Mean first passage times of processes driven by white shot noise

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We consider mean first passage times in systems driven by white shot noise with exponentially distributed jump heights. Simple interpretable results are obtained and the linkage between those results and the steady-state probability density function of the process is presented. The virtual waiting-time or Takács process (constant losses) and the shot noise process with linear losses are analyzed in depth, along with a more complex process with useful implications for the modeling of the soil moisture dynamics in hydrology.

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I. INTRODUCTION

In recent years several papers have dealt with the derivation of exact expressions for the mean first passage times (MFPT's) of specific stochastic processes [1–8]. The particular case of systems driven by white shot noise has also received considerable attention [5–7], both because of its analytical tractability and because of the large number of its possible applications. The main emphasis of this paper will be on the derivation of some interpretable expressions for the MFPT's of stochastic processes driven by white shot noise, in the special case when the jump heights are exponentially distributed.

We will consider processes whose dynamical evolution is given by

$$\frac{ds}{dt} = -\rho(s) + F(t), \tag{1}$$

where $s=s(t)$ is the state variable, t is time, $\rho(s)$ is any function defining the deterministic losses of the process, and $F(t)$ is the random driving process, in the form of white shot noise or white Poisson noise. This latter is defined by a sequence of pulses at random times τ_i, each pulse having an independent random height h_i, i.e.,

$$F(t) = \sum_i h_i \delta(t-\tau_i), \tag{2}$$

where $\delta(\cdot)$ is the Dirac delta function. We assume that the random times $\{\tau_i\}$ form a Poisson sequence, i.e., that the probability distribution of the time intervals $\{t_i=\tau_i-\tau_{i-1};i=1,2,3,\ldots\}$ is $f(t)=\lambda e^{-\lambda t}$, where $1/\lambda$ is the mean interval between two subsequent pulses. Under this assumption the dynamic process (1) is Markovian with respect to s. The probability distribution of the random heights h_i (whose dimension is the same as s) is assumed to be exponential with mean value $(1/\gamma)$ $[f(h)=\gamma e^{-\gamma h}]$. Since we are considering positive jump heights, to guarantee stationarity $\rho(s)$ must also be positive. The extension of the results to the opposite case $[h_i<0$ and $\rho(s)<0]$ is straightforward.

In many practical applications, such as the description of soil moisture in hydrology [9], it is important to consider the case when the state variable s has an upper bound, i.e., $s \leqslant s_b$. The probability distribution of h_i in this case becomes state dependent and reads

$$f'(h,s) = H(s_b-h-s) \times \left[\gamma e^{-\gamma h} + \delta(s_b-h-s) \int_{s_b-s}^{\infty} du\, \gamma e^{-\gamma u} \right], \tag{3}$$

where $H(\cdot)$ is the Heaviside step function. In the following we will use this second formulation which also includes the unbounded case in the limit as $s_b \to \infty$.

Masoliver [5], in the more general framework of non-Markovian processes, obtained closed exact expressions for the MFPT's of dynamic systems driven by white shot noise for the special cases of exponentially distributed and constant jump heights. In this paper we derive exact expressions for the general process (1)–(3). Such expressions agree with those of Masoliver [5], but are more directly derived and written in a much simpler and usable form, thanks to the Markovian nature of the process. Moreover, the linkage between the first passage times and the steady-state probability density function (pdf) of the process is also formally established.

The paper is organized as follows. In Sec. II we detail the dynamics of the system. In Sec. III the derivation of the MFPT's is carried out. The results are then applied to three special forms of loss function in Sec. IV and the conclusions are drawn in Sec. V.

