Spectral oscillations in a frequency-modulation laser operation

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We show the existence of coherent oscillations in the optical frequency spectrum of a laser with internal frequency modulation. These oscillations arise in presence of a slight detuning between the modulation frequency and the cavity axial mode separation and are analogous to collective oscillation modes of a chain of weakly coupled pendula with resonance frequencies slightly varying along the chain. Although spectral oscillations are damped due to the finite gain bandwidth of the laser cavity, they can be observed in the transient dynamics during switch-on of the modulator, or can be resonantly excited by an external forcing.

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Intracavity frequency modulation (FM) of a laser is a well-established mean for the generation of highly coherent frequency-modulated optical signals and ultrashort pulses since the beginning of the laser era [1–3]. Modulation of the optical cavity length at a frequency close to the cavity axial mode separation is known to lead to the generation of a mode-locked pulse train (FM mode-locking) under synchronous modulation, or to the generation of a frequency-modulated optical field (FM regime) for a detuned modulation, with more complex dynamical behaviors connecting these two limiting regimes [2–5]. In recent years there has been a renewed interest, both theoretical and experimental, in the study of FM-operated lasers, mainly motivated by their potential use in optical communication applications [6–9]. This has lead to the development of new laser devices and techniques based on intracavity frequency modulation [7–9], envisaging their potential use in high-bit-rate optical transmission systems [8,9]. At the same time, some effort has been devoted to a deeper understanding of the dynamical aspects involved in FM lasers, including detailed theoretical and experimental studies on noise in FM-operated lasers [6,9,10] and pulse retiming dynamics in FM mode-locked lasers [11].

In this Brief Report we add novel insights into the dynamical behavior of FM-operated lasers, showing the existence of coherent oscillations in the laser spectrum under detuned frequency modulation. In the limit of an infinite gain bandwidth, these oscillations are undamped and manifest themselves as a slow breathing of the Bessel FM laser spectrum at a frequency equal to the detuning frequency. A simple mechanical analogy of this reversible dynamics is elucidated. The finite gain bandwidth of the laser breaks the reversibility of the dynamics, making the spectral oscillations damped but yet observable during transient formation of the FM laser spectrum when the modulator is switched on.

The starting point of the analysis is provided by a rather standard model of intracavity laser frequency modulation in a homogeneously broadened laser with a broad gain bandwidth and a slow relaxation rate for the population inversion [2,4,5]. We consider a ring cavity of length L containing a homogeneously broadened gain medium, a frequency limiter that determines the gain bandwidth of the cavity, and an electro-optic phase modulator that varies periodically the optical cavity length at a frequency ω_m close to the separation $\omega_c = 2\pi c/L$ of the cavity axial modes.

After expanding the normalized intracavity electric field F on the basis of longitudinal ring cavity eigenmodes by setting $F(z,t) = \Sigma_n F_n(t)\exp(-2\pi i n t)\exp(2\pi i n z)$, where z is the longitudinal spatial coordinate along the cavity, scaled to the cavity length L, and t is time normalized to the modulation period $T_m = 2\pi/\omega_m$, the coupled-mode equations for the amplitudes F_n read [2,4]

$$\dot{F}_n = \left[-2\pi i n \gamma + g - l - n^2 \left(\frac{\omega_m}{\omega_g} \right)^2 \right] F_n + \frac{i\Delta}{2}(F_{n+1} + F_{n-1}) \tag{1}$$

$(n = 0, \pm 1, \pm 2, \dots)$, where $\gamma \equiv (\omega_c - \omega_m)/\omega_m$ is the frequency detuning parameter $(|\gamma| \ll 1)$; g is the round-trip saturated gain; l is the cavity loss; ω_g is the spectral bandwidth of the gainline $(\omega_m/\omega_g \ll 1)$, which is assumed to be independent of the gain variable g; Δ is the single-pass modulation index introduced by the phase modulator; and the dot stands for the derivative with respect to time. The saturated gain g obeys the separate equation:

$$\dot{g} = -\gamma_{\parallel} \left(g - g_0 + g \sum_n |F_n|^2 \right), \tag{2}$$

