be compared with full numerical solutions of the governing perturbed NLS equation (2). The approximate ODE's (27) and (14a) and (14b) were solved numerically using a fourth-order Runge-Kutta scheme. The integral in the expression (26) for α was evaluated numerically using the method of Miksis and Ting [20]. The perturbed NLS equation (2) was solved numerically using an extension of the pseudospectral method of Fornberg and Whitham [21]. The extension involved calculating the t derivatives using fast Fourier transforms and propagating in the z direction using a fourth-order Runge-Kutta scheme, this propagation taking place in Fourier space.

When the effect of amplification is added, the parameter σ in the perturbed NLS equation (2) is negative. In this case, the approximate equations (27), which include the effect of radiation damping, possess a steady state. For small γ, it can easily be found that this steady state is given by

$$V=-\Omega z+\frac{3\Omega}{4\gamma\kappa^2}, \quad (29a)$$

$$w=\frac{1}{\kappa} \quad (29b)$$

at first order, where the steady amplitude $\eta=\kappa$ is the solution of

$$\gamma\kappa^6+3\sigma\kappa^4+\frac{27}{16}\frac{\Omega^2}{\gamma}=0. \quad (30)$$

It should be noted that in the steady state $g\neq 0$. This steady state equation for κ is the same as that found in [9,10]. For

$$|\sigma|>\frac{3}{4}\ \Omega^{2/3}\gamma^{1/3} \quad (31)$$

Eq. (30) possesses two solutions, the smaller of which is unstable and the larger of which is stable, again in agreement with [9,10]. The inequality (31) determines the lower energy bound on the existence of stable pulse propagation in the presence of SFF's and amplification [9,10], as for

$$|\sigma|<\frac{3}{4}\ \Omega^{2/3}\gamma^{1/3} \quad (32)$$

Eq. (30) possesses no real solutions.

For a given value of the filter strength γ, a sufficiently high value of the amplification $|\sigma|$ amplifies the dispersive radiation to the point at which a second soliton can form. This formation of a second soliton is obviously undesirable in applications and was termed instability in [7,10]. However, this is not instability in the sense that the evolving pulse loses its coherence. The parameter values for which a second soliton will form were determined in [10] via the perturbed inverse scattering solution of the perturbed NLS equation (2). In the present work equations for mass and energy will be used to determine when a second soliton will form. From Eqs. (3c), (8b), and (27b) it can be seen that the energy of the pulse is

$$E=\frac{\eta^2}{w}-2\eta^4 w. \tag{33}$$

If there is no filtering and amplification, then an initial pulse will evolve to a steady soliton for which $\eta=\kappa$ and $w=1/\kappa$ and whose energy is $E=-\kappa^3$. Hence an initial pulse can evolve to a steady soliton only if its initial energy is negative. If its initial energy is positive, then it will decay into dispersive radiation alone. The borderline case is then an initial pulse with energy $E=0$. From the energy expression (33) we see that for this borderline case the pulse amplitude and width are related by

$$\eta w=\frac{1}{\sqrt{2}}. \tag{34}$$

Let us now consider a pulse of amplitude η and width w which has just enough mass and energy to break up into two solitons. At the boundary between one and two solitons forming, the second soliton will have zero energy. Let us take the final steady amplitude and width of the first soliton to be $\eta=\kappa$ and $w=1/\kappa$. Then

$$E=\frac{\eta^2}{w}-2\eta^4 w=-\kappa^3, \tag{35}$$

since the second soliton has zero energy. From Eqs. (3a) and (8d) it can be seen that the total mass of the two solitons is

$$M=2\eta^2 w=\sqrt{2}\eta+2\kappa \tag{36}$$

on equating the pulse mass to the mass of the final steady soliton with amplitude κ plus the mass of a pulse with zero energy, for which the amplitude-width relation (34) holds. In making this division of the mass and energy, it has been assumed that in the borderline case no mass and energy are taken away by dispersive radiation. On solving for κ from the energy conservation result (35) and then substituting into the mass relation (36), it is found that