II. DYNAMICS OF THE SYSTEM

In the most general case the pdf of the state variable s for the process (1) can be written as

$$p(s,s_0,t) = p_c(s,s_0,t) + \delta(s-s_l) P(s,s_0,t), \tag{4}$$

where s_0 is the starting point of the trajectory, defined by the initial condition $p(s,s_0,t)=\delta(s-s_0)$. The continuous part of the pdf is $p_c(s,s_0,t)$ and $P(s,s_0,t)$ is the time-dependent cumulative density function of s. The atom of probability $\delta(s-s_l)P(s,s_0,t)$ is present in $s=s_l$ if $\rho(s_l) \neq \rho(s_l^-)=0$. The resulting forward differential Chapman-Kolmogorov equations are in this case (e.g., Refs. [10,11]),

$$\frac{\partial}{\partial t}p_c(s,s_0,t)=\frac{\partial}{\partial s}[p_c(s,s_0,t)\rho(s)]-\lambda p_c(s,s_0,t)+\lambda\int_{s_l}^{s}du\,p_c(u,s_0,t)f'(s-u;u)+\lambda P(s_l,s_0,t)f'(s,s_l) \quad (5)$$

for the continuous part of the pdf and

$$\frac{\partial}{\partial t}P(s_l,s_0,t)=-\lambda P(s_l,s_0,t)+\rho(s_l)p_c(s_l,s_0,t) \quad (6)$$

for the atom of probability in $s=s_l$.

Later in this paper we will focus on some analytical relationships between MFPT's and the steady-state pdf of the process. We first summarize the solution of Eqs. (5) and (6) under steady-state conditions. Taking the limit as $t\rightarrow\infty$ of Eqs. (5) and (6) and substituting Eq. (3) in Eq. (5), after some manipulations (see Refs. [11,12]) one obtains the equations valid for the steady-state pdf of s,

$$\frac{d}{ds}[\rho(s)p_c(s)]+\gamma\rho(s)p_c(s)-\lambda p_c(s)=0, \quad (7)$$

$$\lambda P(s_l)=\rho(s_l)p_c(s_l). \quad (8)$$

The general form of the solution for the continuous part of the steady-state pdf is given in [11] as

$$p_c(s)=\frac{C}{\rho(s)}e^{-\gamma s+\lambda\int[du/\rho(u)]}, \quad (9)$$

where C is a constant of integration that can be calculated imposing the condition $P(s_b)=1$ [in the unbounded case, $P(\infty)=1$]. Due to the Markovian nature of the process, the bounded and unbounded cases have the same solution [Eq. (9)], all the differences being embedded in the different values of the constant C (see Ref. [9]).

III. MEAN FIRST PASSAGE TIMES

From the forward equations (5) and (6) it is easy to obtain the corresponding backward or adjoint equations [13]. From the backward equation it is then straightforward to write the differential equation that describes the evolution of the probability density, $g_T(s_0,t)$, that a particle staring from s_0 inside an interval $\{\xi',\xi\}$ leaves for the first time the interval at a time t [13]. This is the usual procedure to obtain an equation for the MFPT statistics when the process is Markovian (e.g., [4,13,14]). For the process under consideration the resulting equation for $g_T(s_0,t)$ is thus

$$\frac{\partial g_T(s_0,t)}{\partial t}=-\rho(s_0)\frac{\partial g_T(s_0,t)}{\partial s_0}+\lambda\int_{s_0}^{\xi}f'(z-s_0,s_0)g_T(z,t)dz-\lambda g_T(s_0,t). \quad (10)$$

One does not need to solve the partial integro-differential equation (10) to obtain the first passage times statistics of the process. In fact, the moments of the probability distribution $g_T(s_0,t)$ are $T_n(s_0)=\int_0^\infty t^n g_T(s_0,t)dt$. Therefore, an expression involving the mean time for exiting the interval $\{\xi',\xi\}$, $T_1(s_0)$ (the subscript 1 is omitted from here on), is obtained from Eq. (10) as

$$-1=-\rho(s_0)\frac{dT(s_0)}{ds_0}+\lambda\int_{s_0}^{\xi}\gamma e^{-\gamma(z-s_0)}T(z)dz-\lambda T(s_0), \quad (11)$$

where the exponential part of the jump distribution, $f'(z-s_0,s_0)$, has been used in the integral on the right-hand side because, in the hypothesis that $\xi<s_b$, the presence of the bound at s_b becomes irrelevant. The integro-differential equation (11) was also obtained by Masoliver [Ref. [5], Eq. (A4)] in a different and more general way.

