Stress transmission through textured granular packings

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We propose a theoretical approach aimed to describe the stress transmission through granular pilings. According to our framework, the stress transmission obeys a diffusive process in random isotropic packings, while it follows a local convection-diffusion equation in textured packings characterized by an anisotropic distribution of the contact orientations. In the latter case, the direction of the convection depends on the angular distribution function of the contact orientations. Our theoretical approach agrees with both experimental and numerical recent evidences, and moreover, succeeds in capturing the role of the pile preparing history.

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I. INTRODUCTION

Recently, an unexpected feature issued from the engineering literature was pointed out by some authors, who raised doubts about the capability of the classical elastoplastic continuum modeling of soil mechanics to describe the mechanical properties of grain collections. It concerned the local minimum of pressure that has been sometimes noticed at the vertical of the apex of conical grain piles, in place of the intuitively expected maximum [1]. In order to account for this "dip" of pressure, Edwards and Oakeshott [2] first proposed that some "arching effects" were able to deflect the weight transmission throughout the material towards the outer edges of the conical heaps. Then Bouchaud, Cates, and Claudin (BCC) [3] proposed a new, nonstandard framework by hypothesizing an *ad hoc* "constitutive" relation $\sigma_{xx}/\sigma_{ZZ}=K=\text{const}$ between the two diagonal components of the stress tensor (in two dimensions). Although this relation closely looks like the classical criterion $\sigma_1/\sigma_2=(1+\sin\phi)/(1-\sin\phi)$ (where σ_1 and σ_2 correspond to the principal stresses) defining the plastic threshold for Mohr-Coulomb materials (of angle of friction ϕ), BCC's constitutive relation is aimed to describe the material behavior *below* plastic failure in a state classically considered as elastic in mechanics or soil engineering textbooks. Injecting BCC's relation in the force balance equation leads straightforwardly to a set of hyperbolic partial differential equations for the stress field. Since the two characteristics are inclined at $\pm\arctan(K^{1/2})$ with respect to the vertical axis, force propagation paths are deflected towards the outer edges, and thus, this formalism succeeds in predicting a stress dip.

Nevertheless, it is worth emphasizing that the occurrence of a stress dip is not a general feature. It is a very particular phenomenon occurring after some very peculiar preparation processes, as clearly reported in the engineering literature [4] and reassessed in recent experimental works [5–7]. Furthermore, note that BCC's model addresses the case of isotropic materials, and deflection of forces appears as an *intrinsic* and *fundamental* effect of the granular state within this frame, *disregarding any textural properties* (viz., isotropy or anisotropy) of piles. Thus, the genericity of the stress dip predicted by such "hyperbolic" models, independently of textural features or constructing history of piles, opposes experimental evidences [5,7]. BCC's model raises, moreover, a number of serious questions. For instance, hyperbolic equations (relating x and z spatial variables) inhibit the propagation of sound, or at least confine it anisotropically to a very narrow region [8], even in the case of *isotropic* materials, which is most unphysical and is in contradiction with all common experiences [9–11]. Next, the predictions of hyperbolic models are so far from those of civil engineering, that buildings or dams based on current elastoplastic modelings would necessarily collapse if the upholders of hyperbolic models were right. For illustration, consider the settling under buildings. Computations are commonly performed in civil engineering within the frame of the classical elastoplastic theory in order to prevent the risk of soil plastic yielding. According to the standard soil mechanics viewpoint, the stress is maximal right below the building, contrary to the hyperbolic model that predicts that the stress maximum loci are laterally shifted from the vertical of the building location; stress field and yield locus predictions are thus drastically unsimilar. At last, note that a BCC-type relation $\sigma_{xx}/\sigma_{ZZ}=K$, is also borne out by perfect fluids (with $K=1$), and following BCC's reasoning, it should be hence inferred that the stress field also obeys a set of hyperbolic partial differential equations in that case, which is obviously inexact.