$$(\eta w)^6-\frac{7}{2}(\eta w)^4+\frac{7}{4}(\eta w)^2-\frac{1}{8}=0, \tag{37}$$

so that $\eta w=1.702$. Therefore for $\eta w>1.702$ an initial pulse will break up into two solitons, both of nontrivial final amplitude. In the preceding analysis for the generation of a second soliton, the effects of amplification and filtering have been ignored. However, the condition $\eta w>1.702$ at some point in the evolution of the pulse for formation of a second soliton will still be valid if it is assumed that, when there is sufficient mass and energy for a second pulse to form, it will do so, and it will then evolve under the influence of amplification and filtering. Much the same assumption was made in [10] based on their perturbed inverse scattering solution. However, the present analysis for predicting the formation of a second soliton can be extended to equations for which there is no inverse scattering solution. The NLS equation (1) possesses an inverse scattering solution. Using this solution, [8] showed that, for a boundary condition of the form $u=\eta\,\mathrm{sech}\,t$, a second soliton will form for the NLS equation when $\eta>1.5$, which is in good agreement with the value $\eta>1.702$ found from the present mass and energy argument.

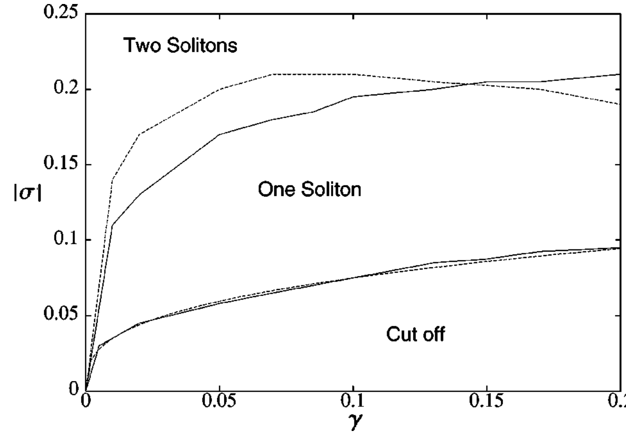

FIG. 1. Number of stable pulses in the $|\sigma|$-γ plane as given by the approximate and full numerical solutions for the soliton boundary condition $\eta=1$, $w=1$, and $V=1$ for the filter sliding rate $\Omega=0.1$. Boundaries from full numerical solution, —; boundaries from approximate equations (27), – – – –.

The approximate equations (27) can now be used to determine when a second soliton will form during the evolution of an initial pulse. The combination ηw is calculated as the pulse evolves and a second soliton is said to form when $\eta w>1.702$. Figure 1 shows a comparison in the $|\sigma|$-γ plane of the boundaries between the regions of one and two solitons as given by the approximate and full numerical solutions. Also shown in this figure is a similar comparison for the region (32) for no stable pulse. It can be seen that there is excellent agreement for the region of no stable pulse, as was also found in [10]. The agreement between the numerical and approximate solutions for the region of two solitons is good in view of the approximations made to derive the approximate boundary $\eta w=1.702$. It can further be seen that the agreement for the region of two solitons decreases as γ increases. This is to be expected as the analysis of Sec. III for calculation of the effect of the shed dispersive radiation was based upon assuming that γ is small. The overall comparison for the region of two solitons is similar to that obtained by Burtsev and Kaup [10].