Differentiating (11) with respect to s_0 and reorganizing the terms, the following second-order differential equation is obtained:

$$\rho(s_0)\frac{d^2T(s_0)}{ds_0^2}+\left(\lambda+\frac{d\rho(s_0)}{ds_0}-\gamma\rho(s_0)\right)\frac{dT(s_0)}{ds_0}+\gamma=0. \quad (12)$$

Equation (12) needs two boundary conditions: the first is obtained from Eq. (11) evaluated at $s_0=\xi$,

$$\rho(\xi)\frac{dT(s_0)}{ds_0}\Big|_{s_0=\xi}=1-\lambda T(\xi). \quad (13)$$

For the second boundary condition one has to consider whether the lower limit ξ' is above or below s_l. In the first case, ξ' is a real absorbing barrier, so that the boundary condition is $T(\xi')=0$ [see Fig. 1(a)]. In contrast, when $\xi'<s_l$, ξ' cannot be reached by the trajectory [see Fig. 1(b)], and the average exiting time from the interval becomes the mean first passage time of the threshold ξ. We will use the notation $T_\xi(s_0)$ to emphasize that in this case the variable depends only on ξ and not on ξ'. In this case the second boundary condition is obtained by setting $s_0=s_l$ in Eq. (11), i.e.,

$$T_\xi(s_l)=\int_{s_l}^{\xi}\gamma e^{-\gamma(z-s_l)}T_\xi(z)dz+\frac{1}{\lambda}. \quad (14)$$

A. MFPT's of a threshold ξ above the initial point s_0

We consider first the case when $\xi'<s_l$ [Fig. 1(b)]. The solution of Eq. (12) with boundary conditions (13) and (14) is [Ref. [5], Eq. (5.34)]

$$T_\xi(s_0)=-\gamma\int_{s_0}\frac{1}{\rho(u)}e^{M(u)}\int_u e^{-M(z)}dz\,du+C_1(\xi)\int_{s_0}\frac{1}{\rho(u)}e^{M(u)}du+C_2(\xi), \quad (15)$$

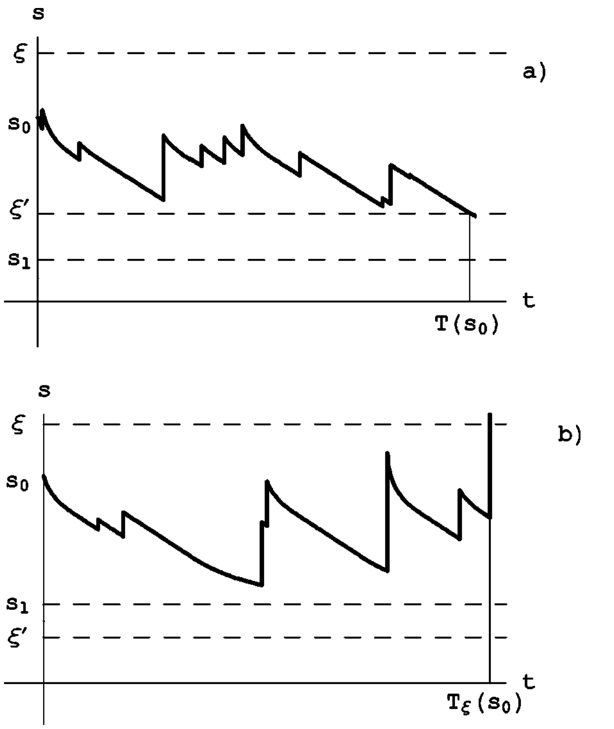

FIG. 1. (a) and (b) trajectories and first passage times when the thresholds ξ' and ξ are both greater than the fixed point s_l (a) and when $\xi' < s_l$ (b).

