Anomalous scaling in two models of passive scalar advection: Effects of anisotropy and compressibility

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The problem of the effects of compressibility and large-scale anisotropy on anomalous scaling behavior is considered for two models describing passive advection of scalar density and tracer fields. The advecting velocity field is Gaussian, δ correlated in time, and scales with a positive exponent ε. Explicit inertial-range expressions for the scalar correlation functions are obtained; they are represented by superpositions of power laws with nonuniversal amplitudes and universal anomalous exponents (dependent only on ε and α, the compressibility parameter). The complete set of anomalous exponents for the pair correlation functions is found nonperturbatively, in any space dimension d, using the zero-mode technique. For higher-order correlation functions, the anomalous exponents are calculated to $O(\varepsilon^2)$ using the renormalization group. As in the incompressible case, the exponents exhibit a hierarchy related to the degree of anisotropy: the leading contributions to the even correlation functions are given by the exponents from the isotropic shell, in agreement with the idea of restored small-scale isotropy. As the degree of compressibility increases, the corrections become closer to the leading terms. The small-scale anisotropy reveals itself in the odd ratios of correlation functions: the skewness factor slowly decreases going down to small scales for the incompressible case, but starts to increase if α is large enough. The higher odd dimensionless ratios (hyperskewness, etc.) increase, thus signaling persistent small-scale anisotropy; this effect becomes more pronounced for larger values of α.

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I. INTRODUCTION

Much attention has been paid recently to a simple model of passive scalar advection by a self-similar Gaussian white-in-time velocity field, the so-called ''rapid-change model,'' introduced by Kraichnan [1]; see, e.g., Refs. [2–9] and references therein. Despite its simplicity, the model reproduces many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. On the other hand, it appears more tractable theoretically: the anomalous exponents have now been calculated on the basis of a microscopic model and within regular expansions in formal small parameters [3–8]. Therefore, passive advection by a ''synthetic'' velocity with prescribed statistics, of practical importance in itself, may also be viewed as the starting point in studying anomalous scaling in turbulence as a whole.

In the original Kraichnan model, the velocity field is taken to be Gaussian, isotropic, incompressible, and decorrelated in time. More realistic models should involve anisotropy and compressibility. Recent studies have pointed out significant differences between the compressible and incompressible cases [10–17]. It is noteworthy that the potential velocity field remains nontrivial in the one-dimensional case, which is more accessible to numerical simulations and allows interesting comparison between the numerical and analytical results; see Ref. [10].

Another important question recently addressed is the effect of large-scale anisotropy on inertial-range statistics of passively advected scalar [18–23] and vector [24–26] fields and the velocity itself [27,28]. These studies have shown that the anisotropy present at large scales has a strong influence on the small-scale statistical properties of the scalar, in dis-

agreement with what was expected on the basis of cascade ideas [18–20]. On the other hand, the exponents describing the inertial-range scaling exhibit universality and hierarchy related to the degree of anisotropy, which gives some quantitative support to Kolmogorov's hypothesis on the restored local isotropy of inertial-range turbulence [17,21–28].

In this paper we analyze the effects of the large-scale anisotropy induced by a random source on a passive scalar by two methods: First, we carry out a zero-mode calculation of the correlation function of the passive scalar with an anisotropic source field and in an isotropic compressible velocity field decorrelated in time. Second, for the same model, we performed a two-loop renormalization-group analysis of the asymptotic behavior of the structure functions of the passive scalar of arbitrary order. This paper is organized as follows. In Sec. II, the zero-mode solution for the correlation function for both passive density and passive tracer is constructed. Two-loop renormalization-group analysis of the structure functions of the passive scalar is carried out in Sec. III with the use of the operator-product expansion. Section IV is devoted to discussion of the results.

