Regular dynamics of low-frequency fluctuations in external cavity semiconductor lasers

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It is commonly believed that the dynamics responsible for low-frequency fluctuations (LFF's) in external cavity semiconductor lasers is stochastic or chaotic. A common approach to address the origin of LFF's is to investigate the dynamical behavior of, and the interaction among, various external cavity modes in the Lang-Kobayashi (LK) paradigm. In this paper, we propose a framework for understanding of the LFFs based on a different set of fundamental solutions of the LK equations, which are periodic or quasiperiodic, and which are characterized by a sequence of time-locked pulses with slowly varying magnitude. We present numerical evidence and heuristic arguments, indicating that the dynamics of LFF's emerges as a result of quasiperiodic bifurcations from these solutions as the pumping current increases. Regular periodic solutions can actually be observed when (1) the feedback level is moderate, (2) pumping current is below solitary threshold, and (3) the linewidth enhancement factor is relatively large.

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I. INTRODUCTION

Nonlinear delay dynamical systems [1] are extensively used in many fields of science and engineering. Such disciplines as population dynamics, epidemiology, financial mathematics, and optoelectronics, to name a few, use such descriptions in their modeling efforts. In recent years prime examples under intense investigation were the MacKay-Glass model [2] of blood flow and the Ikeda equation [3] describing the evolution of the electric field in a ring cavity. Usually the model equations have very simple functional forms, yet this apparent simplicity is deceiving. They display rich and complex dynamics, partially resulting from the high dimensionality that the retarded terms introduce, since the phase space associated with these equations is of infinite dimension.

In optoelectronics, the intrinsic high dimensionality of the delayed optical systems was recently used in communications applications with chaotic wave forms [4]. Typically, optoelectronic systems contain semiconductor lasers, which are preferred over other types of lasers due to their small size and high efficiency. However, the extreme sensitivity of semiconductor lasers to optical feedback, which is inevitable in such systems, makes their operation unpredictable and hard to control. Therefore, the problem of understanding the behavior of the laser under the influence of external optical feedback is of great practical importance. During the past 20 years, in the area of semiconductor laser instabilities the delay equations of Lang and Kobayashi [5] emerged as the premiere model to discuss the behavior of external cavity semiconductor lasers. The focus of this paper is to investigate some intriguing aspects of this dynamics.

Experimentally, the feedback is modeled by an external reflector, which reinjects part of the emitted light back into the main laser resonator. When the pumping current I is well above the lasing threshold of the solitary laser, I_{th}, the introduction of the optical feedback causes a drastic increase in the optical linewidth [6]. The phenomenon is commonly referred to as *coherence collapse* (CC). At the same time, the intensity fluctuations are well randomized, so the average laser intensity remains fairly constant. However, when the pumping current is reduced close to the threshold, the laser intensity starts to exhibit sudden dropouts at irregular time intervals, followed by a gradual recovery. The time scale of these fluctuations (microseconds to nanoseconds) is long compared to the intrinsic time scale of the laser oscillations, so the phenomena are called *low-frequency fluctuations* (LFF's).

The origin of the LFF regime has been debated since it was first observed more than two decades ago [7], with no consensus in sight. Numerous mechanisms proposed to date emphasize either the stochastic or deterministic (chaotic) nature of the phenomenon. In this paper, we present evidence that LFF's can be understood in terms of a particular set of periodic and quasiperiodic solutions that are part of the system's dynamics. We show that such regular solutions gradually emerge from the overall chaotic and stochastic dynamics of the system when the pumping current is lowered close to the lasing threshold. In such parameter regimes, the regular dynamics generates nearly periodic dropout events, which more recently observed in experiments [8]. More importantly, the commonly observed *random* low-frequency fluctuations can be interpreted as chaotic motions over a set of *destabilized* regular solutions, much like a typical scenario of a transition to chaotic dynamics in many nonlinear dynamical systems. These results have significant implications in understanding and applications of external cavity semiconductor lasers. For instance, one might be interested in applying control to eliminate LFF's. Knowing that the underlying dynamics has embedded within itself a regular structure, despite the irregularity of LFFs, one can attempt strategies that are different from the commonly utilized approach of controlling chaos [9].