where $C_1(\xi)$ and $C_2(\xi)$ are integration constants and

$$M(u) = \gamma u - \lambda \int_u \frac{1}{\rho(z)} dz. \tag{16}$$

Equations (15) and (16) present some difficulties of application due to the involved form of the boundary conditions (13) and (14) which define $C_1(\xi)$ and $C_2(\xi)$. In the following we will show that $C_1(\xi)$ and $C_2(\xi)$ can be directly calculated when the jump heights are exponentially distributed. From Eqs. (15) and (16) one may easily determine the value of the integration constant $C_2(\xi)$, which allows us to rewrite Eq. (15) as

$$T_\xi(s_0) = \frac{1}{\lambda} - \frac{C_1(\xi)}{\lambda} e^{M(\xi)} + \frac{\gamma}{\lambda} e^{M(\xi)} \int_\xi e^{-M(x)} dx - C_1(\xi) \int_{s_0}^{\xi} \frac{e^{M(x)}}{\rho(x)} dx + \gamma \int_{s_0}^{\xi} \frac{e^{M(x)}}{\rho(x)} \int_x e^{-M(x')} dx dx'. \tag{17}$$

Equation (17) can now be inserted in Eq. (14). Integration by parts of some terms and reorganization of the terms lead to

$$C_1(\xi)\left[\int_{s_l}^{\xi} e^{-\gamma(u-s_l)} \frac{e^{M(u)}}{\rho(u)} du + e^{-\gamma(\xi-s_l)} \frac{e^{M(\xi)}}{\lambda}\right] = -\frac{1}{\lambda} + \frac{e^{-\gamma(\xi-s_l)}}{\lambda} + \gamma \int_{s_l}^{\xi} e^{-\gamma(u-s_l)} \frac{e^{M(u)}}{\rho(u)} \times \int_u e^{-M(z)} dz du + \frac{\gamma}{\lambda} e^{-\gamma(\xi-s_l)} e^{M(\xi)} \int_\xi e^{-M(u)} du. \tag{18}$$

By noticing from Eq. (16) that

$$\frac{e^{-\gamma u} e^{M(u)}}{\rho(u)} = -\frac{1}{\lambda} \frac{d(e^{-\gamma u} e^{M(u)})}{du}$$

one can proceed with the direct integration of the first term on the left-hand side (lhs) of Eq. (18) and with the integration by parts of the third term on the right-hand side (rhs) of the same equation. A further reorganization of terms leads to

$$C_1(\xi) = \gamma \int_{s_l} e^{-M(u)} du. \tag{19}$$

The value of C_1, which results to be independent of ξ, can be substituted in Eq. (17) yielding

$$T_\xi(s_0) = \frac{1}{\lambda} + \frac{\gamma}{\lambda} e^{M(\xi)} \int_{s_l}^{\xi} e^{-M(u)} du + \gamma \int_{s_0}^{\xi} \frac{e^{M(u)}}{\rho(u)} \int_{s_l}^{u} e^{-M(z)} dz du. \tag{20}$$

Equation (20) represents a first simplification of the result that was given in [5] as a combination of Eqs. (13)–(15). The linkage between the MFPT's and the steady-state pdf of the process allows further simplifications of Eq. (20). Consider Eqs. (9) and (16): one can write $p_c(s) = [C/\rho(s)] e^{-M(s)}$, so that Eq. (20) becomes

$$T_\xi(s_0) = \frac{1}{\lambda} + \frac{\gamma}{\lambda p_c(\xi) \rho(\xi)} \int_{s_l}^{\xi} p_c(u) \rho(u) du + \gamma \int_{s_0}^{\xi} \frac{1}{p_c(u) \rho^2(u)} \int_{s_l}^{u} p_c(z) \rho(z) dz du. \tag{21}$$