II. ZERO-MODE SOLUTION FOR PASSIVE DENSITY AND TRACER

There are two types of diffusion-advection problem for a compressible velocity field [29]. Passive advection of a density field $\theta(x) \equiv \theta(t,\mathbf{x})$ (say, the density of an impurity) is described by the equation

$$\partial_t \theta + \partial_i (v_i \theta) = \nu_0 \partial^2 \theta + f, \qquad (1)$$

while the advection of a ''tracer'' (say, temperature, specific entropy, or *concentration* of the impurity particles) is described by

$$\partial_t \theta + (v_i \partial_i) \theta = \nu_0 \partial^2 \theta + f. \tag{2}$$

Here $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, ν_0 is the molecular diffusivity coefficient, ∂^2 is the Laplace operator, $\mathbf{v}(x)$ is the velocity field, and $f \equiv f(x)$ is an artificial Gaussian scalar noise with zero mean and the covariance

$$\langle f(x)f(x') \rangle = \delta(t-t')C(\mathbf{r}), \quad \mathbf{r} \equiv \mathbf{x} - \mathbf{x}', \tag{3}$$

where $C(\mathbf{r})$ varies noticeably on $r \equiv |\mathbf{r}| \sim L$, the integral turbulence scale. In the presence of a preferred direction specified by a unit vector $\mathbf{n}$, the function $C(\mathbf{r})$ can be written in the form

$$C(\mathbf{r}) = \sum_{l=0}^{\infty} C_l(mr)P_l(z), \quad z \equiv \frac{\mathbf{r} \cdot \mathbf{n}}{r}, \tag{4}$$

where $m \equiv 1/L$, $C_l(mr)$ are coefficient functions such that $C(\mathbf{r})$ becomes constant at $mr=0$ and decays rapidly for $mr \to \infty$, z is the cosine of the angle between $\mathbf{n}$ and $\mathbf{r}$, and $P_l(z)$ are (d-dimensional) Legendre polynomials satisfying the equations

$$(1-z^2)P_l''(z) + z(1-d)P_l'(z) + l(l+d-1)P_l(z) = 0. \tag{5}$$

The anisotropy makes it possible to introduce also a mixed correlator $\langle \mathbf{v}f \rangle \propto \mathbf{n}\delta(t-t')C'(\mathbf{r})$ with some function C' similar to C in Eq. (4). This violates the evenness in $\mathbf{n}$ and gives rise to nonvanishing odd correlation functions of θ, but leads to no serious alterations in the analysis. We shall discuss this case later on, and for the time being we assume $\langle \mathbf{v}f \rangle = \mathbf{0}$.

In the real problem, the field $\mathbf{v}(x)$ satisfies the Navier-Stokes equation. In the simplified model considered in [1–7] it obeys a Gaussian distribution with zero mean and the covariance

$$\langle v_i(x)v_j(x') \rangle = \delta(t-t')K_{ij}(\mathbf{r}) \tag{6}$$

with

$$K_{ij}(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{D_0 P_{ij}(\mathbf{k}) + D_0' Q_{ij}(\mathbf{k})}{(k^2+m^2)^{d/2+\varepsilon/2}} \exp[i(\mathbf{k} \cdot \mathbf{r})], \tag{7}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ and $Q_{ij}(\mathbf{k}) = k_i k_j/k^2$ are the transverse and longitudinal projectors, respectively, $k \equiv |\mathbf{k}|$, D_0 and D_0' are positive amplitude factors, and d is the dimensionality of the coordinate space. For $D_0' = 0$ (the incompressible case) the models (1) and (2) coincide. For $0 < \varepsilon < 2$, the so-called eddy diffusivity

$$S_{ij}(\mathbf{r}) \equiv K_{ij}(\mathbf{0}) - K_{ij}(\mathbf{r}) \tag{8}$$

has a finite limit for $m \to 0$:

$$S_{ij}(\mathbf{r}) = Dr^\varepsilon \left[(d+\varepsilon-1+\alpha)\delta_{ij} - \varepsilon(\alpha-1)\frac{r_i r_j}{r^2} \right], \tag{9}$$

with

$$D = \frac{-D_0\Gamma(-\varepsilon/2)}{(4\pi)^{d/2}2^\varepsilon(d+\varepsilon)\Gamma(d/2+\varepsilon/2)}, \quad \alpha \equiv D_0'/D_0, \tag{10}$$

where Γ is the Euler gamma function (note that D and α are both positive). In the renormalization-group (RG) approach, the exponent ε plays the same role as the parameter $\varepsilon = 4 - d$ does in the RG theory of critical behavior; see Refs. [8], [16]. The relation $D_0/\nu_0 \equiv \Lambda^\varepsilon$ defines the characteristic ultraviolet wave-number scale Λ.