Before we detail our numerical results and analysis, we wish to comment on the role of regular solutions in nonlinear dynamical systems in general. The study of a nonlinear system often begins with an analysis of stationary solutions (fixed points). Their locations and stability determine the structure and properties of chaotic motion in these systems. For example, one of the most common routes to chaos is via the destabilization of periodic orbits in a cascade of period-doubling bifurcations [10]. The resulting chaotic motion is ''pieced together'' from parts of the original periodic motion, and retains the general features and some characteristics of the original orbits. In fact, it is widely accepted that unstable periodic orbits are the ''backbone'' or ''skeleton'' of any chaotic set [11]. The study of the Lang-Kobayashi (LK) model is no exception in this respect. One starts the analysis by identifying external cavity modes (ECM's), and then attempts to interpret the behavior of the LK model at different parameter values in terms of the location and stability properties of the ECM's. This interpretation works reasonably well for studying the destabilization of individual ECM's and for explaining coherence collapse as a process of ''random'' transitions, or hopping, among ECM's. The situation is different for LFF's, whose structure (short pulses, gradual buildup of intensity, and dropouts) is hard to explain solely on the basis of the ECM dynamics. Our belief is that the difficulty in explaining the origin of the LFF regime stems from the inadequacy of the ECM framework. The main goal of this work is to propose a framework for the study of the LFF regime based on a different set of regular solutions, whose existence was discovered in experiments [8]. Since the structure of the solutions is apparently similar to that of the LFF regime, we propose to use them as a basis for the analysis of the LFF dynamics [12].

We stress that the class of regular solutions are intrinsic to the system dynamics, just as periodic orbits are intrinsic and fundamental solutions of any nonlinear dynamical systems. These regular solutions are therefore determined completely by the LK equations, and they appear to be structurally stable, i.e., their existence persists in finite parameter regions. While in many simple nonlinear systems, the creation and evolution of the periodic orbits can be understood very well based on the analysis of a few types of bifurcations such as saddle-node and period-doubling bifurcations, analytical studies of the regular solutions in the LK equations are difficult, if not impossible. In fact, the existence of regular solutions reported in this paper comes mostly from numerical computations. Our confidence in the existence of these solutions is due to the following two facts: (1) the LK equations are believed to be the fundamental equations that model external-cavity semiconductor lasers, and (2) the signature of the regular solutions has been found in experiments [8].

The rest of this paper is organized as follows. In Sec. II, we briefly describe the Lang-Kobayashi model and our numerical procedure. In Sec. III, we present numerical results supporting the existence of the regular dynamics and its relation to the dynamics in the LFF regime. In Sec. IV, we explore some properties of the regular solutions, and make a heuristic argument to explain the numerical results. A discussion of a possible origin of the regular solutions is presented in Sec. V.

II. LANG-KOBAYASHI PARADIGM

Here we discuss the delay differential equations [1] derived by Lang and Kobayashi, [5] that model the external cavity semiconductor lasers. These equations describe the evolution of the complex electric field amplitude $\mathcal{E}(t) = E(t)e^{i\phi(t)}$ and of the excess carrier density $N(t)$. The LK equations can be written in dimensionless forms [13] as follows:

$$\dot{E} = N(t)E(t) + \eta E(t-\tau)\cos[\Delta(t)+\phi_0],$$

$$\dot{\phi} = \alpha N(t) - \eta \frac{E(t-\tau)}{E(t)}\sin[\Delta(t)+\phi_0], \qquad (1)$$

$$T\dot{N} = P - N(t) - [2N(t)+1]E(t)^2,$$

where $\Delta(t) = \phi(t) - \phi(t-\tau)$ is the phase delay during the external cavity round-trip time τ, and time t is measured in the units of the photon lifetime. The semiconductor medium is characterized by the linewidth enhancement factor α and the carrier lifetime T. The external feedback level is represented by η, while $\phi_0 = \omega_0\tau$ is the round-trip phase mismatch, where ω_0 is the emission frequency of the solitary laser. The excess pump current P is proportional to $(I/I_{\mathrm{th}}) - 1$. The influence of spontaneous emission noise can also be accounted for by the addition of a Langevin term [14,15]. Even though the LK model assumes a single-mode operation and neglects multiple reflections from the external mirror, it reproduces many experimentally observed dynamical regimes of external cavity semiconductor lasers.