where g_0 is the small-signal gain due to the pumping and $\gamma_{\parallel}$ is the gain relaxation rate normalized to the modulation frequency $(\gamma_{\parallel} \ll 1)$. Since the mode amplitudes F_n are slowly-varying on the temporal scale of the modulation period, the frequency-domain description provided by Eqs. (1) can be transferred into an equivalent time-domain description by introducing an independent slow-time variable, the round-trip number T, which accounts for the slow evolution of F_n and g at successive round-trips according to Eqs. (1) and (2) (see, e.g., Ref. [12]). After setting $\psi(t,T) \equiv \Sigma_n F_n(T)\exp(-2\pi i n t)$, we obtain the following equations of motion for ψ and g:

$$\begin{aligned} \partial_T \psi &= (g - l)\psi + [\mathcal{D}\partial_t^2 + \gamma\partial_t + i\Delta\cos(2\pi t)]\psi \\ &\equiv (g - l)\psi + \mathcal{L}(t)\psi, \end{aligned} \tag{3}$$

$$\frac{dg}{dT}=-\gamma_{\parallel}(g-g_0-g\mathcal{E}), \tag{4}$$

where $\mathcal{D}\equiv(\omega_m/\omega_g)^2/(2\pi)^2$ is the filtering parameter, $\mathcal{L}(t)\equiv\mathcal{D}\partial_t^2+\gamma\partial_t+i\Delta\cos(2\pi t)$, and $\mathcal{E}(T)\equiv\int_0^1|\psi(t,T)|^2\,dt=\Sigma_n|F_n|^2$ is the energy over one modulation period. The main dynamical features of the FM laser are governed by the eigenmodes $|\alpha(t)\rangle$ and corresponding eigenvalues λ_α of the operator $\mathcal{L}$, which are in turn strongly sensitive to the value of the frequency detuning parameter γ [5,13]. The singular behavior of the system near $\gamma\sim 0$ is responsible for the transition of laser operation from the FM regime under asynchronous modulation to the pulsed mode-locking regime as the condition $\gamma=0$ is approached [5]. In the FM regime the laser emits a field with nearly constant intensity and with a carrier frequency which is sinusoidally swept in time, whereas in the FM mode-locking regime the laser output is a periodic train of mode-locked pulses. In this work we consider the case where the laser is operated in the asynchronous FM regime. For an infinite gain bandwidth ($\mathcal{D}=0$), the eigenmodes of $\mathcal{L}$ reduce to the Bessel normal modes $|\alpha_B(t)\rangle\equiv\exp[-i\Gamma\sin(2\pi t)-2\pi i\alpha t]$ with eigenvalues $\lambda_\alpha=-2\pi i\alpha\gamma$ ($\alpha=0,\pm1,\pm2,\dots$), where $\Gamma\equiv\Delta/(2\pi\gamma)$ is the effective modulation index [2,5]. In addition, the dynamics conserves the energy $\mathcal{E}$, which is fixed by the condition $g=l$. In the frequency domain, the Bessel modes $|\alpha_B(t)\rangle$ correspond to the normal spectral modes $F_n^{(\alpha)}=J_{n-\alpha}(\Gamma)$, which diagonalize the dynamics of Eqs. (1). Since the Bessel modes $|\alpha_B(t)\rangle$ form a complete set of normal modes, the solution to Eq. (3) with the initial condition $\psi(t,0)=\psi_0(t)$ can be written as

$$\psi(t,T)=\sum_\alpha c_\alpha(T)\exp(-2\pi i\gamma\alpha T)|\alpha_B(t)\rangle, \tag{5}$$