The issue of interest is the behavior of various correlation functions in the inertial range specified by the inequalities $\Lambda r \gg 1$, $mr \ll 1$. In the models (1)–(9), odd multipoint correlation functions of the scalar field vanish, while the even equal-time functions satisfy linear partial differential equations; see, e.g., [1–7]. The equation for the equal-time pair correlation function $D(\mathbf{r}) \equiv \langle \theta(t,\mathbf{x})\theta(t,\mathbf{x}') \rangle$ is easily derived from the Schwinger-Dyson equations (see Refs. [16,25]) and has the form (here and below in equal-time functions, we omit time arguments common to all the quantities)

$$2\nu_0\partial^2 D(\mathbf{r}) + [S_{ij}(r)\partial_i\partial_j]D(\mathbf{r}) = C(\mathbf{r}) \tag{11}$$

for the model (2) and

$$2\nu_0\partial^2 D(\mathbf{r}) + \partial_i\partial_j[S_{ij}(r)D(\mathbf{r})] = C(\mathbf{r}) \tag{12}$$

for the model (1), with $C(\mathbf{r})$ from Eq. (4) and $S_{ij}(r)$ from Eq. (9). In the presence of the preferred direction $\mathbf{n}$, the correlation function can be decomposed into Legendre polynomials,

$$D(\mathbf{r}) = \sum_{l=0}^{\infty} D_l(r)P_l(z), \tag{13}$$

where the coefficient functions are sought in the powerlike form

$$D_l(r) \simeq D_l r^{\zeta_l}. \tag{14}$$

Owing to the evenness in $\mathbf{n}$, only even polynomials contribute to Eq. (13).

It is well known (see, e.g., Refs. [3–6]) that the nontrivial inertial-range exponents are determined by the zero modes, i.e., solutions of Eqs. (11) and (12) neglecting both the forcing [$C(\mathbf{r})=0$] and the dissipation ($\nu_0=0$). The homogeneous equations are SO(d) covariant, and the equations for the coefficient functions in Eq. (13) foliate. Substituting the representations (13),(14) into Eqs. (11),(12) and using the relations $\partial_i S_{ij}(r) = \alpha\varepsilon(d+\varepsilon)Dr_j r^{-2+\varepsilon}$, $\partial_i\partial_j S_{ij}(r) = \alpha\varepsilon(d+\varepsilon)(d-2+\varepsilon)Dr^{-2+\varepsilon}$, then gives quadratic equations for the exponents ζ_l in Eq. (14), namely,

$$\zeta_l(\zeta_l+d-2) - l(l+d-2) + \frac{\zeta_l(\zeta_l-1)\varepsilon(\alpha-1)}{(d-1+\alpha+\varepsilon)} = 0 \tag{15}$$

for the tracer, which has two solutions, $\zeta_l = 2 - d - l + O(\varepsilon)$ and

$$\zeta_l = l - \varepsilon \frac{l(l-1)(\alpha-1)}{(d+2l-2)(d-1+\alpha)} + O(\varepsilon^2), \qquad (16)$$

and

$$\zeta_l(\zeta_l + d - 2) - l(l + d - 2)$$
$$+ \varepsilon \frac{\zeta_l(\zeta_l - 1)(\alpha-1) + \alpha(d+\varepsilon)(2\zeta_l + d - 2 + \varepsilon)}{(d-1+\alpha+\varepsilon)} = 0 \qquad (17)$$

for the density, with two solutions, $\zeta_l = -d - l + O(\varepsilon)$ and

$$\zeta_l = l - \varepsilon \frac{l(l-1)(\alpha-1) + \alpha d(d+2l-2)}{(d+2l-2)(d-1+\alpha)} + O(\varepsilon^2). \qquad (18)$$