II. PREPARING PROCESS AND BULK ANISOTROPY

In order to account for the stress transmission in grain packings, and particularly its dependence upon the preparing process, we propose here a new framework by stressing the role of the pile structural properties. After having introduced anisotropy in the probability distribution function of the contact orientations, we establish the stress transmission equation or the basis of a stochastic approach, relating the stress field to the internal texture. Our theoretical frame constitutes a generalization suited for anisotropic random packings of the "*q* model" of Coppersmith *et al.* [12], and succeeds in capturing the role of construction history. In particular, we show why a stress dip may possibly appear (or not appear), depending on the preparing process.

Recently, several noticeable pieces of evidence were obtained from both numerical and experimental studies. In particular, mechanical and structural properties of piles prepared

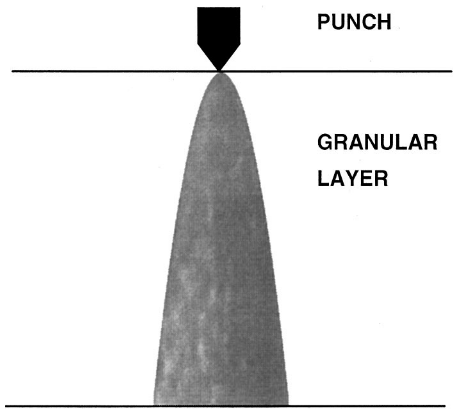

FIG. 1. In the case of a horizontal homogeneous layer prepared by uniform pouring from an extended source; the fabric tensor is isotropic and the response to an extra load located on the free surface follows a simple diffusion process. The strained region (represented in gray) is bounded by a parabola.

according to two well-defined different ways were investigated. For conical piles grown from a central point source, a stress dip was evidenced, associated with an average anisotropy in the contact orientations [13,14]. In contrast, in the case of a layer prepared by uniform pouring from an extended source, a bell-shaped distribution of pressure was measured on ground as a response to an external point overload imposed at the free surface [15]. Moreover, in a very recent experimental report based on photoelastic visualizations, the strained region was shown to be bounded by a parabola (see Fig. 1) as a response to a point load [16]. This last result upholds a diffusive process (equation of parabolic type) for the stress response, in opposition to both the classical elastic theory (elliptic equations) [17] and the hyperbolic proposal of BCC [3]. Besides the remarkable predictive power concerning the contact-force distribution in random packings [12], the q model also perfectly accounts for the diffusive nature of the stress equation [18]. We can provide some theoretical arguments to justify that, fundamentally, the diffusive behavior issued from the q model is actually generic in the case of random (isotropic) packings, provided that there are no long-range structural correlations in grain positions. Our assumption relies on the following remarks. First, it is important to point out that, in the case of isostatic packings, the force transmission factor q can be unambiguously deduced from elementary local geometrical constraints; on the other hand, for the hyperstatic case, the random variable q represents the indeterminacy in the set of friction forces applied onto each grains. Provided that the balances of horizontal forces and torques are both respected, it makes sense to use a stochastic description to access the stress field for all the set of possible pile configurations [19–21]. Second, another important experimental feature is the evidenced *treelike* structure for the stress transmission patterns through the packing. This result was also evidenced from photoelasticity experiments performed on amorphous packings [22]. Hence, random packings are characterized by both the treelike hierarchical structure of the intergrain contact forces, and the absence of geometrical correlation in the packing (over the short geometrical correlation length). This absence of long-range correlation implies a random series of coefficients q along the force transmission tree (coarse grained over a few grain sizes), and consequently a purely diffusive large-scale behavior for the force. It can therefore be concluded that the diffusive behavior is certainly relevant to disordered packings. Nevertheless, we expect that this simple result fails to describe the case of textured disordered piles, characterized by some privileged directions for the average contact orientations. The reason is that biased and long-range correlated q series along the force transmission tree are likely in the latter case. In the following, we will propose a route to access the mechanical response based upon the knowledge of structural correlations, and we will detail how the previous purely diffusive behavior is modified. For the sake of simplicity, we will adopt a scalar model for the force transmission and we shall use, moreover, a mean-field approach in which we only consider force correlations proceeding from the *structural anisotropy*.