Equation (7) can now be used to simplify the above expression. In fact, integration of Eq. (7) and substitution in Eq. (21) yields, after using Eq. (8),

$$T_\xi(s_0) = \frac{P(\xi)}{p_c(\xi) \rho(\xi)} + \int_{s_0}^{\xi} \left[\frac{\lambda P(u)}{p_c(u) \rho^2(u)} - \frac{1}{\rho(u)}\right] du. \tag{22}$$

Note that when the starting point s_0 coincides with the threshold ξ, the integral on the rhs cancels out and the mean crossing time reads

$$T_{\xi}(\xi)=\frac{P(\xi)}{p_c(\xi)\rho(\xi)}. \tag{23}$$

As a consequence, under steady-state conditions the frequency of the upcrossing (or downcrossing) events of the threshold ξ can be obtained from Eq. (23) [15] as

$$\nu(\xi)=p_c(\xi)\rho(\xi). \tag{24}$$

Returning to the MFPT's, some manipulation of Eq. (22) leads to the synthetic expression

$$T_{\xi}(s_0)=T_{s_0}(s_0)+\gamma\int_{s_0}^{\xi}T_u(u)du, \tag{25}$$

which, along with Eq. (23), completely defines the MFPT from s_0 to ξ for $s_0<\xi$. Note that a similar relationship between MFPT's and steady-state pdf's was obtained by Balakrishnan *et al.* [16] for processes driven by Gaussian white noise.

Some important properties of the MFPT's become manifest from this formulation: both $T_{\xi}(\xi)$ and $T_{\xi}(s_0)$ in Eqs. (23) and (25) can be expressed as functions of the ratio $[P(u)/p_c(u)\rho(u)]$, where $P(u)$ is the steady-state cumulative density function calculated at a certain level u, $p(u)$ is the steady-state probability density function at the same level, and $\rho(u)$ is the loss function, again at the level u. We have pointed out before that all the changes induced in $p(s)$ from the presence of the bound at $s=s_b$ are embedded in the constant of normalization C. However, the constant C is present both in $P(u)$ and in $p_c(u)$, so that it cancels out from the expressions of the MFPT. The value of the latter is thus independent of the presence of the bound in $s=s_b$, and, for similar reasons, of the shape of the loss function above the threshold ξ. The opposite is true for the frequency $\nu(\xi)$ that contains the constant C through $p_c(\xi)$ and therefore also depends on the part of the dynamics above ξ.

B. MFPT's of a threshold ξ' below the initial point s_0

We consider now the MFPT's when $\xi'>s_l$ [see Fig. 1(a)] in the special case when $\xi\to\infty$ ($\xi>s_b$ in the bounded case). This variable, that we will call $T_{\xi'}(s_0)$, represents the average time that a particle starting from $s=s_0>\xi'$ takes to arrive to ξ'. Equation (12) needs now to be integrated with the boundary conditions given by Eq. (13) and $T_{\xi'}(\xi'^{+})=0$, where the plus subscript is used to put in evidence the discontinuity $T_{\xi'}(\xi')=P(\xi')/[\rho(\xi')p(\xi')]\neq T_{\xi'}(\xi'^{+})=0$. The procedure to calculate the resulting integration constants is analogous to that used before. The final result is (see also [7])

$$T_{\xi'}(s_0)=\gamma\int_{\xi'}^{s_0}\frac{e^{M(u)}}{\rho(u)}\int_u^{\infty}e^{-M(z)}dz\,du. \tag{26}$$