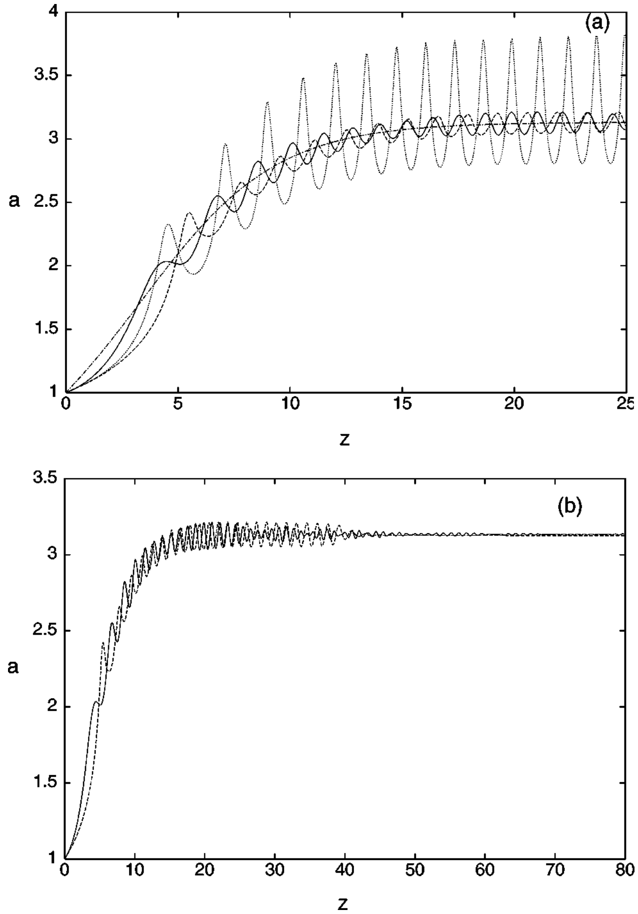

FIG. 2. Comparison between approximate and numerical solutions for soliton boundary condition with $\eta=1$, $w=1$, and $V=0$ with parameter values $\Omega=0.1$, $\gamma=0.03$, and $\sigma=-0.1$. (a) Full numerical solution, —; solution of approximate equations (27) with radiation, – – – –; solution of approximate equations (27) without radiation, $\cdots$; solution of approximate equations (14a) and (14b), ·—·—. (b)Full numerical solution, —; solution of approximate equations (27) with radiation, – – – –.

If the effect of the dispersive radiation shed as the pulse evolves were neglected, so that $\alpha=0$, then the approximate equations (27) would not possess a steady state and there would be a persistent oscillation in η, w, and g about the state given by Eq. (30). As the amplitude and width of the pulse oscillate, dispersion radiation is generated in the shelf under the pulse, which is then amplified and filtered. The steady oscillations in the absence of damping then represent a balance between the amplitude and width oscillations and this radiation in the shelf. The inclusion of the effect of the dispersive radiation shed as the pulse evolves is vital in order to drive the system to a steady state. This shed radiation leaks away from the shelf and allows the system to settle to a steady state. In this regard, Mamyshev and Mollenauer [7] noted that full numerical solutions of the perturbed NLS equation (2) show oscillations in the pulse amplitude, which they attributed to the generation of radiation by the sliding and filtering.

Figure 2(a) shows the evolution of the pulse amplitude as given by the full numerical solution of the governing equation (2), by the present approximate equations (27), both with and without radiation damping, and by the solution of the approximate equations (14a) and (14b) of [9,10] for a case of stable pulse propagation for which the inequality (31) holds. It can be seen that the numerical amplitude shows an oscillation which is also present in the solution of the present approximate equations (27) with radiation loss, and that there is excellent agreement between these two solutions. The main difference between the numerical and approximate solutions is a phase difference, which is expected as equations for the phase are of higher order than the modulation equations (27) [22,23]. It can be further seen that the solution of the modulation equations (14a) and (14b) of [9,10] does not oscillate and in fact gives the mean of the numerical oscillations. This is because their perturbation solution (13) has fixed the width w of the pulse to be the inverse amplitude $1/\eta$, so that the pulse cannot undergo the amplitude-width oscillations exhibited by the numerical and present approximate solutions. The final observation to be made about Fig. 2(a) is that, if the radiation damping in the approximate equations (27) is neglected (i.e., $\alpha=0$), the approximate solution exhibits amplitude-width-shelf oscillations that do not settle to a steady state, as noted in the previous paragraph. The addition of loss due to dispersive radiation allows leakage from the shelf under the pulse so that the pulse can settle to a steady state. It can therefore be concluded that allowing the pulse amplitude and width to vary independently and the inclusion of radiative loss result in better agreement with the full numerical solution. Neither of these effects was included in the perturbation solutions of [9,10].