where $c_\alpha(0)\equiv\langle\alpha_B|\psi_0\rangle$ and $\langle f|g\rangle$ stands for $\int_0^1 f^*(t)g(t)dt$. For $\mathcal{D}=0$, the coefficients c_α are constants and the solution $\psi(t,T)$ is periodic with period equal to the time detuning $T_{\text{det}}\equiv 1/\gamma$, i.e., one has $\psi(t,T+T_{\text{det}})=\psi(t,T)$. This means that the dynamics, besides conserving the energy, is also reversible, in the sense that any initial field distribution is periodically recovered. In particular, we show that there exists a particular family of solutions $|\Phi\rangle$, that we call *spectral oscillons*, corresponding to a breathing Bessel mode, i.e., to a Bessel mode whose modulation index varies periodically between $|\Gamma+\Phi|$ and $|\Gamma-\Phi|$, where Φ is a real number. Since the number of oscillating cavity axial modes is approximately twice the modulation index, a spectral oscillon corresponds to a slow and periodic variation of the spectral extent of the FM laser output (see Fig. 1). As it will be discussed later, the relevance of these dynamical states stems from the fact that they play a major role in the transient dynamics leading to the formation of the FM laser spectrum when the modulator is switched on. The state $|\Phi\rangle$ is defined by a coherent superposition of normal modes with amplitudes proportional to the Bessel functions of the first kind $J_\alpha(\Phi)$, i.e.,

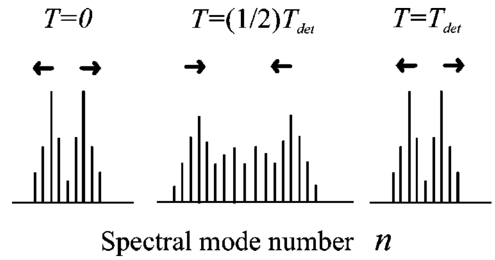

FIG. 1. Schematic of the breathing dynamics of the spectrum for the oscillon mode $|\Phi\rangle$.

$$|\Phi(t,T)\rangle=\sum_{\alpha=-\infty}^{\infty}J_\alpha(\Phi)|\alpha_B(t)\rangle\exp(-2\pi i\alpha T). \tag{6}$$

Taking into account the expression of the normal modes $|\alpha_B(t)\rangle$ and using the identity $\Sigma_\alpha J_\alpha(\Phi)\exp(-2\pi i\alpha t)=\exp[-i\Phi\sin(2\pi t)]$, Eq. (6) can be readily cast in the following form:

$$|\Phi(t,T)\rangle=\exp\{-iW(T)\sin[2\pi t+\phi(T)]\}, \tag{7}$$

where we have set

$$W(T)=\sqrt{\Gamma^2+\Phi^2-2\Gamma\Phi\cos(2\pi\gamma T)}, \tag{8a}$$

$$\sin\phi(T)=-\frac{\Phi}{W}\sin(2\pi\gamma T). \tag{8b}$$

Equation (7) clearly shows that the state $|\Phi\rangle$ is a Bessel mode whose modulation index, $W(T)$, varies periodically between the values $|\Gamma-\Phi|$ and $|\Gamma+\Phi|$ according to Eq. (8a); the slow breathing of the corresponding spectrum is schematically depicted in Fig. 1. We can appreciate the importance of the dynamical states $|\Phi\rangle$ by observing that, for $\Phi=\Gamma$, one has $|\Phi(t,0)\rangle=1$, i.e., the initial state corresponds to the laser oscillating on a single cavity axial mode. The state $|\Gamma(t,T)\rangle$ thus describes the reversible dynamics of the laser spectrum that occurs when the laser oscillates on a single longitudinal mode and the modulator is suddenly switched on. More generally, the state $|\Phi\rangle$ describes the reversible dynamics of the laser spectrum induced by a stepwise change of the modulation depth Δ. It is worth observing that the reversible laser dynamics, attained in the limit of an infinite gain bandwidth, bears a close analogy with the small-amplitude oscillations of a chain of weakly coupled pendula whose resonance frequencies are weakly varied along the chain by, e.g., variation of the pendulum length L_n (see Fig. 2). Denoting by $\omega_n^2=g/L_n$ the resonance frequency of the uncoupled nth pendulum in the chain and by $\Delta\equiv(2k/m)/\omega_0^2$ the normalized coupling strength provided by the springs, assuming a weak linear change of the resonance frequency ω_n^2 along the chain according to $\omega_n^2=\omega_0^2(1+2\pi\gamma n)$, with $\gamma\ll\Delta^{1/2}\ll 1$, we can write the following linearized equations of motion for the small-amplitude oscillation angles θ_n of the pendula:

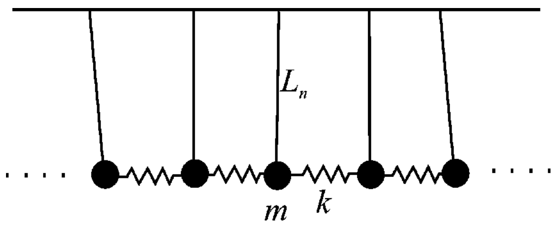

FIG. 2. Mechanical analogy of the laser dynamics in the reversible case. The mass m and spring strength k are constant along the chain of pendula, whereas the length L_n is slightly varying with n, providing a linear change of the resonance frequencies of the pendula along the chain.

$$\ddot{\theta}_n = -\omega_0^2(1+2\pi\gamma n)\theta_n + \omega_0^2\frac{\Delta}{2}(\theta_{n+1}+\theta_{n-1}-2\theta_n). \tag{9}$$

The normal modes $\theta_n^{(\alpha)}$ of oscillation and corresponding resonance frequencies Ω_α of the coupled pendula are obtained in a standard way by looking for a solution of Eqs. (9) in the form $\theta_n^{(\alpha)}(t) = \{\overline{\theta_n^{(\alpha)}}\exp(i\Omega_\alpha t)+\text{c.c.}\}$, with the boundary conditions $\overline{\theta_n^{(\alpha)}} \to 0$ for $n \to \pm\infty$. Using the identity $2nJ_n(x) = x[J_{n-1}(x)+J_{n+1}(x)]$ for Bessel functions J_n, the normal modes are readily calculated as $\overline{\theta_n^{(\alpha)}} = J_{n-\alpha}(\Gamma)$, where $\Gamma \equiv \Delta/2\pi\gamma$, and the resonance frequencies are given by $\Omega_\alpha = \omega_0(1+\Delta+2\pi\gamma\alpha)^{1/2} \simeq \omega_0(1+\Delta/2+\pi\gamma\alpha)$, with $\alpha = 0,\pm1,\pm2,\ldots$. The normal modes of oscillation of the chain thus correspond exactly to the Bessel modes $|\alpha_B\rangle$ found in the laser case, and the coherent superposition $|\Phi\rangle$ of normal modes corresponds, in the mechanical system, to a collective motion of the pendula in which the excitation is periodically transferred between $\sim|\Gamma-\Phi|$ and $\sim|\Gamma+\Phi|$ adjacent pendula in the chain. In particular, the coherent superposition $|\Gamma\rangle$ describes the collective motion of the chain when only one pendulum is initially excited.

When finite cavity gain bandwidth effects are considered, the eigenmodes $|\alpha\rangle$ of $\mathcal{L}$ lose their normal nature and the eigenvalues λ_α become complex-valued $[\text{Re}(\lambda_\alpha)<0]$, with one dominant mode $|0\rangle$ having the lowest damping rate [5]. Due to the existence of a dominant mode, the reversible dynamics of the dissipationless case is broken, and the state $|0\rangle$ is asymptotically and stably reached by the system after transient [13]. However, damped oscillations of the spectral extent, similar to those found in the reversible dynamics, should be transiently observed. In particular, these oscillations should be present in the transient dynamics describing the switch-on of the FM laser. To study the laser dynamics for $\mathcal{D}\neq 0$, we have numerically integrated the coupled-mode equations (1), together with the equation for gain dynamics [Eq. (2)], using a high-accuracy Runge–Kutta method with variable step. As an initial condition, we assumed the solution corresponding the laser operated in the single central longitudinal mode with the modulator switched off, i.e., we assumed $F_n(0)=0$ if $n\neq 0$, $g=l$, and $|F_0| = (C-1)^{1/2}$, where $C\equiv g_0/l$ is the pump parameter. The spectral width of the oscillating field is measured through the variance $\Delta n \equiv (\Sigma_n n^2|F_n|^2/\Sigma_n|F_n|^2)^{1/2}$. A typical behavior of the slow-evolution of Δn at successive round-trips is shown in Fig. 3(a). The normalized laser parameters chosen in the simulations are typical for operational conditions of FM solid-state lasers [5,7]; in particular, notice that the frequency detuning has been chosen larger than the gain relaxation rate. After the modulator is switched on at $T=0$, the spectral oscillations are damped out due to the spectral filtering, with the appearance of higher-order harmonic components (in particular the second harmonic) in the frequency response [see the insets in Fig. 3(a)]. The existence of higher-order harmonic frequency components can be understood by observing that, owing to the nonvanishing value of $\mathcal{D}$, the coefficients c_α in the expansion (5) become time dependent and their dynamical equations read