The standard arguments [3–6] show that only the second solution $\zeta_l = l + O(\varepsilon)$ is "admissible." It has the form

$$\zeta_l = [2(-1+d+\alpha+\alpha\varepsilon)]^{-1}$$
$$\times \{ -2 + 3d - d^2 + 2\alpha - d\alpha + \varepsilon - d\varepsilon + \alpha\varepsilon$$
$$+ [(2 - 3d + d^2 - 2\alpha + d\alpha - \varepsilon + d\varepsilon - \alpha\varepsilon)^2$$
$$- 4l(-2 + 3d - d^2 + l - dl + 2\alpha - d\alpha - l\alpha$$
$$+ 2\varepsilon - d\varepsilon - l\varepsilon)(-1 + d + \alpha + \alpha\varepsilon)]^{1/2} \} \qquad (19)$$

for the tracer and

$$\zeta_l = [2(-1+d+\alpha+\alpha\varepsilon)]^{-1} \{ -2 + 3d - d^2 + 2\alpha - d\alpha$$
$$+ \varepsilon - d\varepsilon + \alpha\varepsilon - 2d\alpha\varepsilon - 2\alpha\varepsilon^2 + [(2 - 3d + d^2 - 2\alpha$$
$$+ d\alpha - \varepsilon + d\varepsilon - \alpha\varepsilon + 2d\alpha\varepsilon + 2\alpha\varepsilon^2)^2$$
$$- 4(-1 + d + \alpha + \alpha\varepsilon)(-2l + 3dl - d^2l + l^2 - dl^2$$
$$+ 2l\alpha - dl\alpha - l^2\alpha + 2l\varepsilon - dl\varepsilon - l^2\varepsilon - 2d\alpha\varepsilon$$
$$+ d^2\alpha\varepsilon - 2\alpha\varepsilon^2 + 2d\alpha\varepsilon^2 + \alpha\varepsilon^3)]^{1/2} \} \qquad (20)$$

for the density. For $\alpha = 0$, these solutions coincide with each other and with the exponents obtained earlier in Refs. [3,9] for the incompressible case.

The exponents (19) and (20) exhibit a hierarchy related to the degree of anisotropy:

$$\zeta_l > \zeta_{l'} \quad \text{if} \quad l > l', \qquad (21)$$

i.e., the smaller the index l, the smaller the exponent and, consequently, the more important the contribution to the inertial-range behavior. The leading term is given by the exponent ζ_0 from the "isotropic shell." This behavior is illustrated by Figs. 1 and 2.

Although the hierarchy holds for all values of $\alpha(\partial\zeta_l/\partial l > 0)$, the corrections become closer to leading terms as α increases: $\partial^2\zeta_l/\partial l\partial\alpha < 0$. This behavior is illustrated by Fig. 3.

Since Eq. (2) is invariant with respect to the shift $\theta \to \theta + \text{const}$, the relevant quantities for the tracer are the so-called structure functions,

$$S_n(\mathbf{r}) = \langle [\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}')]^n \rangle, \quad \mathbf{r} = \mathbf{x} - \mathbf{x}', \qquad (22)$$

with the Legendre decomposition

$$S_n(\mathbf{r}) = \sum_{l=0}^{\infty} S_{nl} r^{\zeta_{nl}} P_l(z) \qquad (23)$$

with some numerical coefficients s_{nl}. Comparison with Eqs. (13) and (14) gives $\zeta_{2l} = \zeta_l$ with ζ_l from Eq. (19) for all $l > 0$. For $l = 0$, the constant term with $\zeta_0 = 0$ drops out from the difference in Eq. (22), and the behavior of the isotropic shell is determined by the subleading exponent $(2 - \varepsilon)$; see, e.g., Ref. [17]. Note that the hierarchy relations (21) remain valid also for $S_2 = [D(\mathbf{0}) - D(\mathbf{r})]/2$.

For the density case, the exponent

$$\zeta_0 = \frac{-\varepsilon(d+\varepsilon)\alpha}{(d-1)+\alpha(1+\varepsilon)} < 0$$

[the square root in Eq. (20) is taken explicitly] gives the leading contribution for both the pair correlation function $D(\mathbf{r})$ and the structure function S_2 in Eq. (22), in agreement with the exact solution of [16] (for $d = 1$, see Ref. [10]). Note that for this case the anomalous scaling emerges already for the pair correlation function, as in the model of a passively advected magnetic field studied in Ref. [30].