Figure 2(b) shows the evolution of the pulse amplitude as given by the full numerical solution and by the present approximate equations for the same parameter values as in Fig. 2(a) for a larger range of z. It can be seen that the numerical amplitude shows long term oscillations which are mirrored by the approximate solution. The approximate solution shows excellent agreement with the numerical solution in terms of both the final steady amplitude and the value of z at which the oscillations have essentially died out.

Let us now consider the evolution of a pulse in the absence of amplification. Figure 3(a) shows a comparison between the solutions of the present approximate equations (27), the approximate equations (14a) and (14b) of [9–11], and the full numerical solution of the perturbed NLS equation (2). The present approximate equations were solved both with and without the effect of shed dispersive radiation. The parameter values used were the same as those of Malomed and Tasgal [11], $\Omega=0.1$, $\gamma=0.09$, and $\sigma=0.046$. The boundary pulse was taken as a NLS soliton with $\eta=1$ and $w=1$ and the initial velocity was $V=0$. As can be seen from the figure, the solution of the present approximate equations is closer to the numerical solution than the solution of the approximate equations of [9–11], especially for larger z. It can also be seen that for the NLS soliton boundary condition the shed dispersive radiation makes little difference to the evolution of the pulse. This is because the nonzero loss σ quickly damps the dispersive radiation so that it has essentially no effect on the evolution of the pulse. Indeed, the approximate solution is slightly closer to the full numerical

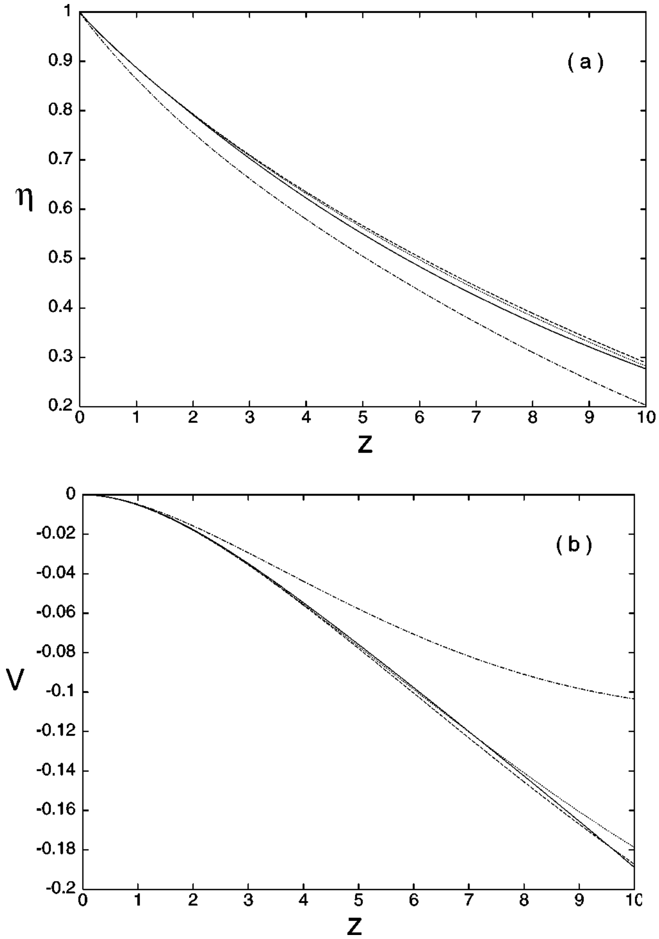

FIG. 3. Comparison between approximate and numerical solutions for soliton boundary condition with $\eta=1$, $w=1$, and $V=0$ with parameter values $\Omega=0.1$, $\gamma=0.09$, and $\sigma=0.046$. Full numerical solution, —; solution of approximate equations (27) with radiation, – – – –; solution of approximate equations (27) without radiation, · · ·; solution of approximate equations (14a) and (14b): ·—·—. (a) Amplitude η as a function of distance z. (b) Velocity V as a function of distance z.