$$\frac{dc_\alpha}{dT} = (g-l)c_\alpha + \sum_\beta c_\beta\langle\alpha_B|\mathcal{D}\partial_t^2|\beta_B\rangle\exp[-2\pi i\gamma(\beta-\alpha)T] \tag{10}$$

with $c_\alpha(0) = J_\alpha(\Gamma)$. Equations (10) are supplemented by Eq. (4) for the gain dynamics with $\mathcal{E} = \Sigma_\alpha|c_\alpha|^2$. Since the sum on the right-hand side in Eq. (10) is nonvanishing for $\beta=\alpha,\alpha$

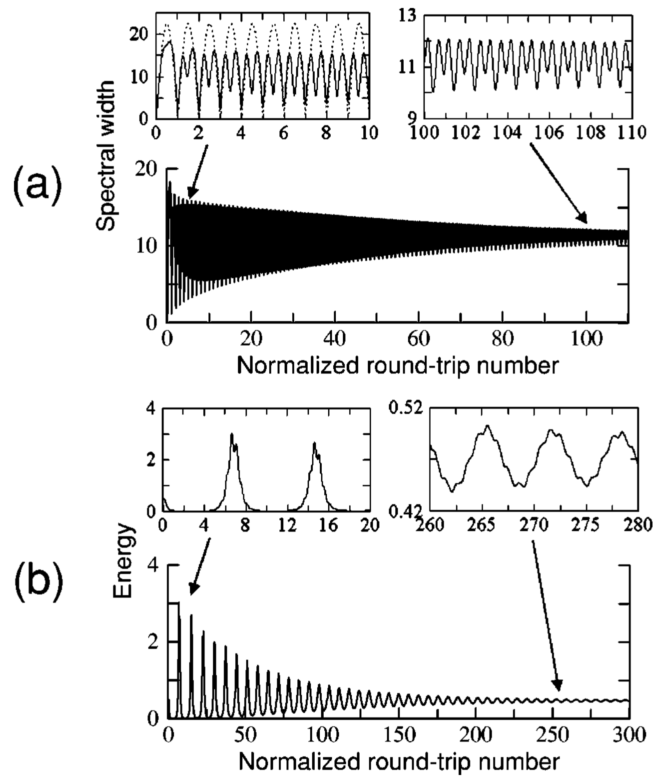

FIG. 3. Evolution of the spectral width Δn (a) and energy $\mathcal{E}$ (b) of the laser field versus the normalized round-trip number T/T_{det}, as obtained by numerical simulations of Eqs. (1) and (2), for parameter values $\gamma=10^{-3}$, $\mathcal{D}=2\times10^{-7}$, $\Delta=0.1$, $C=1.5$, and $\gamma_\parallel = 2\times10^{-5}$. The number of cavity axial modes used in the simulation is $N=101$. The insets in the figures are enlargements showing the dynamical evolution of Δn and $\mathcal{E}$ on the time scale of the time detuning T_{det}. The dashed curve in the inset on the left-hand side in Fig. 3(a) is the behavior of Δn as predicted in the undamped case (i.e., for $\mathcal{D}=0$).