III. TWO-LOOP RENORMALIZATION-GROUP ANALYSIS OF STRUCTURE FUNCTIONS

The higher-order structure functions can be studied using the field theoretic renormalization group and operator product expansion. A detailed exposition of these techniques and practical calculations can be found in Refs. [8,16,17, 21,25]; below we confine ourselves to only the necessary information.

The field theoretic models corresponding to the stochastic equations (1) and (2) are multiplicatively renormalizable; the corresponding RG equations have infrared stable fixed points. In particular, this leads to the following representations for the structure functions in the model (2) in the inertial range ($\Lambda r \gg 1$, $mr \ll 1$):

$$S_n = D_0^{-n/2} r^{n(1-\varepsilon/2)} \sum_a C_a(r,z)(mr)^{\Delta_a}. \qquad (24)$$

Here $C_a(r,z)$ are coefficients analytical in m and finite for $m \to 0$, and Δ_a are the critical dimensions of the composite operators entering in the operator product expansion.

The leading zero-mode contribution in the lth shell for S_n is determined by the critical dimension Δ_{nl} of the irreducible
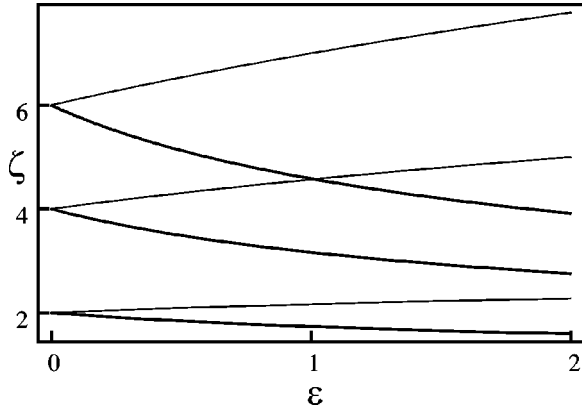
FIG. 1. Behavior of the exponents ζ_l ($l=2$, 4, and 6 from the bottom to the top) from Eq. (19) vs ε in three dimensions for $\alpha=0$ (thin lines) and $\alpha=\infty$ (thick lines).

FIG. 2. Behavior of the exponents ζ_l ($l=0$, 2, 4, and 6 from the bottom to the top) from Eq. (20) vs ε in three dimensions for $\alpha=0$ (thin lines) and $\alpha=\infty$ (thick lines).

traceless lth rank tensor operator built of n fields θ and minimal possible number of derivatives. For $l \leq n$ such an operator has the form

$$\partial_{i_1}\theta \cdots \partial_{i_l}\theta(\partial_i \theta \partial_i \theta)^p + \cdots, \quad n=l+2p. \quad (25)$$

Here the dots stand for the appropriate subtractions involving the Kronecker δ symbols, which ensure that the resulting expressions are traceless with respect to contraction of any given pair of indices, for example, $\partial_i\theta\partial_j\theta - \delta_{ij}\partial_k\theta\partial_k\theta/d$, $\partial_i\theta\partial_j\theta\partial_k\theta - (\delta_{ij}\partial_k\theta + \delta_{ik}\partial_j\theta + \delta_{jk}\partial_i\theta)/(d+2)$, and so on.

The exponents $\zeta_{nl} = n(1-\varepsilon/2)+\Delta_{nl}$ are calculated in the form of series in ε, where $\zeta_{nl} = n + \Sigma_{k=1}^{\infty}\zeta_{nl}^{(k)}\varepsilon^k$. In the first order in ε,
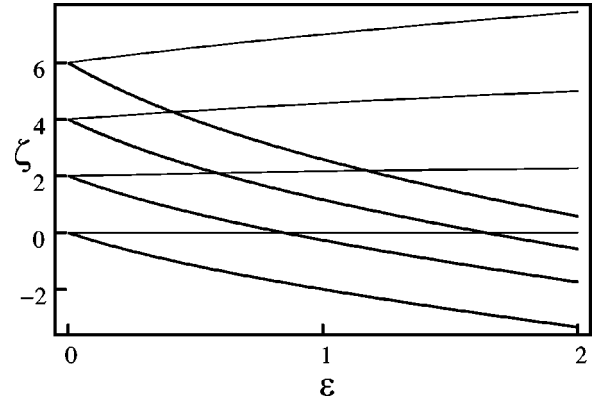
$$\zeta_{nl}^{(1)} = -\frac{1}{(d+2)}\left\{\frac{(n-l)(d+n+l)}{2} + \frac{l(l-1)(\alpha-1)+\alpha(n-1)(n+l-2)}{(d-1+\alpha)}\right\}. \quad (26)$$