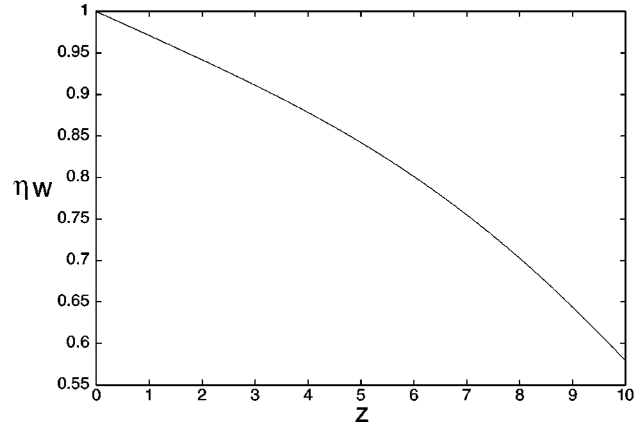

FIG. 4. The product of amplitude and width ηw as given by the approximate equations (27) with radiation for soliton boundary condition with $\eta=1$, $w=1$, and $V=0$ and parameter values $\Omega=0.1$, $\gamma=0.09$, and $\sigma=0.046$.

solution when the effect of the shed dispersive radiation is neglected. This counterintuitive result is due to the errors made in the derivation in Sec. III of the effect on the pulse of the dispersive radiation and again implies that the dispersive radiation can be neglected for nonzero fiber losses σ for a soliton boundary condition.

Figure 3(b) shows a comparison between the pulse velocity as given by the approximate and numerical solutions for the same parameter values as in Fig. 3(a). This velocity comparison shows a marked difference between the approximate solution of the present work and that of [9–11]. It can be seen that the velocity as given by the approximate equations of [9–11] approaches a steady value, while the velocity as given by the present approximate equations continues to decrease as z increases, in agreement with the full numerical solution. In addition, the velocity as given by the approximate equations (27) is in good agreement with the numerical velocity. As for the amplitude comparison of Fig. 3(a), the inclusion of shed radiation makes little difference to the approximate velocity due to the nonzero fiber loss σ damping the dispersive radiation.

As radiation loss makes little difference for NLS soliton boundary conditions, it cannot explain the difference between the solution of the approximate equations (27) and the equations (14a) and (14b) of [9–11]. Therefore the only explanation for the difference must be in the trial functions used. The trial function used in [9–11], Eq. (13), does not allow the pulse amplitude and width to vary independently. Rather, they are restricted to be inversely proportional, as in a NLS soliton. Figure 4 shows the product of amplitude and width ηw as given by the solution of the approximate equations (27) with radiation for the same parameter values as in Fig. 2. Notice that the amplitude and width are clearly not inversely proportional, in contrast to the assumption made in [9–11]. The trial function used in the present work, Eq. (7), allows for independently varying pulse amplitude and width. As found previously in the case of amplification, this added degree of freedom results in better agreement with full numerical solutions.