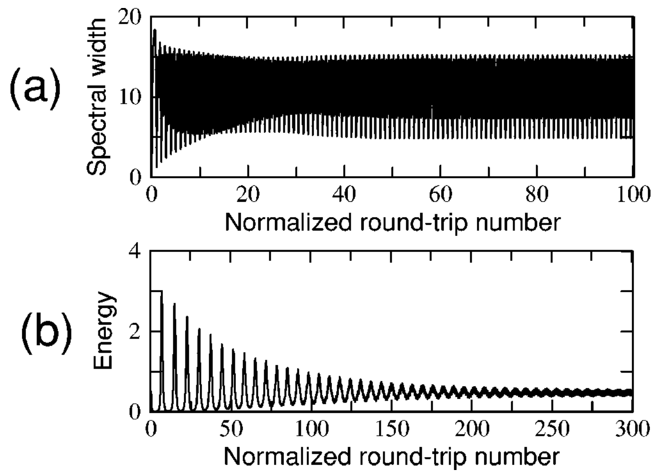

FIG. 4. Evolution of the spectral width Δn (a) and energy $\mathcal{E}$ (b) of the laser field in presence of a periodic stepwise modulation of Δ with period $T_{\rm det}$: $\Delta=0.1$ for $nT_{\rm det}<T<(n+1/2)T_{\rm det}$ and $\Delta=0.101$ for $(n+1/2)T_{\rm det}<T<(n+1)T_{\rm det}$ $(n=0,1,2,3,\ldots)$; the other parameter values are the same as in Fig. 3.

$\pm1,\alpha\pm2$, higher-order beating terms at harmonics of the detuning frequency are generated in the dynamical evolution. In addition, when the modulator is switched on, due to the finite gain bandwidth the initial field energy stored in the cavity is larger than its steady-state value in the FM mode of operation, so that damped relaxation oscillations due to the gain dynamics are excited, as shown in Fig. 3(b). Notice also that the behavior of the field energy shows, besides the spiking behavior typical of the relaxation oscillations, also a superimposed faster modulation at the detuning frequency which is a signature of the spectral oscillations [14] [see the insets in Fig. 3(b)]. The numerical results shown in Fig. 3 are derived by a numerical analysis of Eqs. (1) and (4), however we checked that the same results are obtained by integration of Eqs. (10) and (4), i.e., by projecting the field $\psi(t,T)$ on the basis of Bessel modes $|\alpha_B(t)\rangle$ instead of that of the longitudinal ring cavity eigenmodes. As a final remark, we mention that sustained spectral oscillations may be resonantly excited by external forcing, such as by a periodic variation of the modulation index Δ at the detuning frequency. As an example, Fig. 4 shows the resonant excitation of spectral oscillations, for the same parameter values as in Fig. 3, as obtained by a small-amplitue stepwise modulation of Δ at a frequency equal to the frequency detuning γ.

In conclusion, we have shown the existence of coherent oscillations in the spectral output of an FM-operated laser and discussed their relevance in the transient dynamics leading to the formation of the FM laser spectrum. In the limit of an infinite gain bandwidth, these oscillations are undamped and the spectral dynamics bears a close analogy with the reversible dynamics of a chain of weakly coupled pendula with varying frequency. It is the dissipative nature of the laser dynamics, due to the finite gain bandwidth, that makes the spectral oscillations damped. The present analysis provides new important insights into the dynamics of FM-operated lasers and may be helpful for a deeper understanding and for the control of complex dynamical and noisy behaviors observed in FM-operated lasers [6,9,14].

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[13] The degree of freedom provided by the gain dynamics is responsible solely for the occurrence of laser relaxation oscillations during transients; however these evolve on the slow time scale T and hence do not influence the spectral and temporal properties of the laser field on the fast time scale t. In addition, after expansion of $\psi(t,T)$ on the basis of the eigenmodes $|\alpha\rangle$, it can be shown that the relaxation oscillations are always damped in the autonomous case.
[14] Intensity noise at the detuning frequency and harmonics is typically observed in FM-operated lasers, and represents one of the major limitations for their use in optical communication applications [9]. In addition, the noise strength at these frequency components is strongly enhanced when a portion of the FM spectrum of the laser field, e.g., the lateral Stokes or anti-Stokes bands of the FM spectrum, is filtered externally to the laser cavity (see Ref. [9]). This enhancement effect may find a simple and elegant explanation in the breathing dynamics of the FM spectrum.