For the incompressible model, the exponent from the isotropic shell ($l=0$) was obtained in Refs. [3,4] to the order $O(1/d)$ and in Refs. [5,6] to the order $O(\varepsilon)$; the results for $n=3$ are given in Ref. [7]. The general case can be found in Ref. [21]; see also [22,31]. For general $\alpha>0$, the exponent ζ_{n0} was found in Ref. [14] [see also [15]; in the notation of those papers, $\wp = \alpha/(d-1+\alpha)$]. The result for general n,l is given in Ref. [21] (for more details, see [17]).

We have performed the two-loop calculation of the exponents ζ_{nl} and obtained:

$$\zeta_{nl}^{(2)} = \frac{[-d(d+1)+d(d^2-2d-4)\alpha+(3d+4)\alpha^2](d\kappa_1-\kappa_2)}{d(d+2)^3(d-1+\alpha)^2} + \frac{(n-2)}{4d(d+2)^3(d+4)(d-1+\alpha)^2}$$

$$\times(8d[-(d^2+5d+10)-(d-2)(d+4)\alpha+(2d^2+7d+2)\alpha^2]\kappa_1$$

$$+4[(d+1)(3d^3+17d^2+20d-24)-(d+4)(d^3+7d^2-2d-4)\alpha+(d+1)(5d^2+8d-24)\alpha^2]\kappa_2$$

$$+3(d+2)h(d)\{4d[3+(d-4)\alpha-(d-1)\alpha^2]\kappa_1+[-3(d+1)(d^2+5d-4)$$

$$+2(d^3+10d^3-d-4)\alpha-3(2d^3-3d-4)\alpha^2]\kappa_2\}), \quad (27)$$

where we have written $\kappa_1 = n(n-1)$, $\kappa_2 = (n-l)(n+l-2+d)$, $h(d)=F(1,1,d/2+1;1/4)$, and $F(a,b;c;z)$ is the hypergeometric function; see, e.g., Ref. [32]. For integer space dimension d one has $h(1)=2\pi/(3\sqrt{3})$ and $h(2)=4\ln(4/3)$, and the others can be obtained from the recursion relation

$$3h(d)+\frac{d}{d+2}h(d+2)=4, \quad (28)$$

valid for all d.

For the incompressible case $\alpha=0$, Eq. (27) becomes

$$\zeta_{nl}^{(2)} = -\frac{(d+1)(d\kappa_1-\kappa_2)}{(d+2)^3(d-1)^2} + \frac{(n-2)}{4d(d+2)^3(d+4)(d-1)^2}$$

$$\times\{-8d(d^2+5d+10)\kappa_1$$

$$+4[(d+1)(3d^3+17d^2+20d-24)]\kappa_2$$

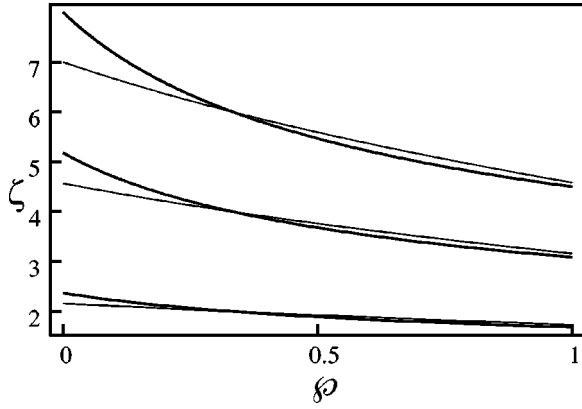$$+9(d+2)h(d)[4d\kappa_1-(d+1)(d^2+5d-4)\kappa_2]\}. \quad (29)$$

FIG. 3. Behavior of the exponents ζ_l ($l=2$, 4, and 6 from the bottom to the top) from Eq. (19) vs $\wp = \alpha/(d-1+\alpha)$ for $\varepsilon=1$, $d=3$ (thin lines), and $d=2$ (thick lines).