Let us now examine the evolution of a non-NLS soliton boundary condition. As large loss and filter strength act as damping, killing off most dynamic, evolutionary behavior, we shall take small values for σ and γ. Figure 5(a) shows a comparison between the pulse amplitude η as given by the present approximate equations (27), both with and without radiation damping, and by the full numerical solution of the perturbed NLS equation (2). The boundary condition is a non-NLS soliton pulse with $\eta=1.25$ and $w=1$. The initial velocity was taken as $V=0.1$ and the parameter values $\sigma=0$, $\gamma=0.01$, and $\Omega=0.1$ were chosen. The pulse is therefore propagating into a lossless fiber with a SFF, so that dispersive loss is expected to have an effect on the pulse evolution. No comparison was made with the solution of Eqs. (14a) and (14b) as these approximate equations are valid only for a NLS soliton boundary condition. It can be seen from the amplitude comparison shown in Fig. 5(a) that incorporating radiation loss gives an approximate amplitude in better agreement with the full numerical amplitude. The radiation loss acts as a damping, without which the pulse

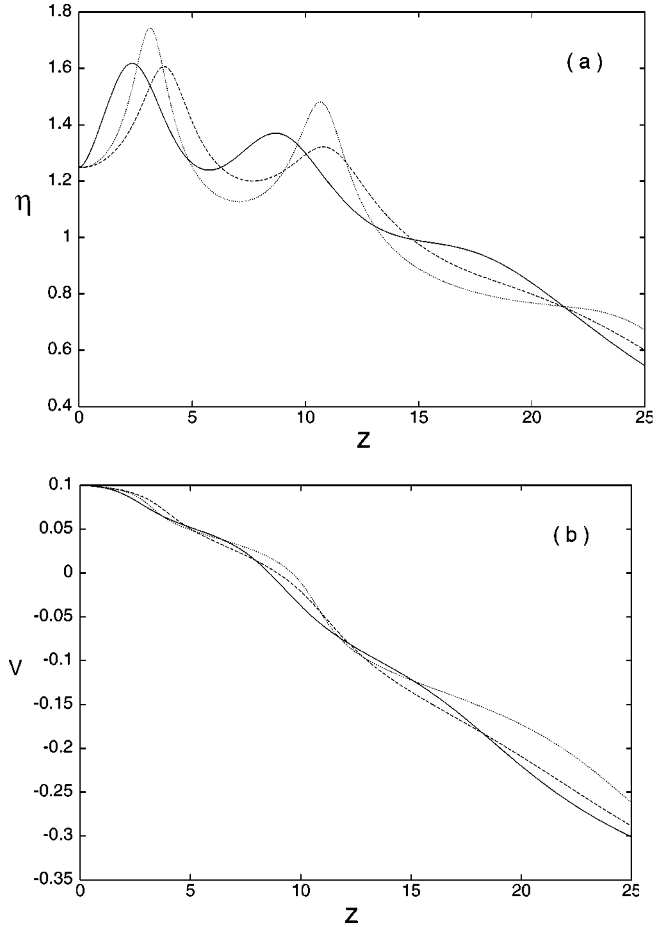

FIG. 5. Comparison between approximate and numerical solutions for nonsoliton boundary condition with $\eta=1.25$, $w=1$, and $V=0.1$ with parameter values $\Omega=0.1$, $\gamma=0.01$, and $\sigma=0.0$. Full numerical solution, —; solution of approximate equations (27) with radiation, – – – –; solution of approximate equations (27) without radiation, $\cdots$. (a) Amplitude η as a function of distance z. (b) Velocity V as a function of distance z.

amplitude is overestimated at every oscillation. However, it can be seen that the amplitude of the oscillations of the approximate solution is decaying slightly faster than that of the numerical solution, so that the radiation damping has been overestimated. There is again a phase difference and a period difference between the approximate and numerical amplitude oscillations which is due to the assumption that the shelf forms instantaneously. The phase of the amplitude oscillation is a higher order effect, and, while methods exist to determine equations for this phase [22,23], these methods do not determine the initial phase, which is of importance here. In this regard, it should be noted that the amplitude of the pulse oscillation is in good agreement with the numerical amplitude.