During the pile growth, bulk anisotropy may be systematically induced by the construction process itself [23]. For grain collections, the dependence of the equilibrium state on the preparation method was probably first realized by Darwin more than a century ago [24]. This memory effect imprinted into the bulk is often qualified as ''material fabric'' in civil or chemical engineering. Various engineering studies have focused on this aspect [23], and several alternative descriptions were devised in order to characterize the bulk anisotropy. Since Satake [25] it is usual to introduce the second-rank ''fabric'' tensor in the following manner. For a random packing of (possibly unequal) spheres, consider the unit vector $\mathbf{n}$ normal to the contact planes between two adjacent spheres. The fabric tensor $\mathbf{F}$ is defined from the averaging of the dyadic $\mathbf{n}\otimes\mathbf{n}$ over all contacts achieved in the bulk, $\mathbf{F}=(1/\text{number of contacts})\Sigma_{\text{contacts}}\mathbf{n}\otimes\mathbf{n}$. Such a definition of a tensorial quantity is of interest in order to deal with frame-invariant properties, and in the perspective of linearized theories [26]. Another characterization of the fabric anisotropy can be defined from the distribution $p(\theta)$ of the contact orientations, defined from the probability $p(\theta)d\theta$ of a particle to have a contact vector $\mathbf{n}$ pointing between the directions θ and $\theta+d\theta$. The normalization factor $z=\int_0^{2\pi}p(\theta)d\theta$ proceeds then from the average coordination number z. Therefore, the probability $p(\theta)$ expands in a Fourier series as

$$p(\theta)=\frac{z}{2\pi}[1+A\cos 2(\theta-\theta_a)+B\cos 4(\theta-\theta_b)+\cdots]. \quad (1)$$

Actually, as pointed out by Radjai and Roux [14], the major principal direction of the fabric tensor $\mathbf{F}$ precisely identifies with the direction θ_a, and the coefficient A is equal to $2(F_1-F_2)$ (where F_1 and F_2 are, respectively, the major and minor eigenvalues of $\mathbf{F}$). Both experimental and numerical investigations recently accessed such probability distribution functions $p(\theta)$ [13–15]. For piles prepared by uniform random deposition of grains the fabric tensor is isotropic ($F_1=F_2=\frac{1}{2}$). More precisely, the contour of the histogram $p(\theta)$, expressed in polar coordinates (p,θ) displays four

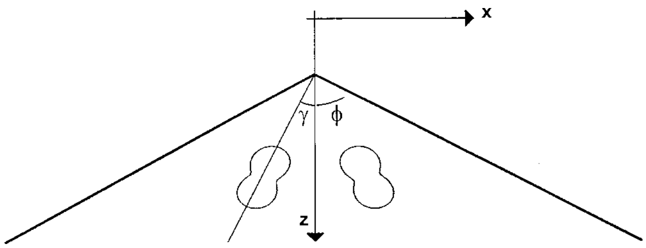

FIG. 2. A conical pile prepared by pouring from a central point source displays an anisotropic distribution for the contact orientations. In this figure we represent the two symmetrical histograms of the probability $p(\theta)$ of the contact orientations, which keep the memory of the construction process, and results from the past surface avalanches.