The results for $l=0$ and 2 were obtained earlier in Ref. [8]. In that paper, they were expressed in terms of the functions $h_1(d)=h(d+2)$ and $h_2(d)=h(d+4)$, which can be reduced to the form (29) using the relation (28).

For $d=1$, only the exponents ζ_{n0} and ζ_{n1} make sense (traceless operators of the rank $l\geq 2$ vanish identically). They are independent of α (the velocity is purely potential) and have the form ($l=0$ if n is even and $l=1$ if n is odd)

$$\zeta_n = n + n\frac{\varepsilon}{2} - \frac{n^2\varepsilon}{2} + \frac{n(n-1)(n-2)\varepsilon^2\pi}{6\sqrt{3}} + O(\varepsilon^3).$$
(30)

The expression (30) is in agreement with the $O(\varepsilon)$ result of Ref. [10] and the $O(\varepsilon^2)$ result of Ref. [16] (in the latter, the density case was studied, but in one dimension these models can be related by the replacement $\theta \to \partial\theta$).

For $l>n$, the leading operators contain more derivatives than fields, the corresponding exponents behave as $\zeta_{nl}=l-n+O(\varepsilon)$, and the corresponding terms in representation (23) rapidly decrease for $mr\to 0$.

Now let us return to the density model (1). It is not invariant with respect to the shift $\theta\to\theta+$const, the operators θ^n have nontrivial critical dimensions,

$$\Delta_n = n\left(-1+\frac{\varepsilon}{2}\right) - \frac{\alpha n(n-1)d\varepsilon}{2(d-1+\alpha)}$$
$$+ \frac{\alpha(\alpha-1)n(n-1)(d-1)\varepsilon^2}{2(d-1+\alpha)^2}$$
$$+ \frac{\alpha^2 n(n-1)(n-2)dh(d)\varepsilon^2}{4(d-1+\alpha)^2} + O(\varepsilon^3), \quad (31)$$

and different terms in the structure functions have different scalings (see Ref. [16]). The relevant quantities are then the equal-time pair correlation functions of the powers of θ, which have the form

$$\langle \theta^n(\mathbf{x})' \rangle \propto \nu_0^{-(n+p)/2}\Lambda^{-(n+p)}(\Lambda r)^{-\Delta_n - \Delta_p}F_{np}(\mathbf{r})(Mr)^{\Delta_{n+p}}.$$
(32)

The leading terms in the Legendre decompositions for the scaling functions,

$$F_{np}(\mathbf{r}) = \sum_{l=0}^{\infty} C_l^{(np)}(mr)P_l(z),$$
(33)

are given by the contribution from the scalar operators θ^{n+p} without derivatives, $C_0^{(np)} \propto (mr)^{\Delta_{n+p}}$.

The operators that determine corrections ($l>0$) to the leading term with $l=0$ necessarily contain l derivatives and, in contrast to the tracer case, the correction exponents differ from the leading ones by a few units, $l+O(\varepsilon)$. In particular, the leading $l=2$ correction is related to the operator

$$[\partial_i\theta\partial_j\theta - \delta_{ij}(\partial_k\theta\partial_k\theta)/d]\theta^{n+p-2},$$

whose dimension equals

$$2+(n+p)(-1+\varepsilon/2)-\frac{\varepsilon}{(d-1+\alpha)}$$
$$\times\left\{\frac{(n+p)(n+p-1)\alpha d}{2} + \frac{2(\alpha-1)}{(d+2)}\right\} + O(\varepsilon^2).$$

Note also that the exponents ζ_l in Eqs. (19) and (20) are related to the composite operators with two fields θ and l free indices, which (up to total derivatives and subtractions with the δ symbols) reduce to the form $\theta\partial_{i_1}\cdots\partial_{i_1}\theta$; the one-loop calculation confirms the $O(\varepsilon)$ results (16) and (18).