Figure 5(b) shows the velocity V of the pulse as given by the solution of the approximate equations (27), both with and without radiation damping, and by the full numerical solution of the perturbed NLS equation (2). The boundary and parameter values are as for Fig. 5(a). It can be seen that the inclusion of radiative loss is necessary in order to obtain good agreement with the numerical solution, particularly for large z. As for the amplitude oscillations of Fig. 5(a), there is again a phase and period difference between the numerical and approximate velocity oscillations.

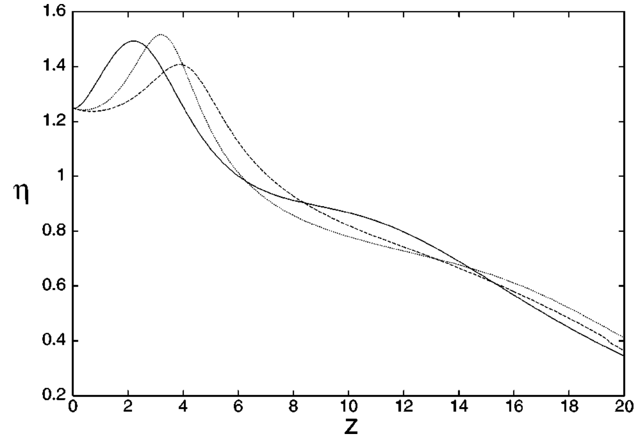

FIG. 6. Comparison between approximate and numerical solutions for the pulse amplitude η for the nonsoliton boundary condition with $\eta=1.25$, $w=1$, and $V=0.1$ with parameter values $\Omega=0.1$, $\gamma=0.03$, and $\sigma=0.005$. Full numerical solution, —; solution of approximate equations (27) with radiation, – – – –; solution of approximate equations (27) without radiation, $\cdots$.

Using the same parameters values as Malomed and Tasgal [11], Fig. 6 shows a comparison between the pulse amplitude η as given by the solution of the present approximate equations (27), both with and without radiation loss, and the full numerical solution for the non-NLS soliton boundary condition with $\eta=1.25$, $w=1$, and $V=0.1$. The parameters used were $\Omega=0.1$, $\gamma=0.03$, and $\sigma=0.005$ [11]. As for the comparison shown in Fig. 3, adding radiation loss causes little change in the agreement with the full numerical solution. This is again due to the damping of the radiation by the fiber loss ($\sigma\neq 0$). It can also be seen that radiation loss overestimates the amplitude damping near the boundary $z=0$. This is not surprising as the radiation loss was derived for large z behavior [13]. Without radiation loss, the amplitude oscillation has too large an amplitude near $z=0$.

V. CONCLUSIONS

The evolutionary behavior of pulses in nonlinear optical fibers including fiber loss, amplification, and SFF's has been examined. As exact solutions of the governing perturbed NLS equation (2) do not exist, approximate methods were used to analyze the pulse evolution. The approximate method of [13] was extended and approximate evolution equations for the pulse were derived using conservation and moment equations. Major benefits of this method are that it allows the amplitude and width of the pulse to evolve independently and that it incorporates the effect of the dispersive radiation shed as the pulse evolves. Excellent agreement was found between solutions of these approximate equations and full numerical solutions of the governing perturbed NLS equation (2). Based on a mass and energy argument, a condition was also found for an evolving pulse to split into two pulses. Good agreement was found between this condition and full numerical solutions of the governing perturbed NLS equation (2). Furthermore, comparisons were made with solutions obtained from the perturbation equations of [9–11]. It was found that the present approximate equations give better agreement with full numerical solutions and that the previous perturbation equations miss important features of the pulse evolution. This is because (i) the present method allows for independent amplitude and width oscillations of the pulse and (ii) the present method includes the dispersive radiation shed by the pulse as it evolves. The perturbation equations of [9–11] do not include these effects. Finally, it was concluded that in the absence of amplification and when fiber loss is present, the damping effect of shed dispersive radiation is negligible due to the damping of this radiation.

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