lobes but as we will show further, this fourth order anisotropy does not play any role in the linear mechanical response and the medium behaves as isotropic regarding stress transmission. On the other hand, an important textural effect was recognized after the construction of conical piles by pouring from a point source [13]. In the latter case, $p(\theta)$ displays a two-lobed contour (see Fig. 2), and we can approximate $p(\theta)$ by the first two terms of its Fourrier expansion [Eq. (1)]. The direction θ_a corresponding to the maximum of the distribution $p(\theta)$ appears to be oriented at 45° with respect to the free surface. Such fabric anisotropy can be explained by considering the following simple argument. The growth of the conical pile resulted from the flow of surface avalanches, whose matter was supplied from the point source. The intermittent surface flows (of typically 10 or 20 grain-size thickness) led to an internal stratification, which has, for instance, been recently visualized by means of colored particles [27–29]. Those successive surface avalanches corresponded to shallow shearing flows, and therefore the stress principal axes were oriented at 45° with respect to the free surface. Then we can understand why contacts were preferably created during the pile construction along the orientation associated with the maximum of compression. Since the free surface slope of the growing pile is equal to the angle of friction ϕ of the medium, we easily recover that the preferential direction for contact orientations is oriented at $\gamma = 45° - \phi$ with respect to the (vertical) symmetry axis of the cone. Typical values of γ are 18° for glass spheres (ballotini, $\phi = 27°$) and 8° for sand (Fontainebleau sand, $\phi = 37°$).

III. GENERALIZATION OF THE *q* MODEL TO ANISOTROPIC MATERIAL

In order to establish the mechanical behavior of textured pilings, we study the response to a localized vertical extra force $\mathcal{F}_Z(x=0,y=0) = F_0\,\delta(0,0)$ oriented downwards [note that the point (0,0) is not necessarily located on the free surface of the pile]. The response to this point force probes the Green's function for the stress equation (provided that the partial differential equation describing the stress field throughout the discrete medium is linear). Of course, a typical contact force induced by an external loading consists both of a compressive and a frictional component. We just note $\mathcal{F}_Z(x,y)$ the sum of these two parts.

Within the discrete framework of the q model (which deals with on-lattice grains), the force exerted by the adjacent upper grains (belonging to the layer j) onto a particle in contact beneath (belonging to the layer $j+1$) reads (in two dimensions)

$$\mathcal{F}_Z(i,j+1) = \sum_{i'} q_{\{(i',j)\rightarrow(i,j+1)\}}\mathcal{F}_Z(i',j), \tag{2}$$

where i (j) labels the horizontal position (the depth). In Eq. 2, force transmission factors $q_{\{(i',j)\rightarrow(i,j+1)\}}$ are independent random variables.

Now consider two grains in contact, with a contact vector $\mathbf{n}$ oriented at angle θ with respect to the vertical axis (oriented downwards). In the spirit of the q model, we just require the vertical projections of (both repulsive and frictional parts of) the forces exerted onto each grain to be balanced. The occurrence of such contacts is weighted by the probability $p(\theta)$ and the vertical (horizontal) projection of the distance l joining the center of masses is $l\cos\theta$ ($l\sin\theta$). After averaging on all possible pile configurations, the force transmission rule (2) reads

$$\mathcal{F}_Z(x,z) = \int_{-\pi/2}^{+\pi/2} d\theta\, p(\theta) q(x - l\sin\theta, z - l\cos\theta \rightarrow x,z) \times \mathcal{F}_Z(x - l\sin\theta, z - l\cos\theta) \tag{3}$$

and the force balance equation obviously leads to the sum rule

$$\int_{-\pi/2}^{+\pi/2} d\theta\, p(\theta) q(x,z \rightarrow x + l\sin\theta, z + l\cos\theta) = 1. \tag{4}$$

According to the original q model, we assume the q distribution to be constant over the interval [0,1] and independent of θ. We deduce then from Eq. (5) that the mean value $\overline{q}$ is equal to $2/z$ (the final result should be qualitatively unaffected by considering a nonconstant distribution for q).