Since the leading terms of the even functions (23) are determined by the exponents of the isotropic shell (i.e., those related to scalar composite operators), the inertial-range behavior of the former is the same as in the isotropic model. This gives quantitative support both to Kolmogorov's hypothesis on the restored local isotropy of the inertial-range turbulence and to the universality of anomalous exponents with respect to the way in which the turbulence is excited.

On the contrary, the small-scale anisotropy reveals itself in odd correlation functions (which are nonzero in the presence of a mixed correlator $\langle \mathbf{v}f \rangle$ or the constant mean gradient of the scalar field). It follows from the above analysis that the dimensionless ratios $\mathcal{R}_n \equiv S_{2n+1}/S_2^{(2n+1)/2}$ for the tracer case in the inertial range have the form

$$\mathcal{R}_n \propto (mr)^{\Delta_{2n+1,1}},$$
(34)

where $\Delta_{2n+1,1}=\zeta_{2n+1,1}-(2n+1)(2-\varepsilon)$ is the critical dimension of the *vector* composite operator (25) built of $2n+1$ scalar gradients. From Eq. (26) we find, to the first order in ε,

$$\Delta_{2n+1,1}=\varepsilon[(d-1+\alpha)(d+2-4n^2)-8\alpha n^2]/2(d+2)$$
$$\times(d-1+\alpha).$$
(35)

For small α, the skewness factor $\mathcal{R}_1$ decreases for $mr\to 0$, but more slowly than expected on the basis of cascade ideas (the latter suggest that the odd ratios should vanish for small mr, where the turbulence is expected to become isotropic), while the higher-order ratios increase, thus signaling the

persistence of small-scale anisotropy. When α increases, $\mathcal{R}_1$ also becomes divergent for $mr \to 0$ provided α is large enough [namely, $\alpha > (d-1)(d+2)/(10-d) + O(\varepsilon)$], while the higher-order ratios diverge even faster.

It was argued in Refs. [13–15] that the anomalous scaling regime in the models at hand breaks down if ε and α are both large enough [$\wp \equiv \alpha/(d-1+\alpha) > d/\xi^2$] and an inverse energy cascade with no anomalous scaling takes place. This effect obviously cannot be detected within the ε expansion. It is noteworthy that the exact nonperturbative exponents (19) and (20) show no hint of anomaly at the threshold $\wp = d/\xi^2$, in contrast to the exact exponents for the magnetic case which become complex when the anomalous scaling regime breaks down; see Refs. [30,24].

IV. CONCLUSION

To conclude, we have studied the effects of compressibility and large-scale anisotropy on the anomalous scaling behavior in two models that describe passive advection of scalar tracer and density fields. The advecting velocity field is Gaussian and δ correlated in time, and its spatial correlations scale with a positive exponent ε. Explicit inertial-range expressions for the scalar correlation functions have been obtained; they are represented by superpositions of power laws with nonuniversal amplitudes and universal anomalous exponents (dependent only on ε and α, the compressibility parameter). The complete set of anomalous exponents for the pair correlation functions has been found nonperturbatively, in any space dimension d, using the zero-mode approach. For higher-order correlation functions, the anomalous exponents have been calculated to $O(\varepsilon^2)$ using RG techniques. As in the incompressible case, the exponents exhibit a hierarchy related to the degree of anisotropy; the leading contributions to the even correlation functions are given by the exponents from the isotropic shell, in agreement with the idea of restored small-scale isotropy.

This picture seems rather general, being compatible with that established recently for Navier-Stokes turbulence [28], passive scalar advection by the two-dimensional Navier-Stokes field [23], and passive advection of scalar [21] and vector [24,25] fields by a white-in-time incompressible synthetic velocity field.

As the degree of compressibility increases, the corrections become closer to the leading terms; cf. Ref. [26] for the passive advection of a magnetic field.

In contrast, the small-scale anisotropy reveals itself in the odd ratios of correlation functions: the skewness factor slowly decreases down to small scales for the incompressible case [7], but begins to increase if α is large enough. The higher-order odd dimensionless ratios (hyperskewness, etc.) increase, thus signaling the persistent small-scale anisotropy; cf. Refs. [17,21,23]. This effect becomes even more pronounced for larger values of α; cf. Ref. [26] for the magnetic case.

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