Based on the LK model, the CC regime has been interpreted as a manifestation of chaotic dynamics [16]. Coupling of the single-mode laser with the external cavity adds a series of new *external cavity modes*. These are the stationary solutions of Eq. (1), which have the forms $E(t) = E_s$, $\phi(t) = \omega_s t$, and $N(t) = N_s$, where

$$E_s^2 = \frac{P - N_s}{2N_s + 1} > 0, \quad N_s = -\eta\cos(\Delta_s + \phi_0), \qquad (2)$$

and $\Delta_s \equiv \omega_s\tau$ are determined implicitly by the equation $\Delta_s = -\eta\tau\sqrt{1+\alpha^2}\sin(\Delta_s + \phi_0 + \tan^{-1}\alpha)$. The stationary solutions appear in pairs, and their number is proportional to $C = \eta\tau\sqrt{\alpha^2+1}$, so that it grows with the increasing feedback level and/or the external cavity length. One of the solutions in each pair is intrinsically stable, and is therefore identified with an ECM, while the other is unstable and often called the *antimode*. From the standpoint and dynamical systems, the mode-antimode pairs are created after saddle-node bifurcations, and the antimodes are located on the basin boundaries separating different ECMs. Stable upon creation, each ECM becomes unstable at a slightly higher feedback, and is replaced by a limit cycle due to the Hopf bifurcation. A further increase in the feedback level leads to a chaotic attractor along either a quasiperiodic or period-doubling route. At yet higher feedback, attractors corresponding to different ECM's begin to merge via basin boundary crises, and a large attractor appears on the ruins of many single ECM attractors. As the ECM's span a wide range of frequencies, the system evolution on this large attractor corresponds to the loss of coherence of the emitted light.

When the laser is biased well above solitary threshold, the trajectory visits vicinities of different ECM's in an essentially random fashion. However, when the pumping current is lowered close to the threshold, the system enters the LFF regime, in that the evolution of the system acquires a definite direction in phase space toward ECM's with higher gain. This drift leads to a gradual increase in the output power (buildup), followed by a relatively fast decrease of the total intensity (dropout) when the system returns to lower gain ECM's. Early theoretical investigations of the nature of the LFF attributed random dropout events to the influence of spontaneous emission noise [14,17]. It was soon realized, however, that the LFF regime is present in the LK model even when the noise term is omitted, thus suggesting the deterministic nature of the LFF phenomenon. This led to the interpretation of the LFF as a chaotic itinerancy with a drift (buildup) followed by a switching to the antimode controlled dynamics (dropout), caused by the boundary crisis of the high-gain ECM attractors [18,19]. Recently the noise-induced dropout scenario was revived again in conjunction with the notion of excitability [20]. Another area of current debate is centered around the role of multimode behavior in the LFF regime [21–23].

In spite of the fact that noise influences the statistics of time intervals between successive dropouts [15], while the presence of multiple modes changes the intensity distribution during the buildup process in certain cases [23], the LFF regime is easily identified in the numerical simulations of the fully deterministic LK model [19,24,25]. Therefore, the LK model retains the essence of the dynamics in the LFF regime, and thus contain clues as to the origin of this phenomenon.