After having expanded the last right-hand factor of Eq. (3) as a Taylor series according to the small parameter l,

$$\begin{aligned}\mathcal{F}_Z(x - l\sin\theta, z - l\cos\theta) &= \mathcal{F}_Z(x,z) - l\sin\theta\frac{\partial\mathcal{F}_Z}{\partial x} - l\cos\theta\frac{\partial\mathcal{F}_Z}{\partial z} + \frac{l^2}{2}\sin^2\theta\frac{\partial^2\mathcal{F}_Z}{\partial x^2}\\ &\quad + \frac{l^2}{2}\cos^2\theta\frac{\partial^2\mathcal{F}_Z}{\partial z^2} + \frac{l^2}{2}\sin 2\theta\frac{\partial^2\mathcal{F}_Z}{\partial x\partial z},\end{aligned} \tag{5}$$

its integration following the variable θ yields the equation

$$\frac{2l}{\pi}\left[1 + \frac{A}{3}\cos 2\theta_a\right]\frac{\partial\mathcal{F}_Z}{\partial z} = -\frac{4l}{3\pi}A\sin 2\theta_a\frac{\partial\mathcal{F}_Z}{\partial x} + \frac{l^2}{4}\left[1 - \frac{A}{2}\cos 2\theta_a\right]\frac{\partial^2\mathcal{F}_Z}{\partial x^2}. \tag{6}$$

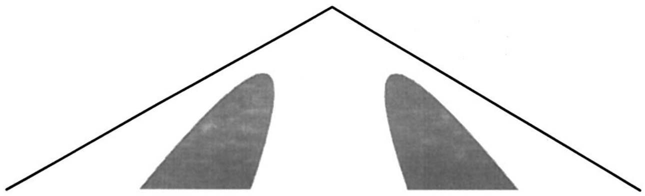

FIG. 3. In the case of conical piles prepared by pouring from a central point source, the anisotropy of the internal texture tends to deflect the weight transmission towards the outer edges, and can lead to a central ''pressure dip'' on ground. The gray areas represent the strained regions as responses to the two symmetrical extra point loads. Taking into account structural properties brings a foundation to the deflection effect first suggested by the authors of Ref. [2].

We recognize here a convection-diffusion equation. In the isotropic case ($A=0$) the above equation reduces itself to a simple diffusion equation, as in the original q model and the diffusion coefficient equals $D=\pi l/8$. This simple diffusive behavior was experimentally recognized by Da Silva and Rajchenbach [16]. Note that in the presence of an internal fabric, the texture anisotropy alters the diffusion coefficient because it modifies both the mean steps along the x and z directions. In the limit $A \ll 1$, the convection ''velocity'' dx/dz is given by $(2A/3)\sin 2\theta_a$. As a consequence, the effect of the fabric characterizing a conical pile prepared by pouring from a point source is to deflect the weight transmission towards the outer edges (Fig. 3). Note that the angle of deflection $\psi=\arctan[(2A/3)\sin 2\theta_a]$ (relative to the vertical direction) is smaller than the angle of repose of the free surface, which is a necessary condition to ensure the heap stability. Therefore, provided that the blurring effect owing to the diffusion is small compared to the deflection angle, the mechanism that we point out is relevant to create a pressure dip.

IV. CONCLUSION

As a conclusion, the present work points to the link between internal texture and stress field in granular pilings and sheds lights on the role of the construction process. The origin of the pressure dip, which sometimes appears on ground at the vertical of the apex of conical piles is explained. By the means of simple approximations (forces are considered scalar and the Fourier expansion of the probability distribution of the contact orientations is limited to the second order), our description provides foundations and provides a simple ansatz to carry out a mathematical treatment of the deflection of force paths first suggested by Edwards and Oakeschott [2] by stressing the effect of textural properties. In that, it contrasts with the previous hyperbolic models, which considered the force path deflection as an intrinsic effect of the granular matter, independently of structural properties. Finally, the predictions of our model fully agree with both numerical and experimental evidence provided by different procedures of pile construction and previously reported in literature.

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