III. EVIDENCE OF THE REGULAR DYNAMICS

We now present numerical evidence of the existence of particular types of periodic or quasiperiodic solutions of the LK equations, and conjecture that these solutions are responsible for the drift dynamics in the LFF regime. We use the fourth-order Adams-Bashford-Moulton predictor-corrector method [26]. Figure 1(a) shows time dependence of an electric field $E(t)$ obtained from numerical integration of Eq. (1) with pumping near a solitary threshold. An important aspect of the laser operation is that slow variations of the field $(\sim 10\tau)$, which are identified with LFF's, are in fact an envelope for a sequence of narrow pulses. These pulses have been first predicted from numerical solution of the LK model [19] and later observed experimentally with streak cameras [27,21,22]. The authors of Ref. [19] interpreted the pulsations as a form of ''mode locking.'' Indeed, as shown in Fig. 1(b), the pulses occur roughly at the phase delays Δ_s of the ECM's [28]. In order to study the LFF phenomenon from a dynamical system standpoint, while taking into consideration

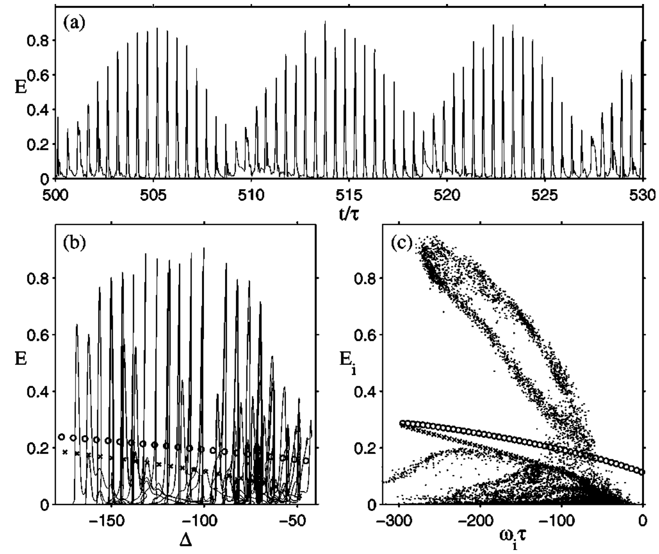

FIG. 1. (a) Numerical solution of Eq. (1) in the LFF regime: $\alpha=6$, $T=300$, $\tau=700$, $\eta=0.07$, $\phi_0=0$, and $P=0.001$. (b) The same solution is shown in the configuration space of $E(t)$ vs $\Delta(t)$. (c) Poincaré section ($\dot{E}=0$, $\ddot{E}<0$) of the same solution. The circles and crosses in (b) and (c) show the location of the ECM's and the antimodes, respectively.

the pulsating behavior of the laser, we construct the Poincaré surface of section at $\dot{E}=0$, $\ddot{E}<0$. That is, we characterize the system evolution by recording a sequence of local maxima of the electric field, $E_i=E(t_i)$, as well as of the excess carrier density, $N_i=N(t_i)$, and the *instantaneous frequency*, $\omega_i=\dot{\phi}(t_i)$. We use ω_i instead of the phase delay coordinate $\Delta_i=\Delta(t_i)$, since it better characterizes the dominant frequency of the emitted light in the pulse. The plot of E_i versus $\omega_i\tau$ in Fig. 1(c) reveals a definite pattern traced by the large pulses. To explore the nature of the dynamics, we have constructed the Poincaré sections at decreasing values of the pumping parameter P, as shown in Fig. 2. The remarkable result is that as the pumping parameter is lowered below the threshold, the pattern becomes better defined, and at $P=-0.012$ the system dynamics is essentially quasiperiodic, i.e., the tips of the pulses E_i trace out a line in the configuration space. We find that the largest period associated with the quasiperiodic motion is extremely large ($100–400\tau$) and at $P=-0.014$ tends to infinity, so that the system evolution is reduced to a periodic motion. For $P<-0.014$ the motion again becomes quasiperiodic. The sequence of plots in Figs. 2(a)–2(d) thus represents a general *bifurcation* scenario in which the pumping parameter P serves as a bifurcation parameter. Thus, as P is increased, the periodic solution [Fig. 2(d)] undergoes a Hopf bifurcation to generate a quasiperiodic solution [Fig. 2(c)], which then becomes chaotic [Figs. 2(b) and 2(a)] through the quasiperiodic route to chaos [29].

It is important to note that the quasiperiodic behavior emerges gradually, so that its appearance cannot be interpreted as a window of spontaneous stabilization of the otherwise unstable quasiperiodic-periodic orbit. Instead it is more plausible to assume that, with decreasing pumping, the laser settles into a more regular regime as a means of coping

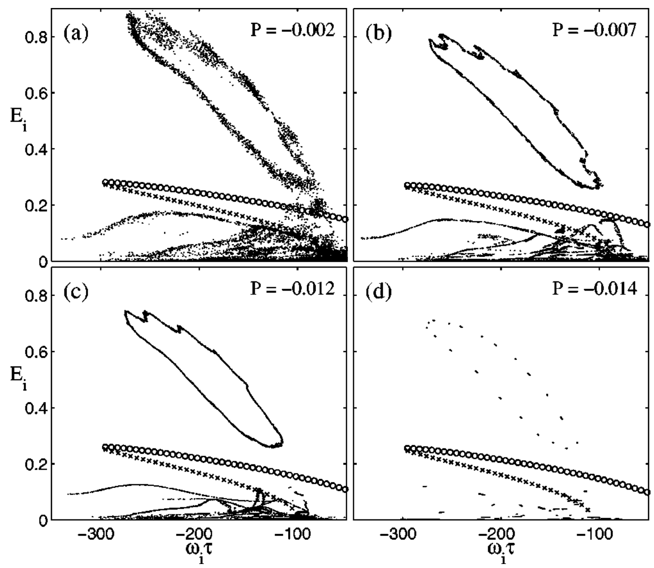

FIG. 2. Emergence of quasiperiodic (c) and periodic (d) LFF's with decreasing pumping parameter *P*. The other parameters are the same as in Fig. 1. Each plot contains 10 000 points.

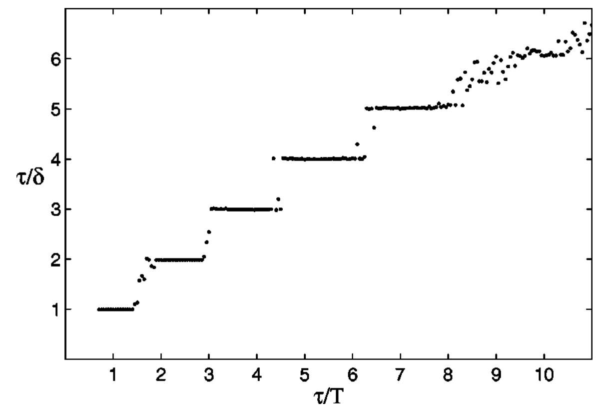

FIG. 3. Dependence of the average interval δ between large pulses on the ratio τ/T. A pulse is considered ''large'' when its peak is above the energy surface of the ECM's, as given by Eq (2): $E_i^2 > (P-N_i)/(2N_i+1)$. The system parameters are $\alpha=6$, $\tau=700$, $\eta=0.07$, $\phi_0=0$, and $P=-0.012$. Each dot represents a median value of 3000 time intervals.

with insufficient influx of the pumping current. Indeed, in the CC regime, the laser operates extremely erratically, readily expending the energy pumped into it. When the pumping is lowered close to the threshold, the laser has to find a more efficient operation mode. One of the possibilities would be to settle into one of the ECM's. However, a large value of the linewidth enhancement factor α in semiconductor lasers induces a strong coupling between the ECMs, which prevents the system from operating in a single ECM state [30].

Our numerical results, as well as recent experiments [8], suggest that there exists a different type of efficient operation regime, which is characterized by a sequence of narrow pulses with slowly varying magnitude and a wide frequency band. The system evolution in this regime is strikingly similar to that of the LFF's; that is, both LFF's and regular solutions are characterized by a drift toward ECM's with higher gain, followed by a return to lower gain ECM's. We stress here that the comparison between our numerical results and the experimental findings in Ref. [8] is specific for the regular solutions *only*. In particular, a similarity is observed if one compares Fig. 1 with Fig. 1(c) in Ref. [8]. However, there are notable differences between the temporal shapes of the time evolution of the laser fields in our numerical plot and in experiments. Such a comparison, of course, is meant to be qualitative only, as there are complicating factors such as spontaneous emission in experimental lasers which were not included in our computations. The focus of our paper is to look for deterministic dynamical structures of LFF's.

The dynamical features that distinguish the LFF regime from the efficient regime is that the former is chaotic and admits irregularities similar to the CC regime, such as inverse switching [18], while the latter is very regular, and is either periodic or quasiperiodic. Thus the LFF regime shares similarities with both the CC regime and the efficient regime of lasers operating below the solitary threshold. Based on this evidence, we view the LFF's, which occur at low pumping, as an intermediate regime between the CC regime and the efficient regular regime.

With pumping current increasing from below the solitary threshold, the sequence of transitions appears as follows. When the pumping is very low, lasing is possible only in a sequence of short pulses with a drift, i.e., the efficient regime shown in Figs. 2(c) and 2(d). As the pumping increases, the operation in the efficient regime becomes unstable and bifurcates into chaos via a quasiperiodic route. The resulting regime is that of the LFF's, which are irregular, but still retain the global drift dynamics characteristic of the efficient regime. With higher and higher pumping currents, the amount of irregular behavior increases, while the drift dynamics becomes less and less apparent, until it disappears when the laser enters the CC regime. These observations are consistent with experiments exploring the parameter space in the direction of varying pumping [31].

IV. SOME PROPERTIES OF THE REGULAR SOLUTIONS

Further exploring the nature of the quasiperiodic solutions, we find that the average time interval between large pulses in Fig. 1(a) is about $\delta=0.504\tau$, which corresponds to approximately two pulses per round-trip time, and shows no significant dependence on P, η, α, or ϕ_0. On the other hand, depending on the value of the carrier lifetime parameter T, the system exhibits a sequence of *time-locked* states containing an integer number of pulses per τ, as shown in Fig. 3. Note that the time-locked behavior of the fast pulsations was observed experimentally by Vaschenko *et al.* [21], and described as a ''marked pseudoperiodicity at the round-trip time of the optical field in the external cavity.''

The time-locking of the pulses can be easily explained based on the LK model in Eq. (1). From the third equation we see that $N(t)$ is always negative when $P<0$. This means that, when pumped below threshold, only the second term in the first equation contributes to the formation of a new pulse. That is, a new pulse is formed at the moment when the energy from a previous pulse returns into the laser cavity after the round-trip time. Therefore, the time-locked state develops because the process of *triggering* the laser emission by pulses reflected from the external cavity is naturally locked to the round-trip time. The dependence of the number of pulses per τ on the carrier lifetime T can be understood based on the evolution of the excess carrier density $N(t)$. The evolution of $N(t)$ during the pulse is governed essentially by the last term in the third equation of Eqs. (1), $T\dot{N} \approx -E^2$, which causes a rapid decrease of $N(t)$. In order to be able to sustain the next pulse, the carrier density needs time to recover. The recovery process, during which $E\approx 0$, is described by the exponential law $N(t)\approx P-N_0 e^{-t/T}$, so the number of pulses per τ must be inversely proportional to T. Note that, as shown in Fig. 3, the time-locked states do not form for $\tau/T>8$. The reason is that for six or more pulses per τ the interval between pulses becomes comparable to their duration, which leads to a breakdown of the triggering mechanism.

The triggering mechanism can also explain the apparent ''mode locking'' of the pulses in the phase delay variable $\Delta(t)$. Indeed, efficient triggering requires that $\cos[\Delta(t)+\phi_0] \approx 1$ in the first equation of Eqs. (1), which means that $\Delta(t) \approx \Delta_n = 2\pi n - \phi_0$ during the pulse, where n is an integer. Numerical solution of the LK model shows that the value of $\Delta(t)$, during large pulses, is much closer to Δ_n than to Δ_s. Therefore, the mode locking cannot be associated with individual ECM's, but rather reflects a complex *simultaneous* influence of many ECM's on the system evolution [28].

V. DISCUSSIONS

In conclusion, we have discovered a particular type of quasiperiodic or periodic solutions of the LK model, and provide evidence that these solutions are responsible for the drift toward the higher gain ECM's followed by the return dynamics (often referred to as the ''Sisyphus effect'') in the LFF regime.

Finally, we wish to discuss some observations as to the possible origins of the regular solutions in the LK model. Since these solutions are much more complicated than the ECM's, it does not appear feasible at present to derive them analytically from the LK model without any approximations. Therefore, we will refer to a recently proposed *truncated* LK model [32], which is constructed by replacing $E(t-\tau)$ and $\dot{\phi}(t)$ in Eq. (1) with truncated Taylor series expansions in terms of $E(t)$ and $\Delta(t)$, respectively: $E(t-\tau)=E(t)$ and $\dot{\phi}(t)=\Delta/\tau+\dot{\Delta}/2$. The result is a system of three *ordinary* differential equations

$$\dot{E}=NE+\eta E\cos(\Delta+\phi_0),$$

$$\dot{\Delta}/2=-\Delta/\tau+\alpha N-\eta\sin(\Delta+\phi_0), \tag{3}$$

$$T\dot{N}=P-N-(2N+1)E^2,$$

which is much easier to explore both analytically and numerically. Even though the above expansions cannot be justified for large values of τ, the truncated system retains a very important feature in that it has the same stationary solution as the original LK system. Moreover, the ECM's in the truncated system also destabilize via Hopf bifurcations, and are replaced by the limit cycles around individual ECM's. Most importantly, when the feedback is further increased, the limit cycles begin to merge via a gluing bifurcation, creating a large limit cycle revolving around many ECM's. This cycle is very similar to the one shown in Fig. 2(d). Even though the small dimensionality of the truncated system cannot support the rich variety of regimes present in the full LK model, we do see a transition from periodic to chaotic behavior when the pumping parameter P is increased. The apparent similarity between the quasiperiodic or periodic LFF solutions in the LK model and the large cycles of the truncated model suggests that the regular LFF solutions also emerge in a sequence of gluing bifurcations. We intend to explore this subject in more detail in our future publications.

ACKNOWLEDGMENT

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ECM's, and it is unlikely that we will ever be able to derive them analytically from the LK model without any approximations. However, this does not mean that they are less fundamental than the ECM solutions. Numerical simulations show that the new solutions are structurally stable and exist in a wide range of parameter values. We believe that the study of the origin and properties of these solutions will facilitate the analysis and understanding of the LFF regime.

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[28] Note that ''mode locking'' of pulses occurs in the phase delay variable $\Delta(t)$, and thus cannot be interpreted as each pulse being emitted at a steady frequency associated with a particular ECM. Our computations show substantial deviations of the instantaneous frequency during the pulse away from the ECM frequency suggested by Δ_s.

[29] Ruelle and Takens considered four frequency quasiperiodic flows on the torus T^4. They showed that it was possible to make arbitrarily small, but carefully chosen, smooth perturbations to the flow so that the flow becomes chaotic [D. Ruelle and F. Takens, Commun. Math. Phys. **20**, 167 (1971); **23**, 343 (1971)]. Newhouse, Ruelle, and Takens subsequently showed that the same held for a three-frequency quasiperiodic flow on the torus T^3 [S. Newhouse, D. Ruelle, and F. Takens, *ibid.* **64**, 35 (1978)].

[30] In our simulations we are able to reach the quasiperiodic and periodic regime only for $\alpha \gtrsim 5$. Otherwise, the coupling between ECM's is not strong enough, and, as the pumping is lowered below threshold, the system dynamics becomes localized around one of the ECM's before it reaches the quasiperiodic and periodic regime. However, we stress that regular solutions exist in the same parameter range as the LFF's. The difference is that for smaller α they are most likely to be *unstable* with respect to other solutions (such as ECM's, or chaotic solutions). The fact that we do not observe them in the simulation, does not mean that they do not exist, but rather that they are simply unstable. The only reason it is easier to observe them at larger α is that they are stable in this parameter range.

[31] T. Heil, I. Fischer, and W. Elsässer, Phys. Rev. A **58**, R2672 (1998).

[32] G. Huyet, P. A. Porta, S. P. Hegarty, J. G. McInerney, and F. Holland, Opt. Commun. **180**, 339 (2000).