Splitting the integral on the rhs in the part below and above the bound s_b and considering again the relationship between $M(u)$ and the steady-state pdf, one obtains

$$\begin{aligned}T_{\xi'}(s_0)&=\int_{\xi'}^{s_0}\frac{1}{\rho^2(u)p_c(u)}[\lambda-\lambda P(u)+p_c(u)\rho(u)]du\\&=T_{s_0}(s_0)-T_{\xi'}(\xi')+\frac{1}{\nu(\xi')}-\frac{1}{\nu(s_0)}\\&\quad+\gamma\int_{\xi'}^{s_0}\left(\frac{1}{\nu(u)}-T_u(u)\right)du,\end{aligned} \tag{27}$$

where $T_{s_0}(s_0)$, $T_{\xi'}(\xi')$, and $T_u(u)$ are calculated from Eq. (23), $\nu(\xi')$, $\nu(s_0)$, and $\nu(u)$ from Eq. (24).

Differently from $T_{\xi}(s_0)$, $T_{\xi'}(s_0)$ depends on the presence of the bound at $s=s_b$ and on the shape of $\rho(s)$ for $s>s_0$. This is clear from the presence in Eq. (27) of $\nu(\xi')$, $\nu(s_0)$, and $\nu(u)$, which in turn contain the normalization constant C. In fact, the trajectory from s_0 to ξ' can take any value above ξ' with the presence of the upper bound decreasing the first passage time of ξ' for all those trajectories that would have taken values above s_b in unbounded conditions.

IV. APPLICATIONS

We will consider in the following three special cases of particular physical importance. The first is the well known virtual waiting time or Takács problem, with $\rho(s)=\beta$, $s\geqslant 0$; the second is the shot noise process with linear losses, e.g., $\rho(s)=\beta s$; in the third case a piecewise loss function and the bound in $s_b=1$ are considered. This latter choice is important to outline the procedure of analysis of the MFPT's when more complicated forms of $\rho(s)$ need to be used. The special case when s is the relative soil moisture content forced by a stochastic rainfall input [9,12,17–19] will be used as an example of an important application.

A. The virtual waiting-time process

The virtual waiting-time process is a very well studied one, since Takács [20] pointed out its importance in queuing and storage contexts. The loss function for this process is $\rho(s)=\beta$, $s\geqslant 0$, and s can be, for example, the total time it would take to serve all costumers in an office at time t (if $\beta=1$ we have a single server $M/M/1$ queue) or the time-dependent amount of water in a reservoir depleted at constant rate β. The steady-state probability density function for this process is [see Eq. (9)]

$$p_c(s)=\frac{C}{\beta}e^{-s[\gamma-(\lambda/\beta)]} \tag{28}$$

with an atom of probability in $s=0$, $P(0)=C/\lambda$. In the unbounded case the condition of stationarity of the process is $\gamma>\lambda/\beta$ and the constant of integration is $C=[\lambda(\gamma\beta-\lambda)/\gamma\beta]$, in the bounded case the process is always stationary and the constant of integration is

$$C=\frac{\lambda(\gamma\beta-\lambda)}{\gamma\beta-\lambda e^{-s_b[\gamma-(\lambda/\beta)]}}.$$

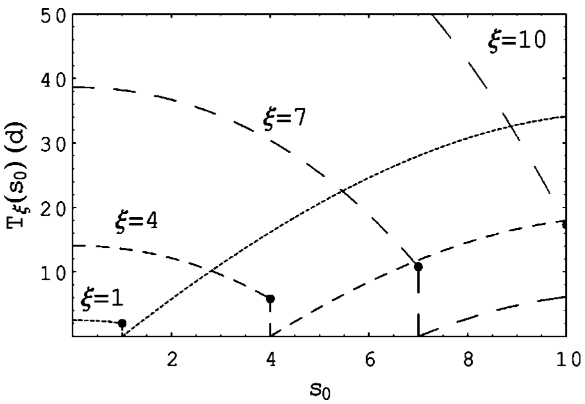

FIG. 2. MFPT's of ξ (ξ'), $T_\xi(s_0)$, as a function of the initial location s_0 for the Takács process. Four curves for different thresholds ξ are shown with dashed segments of different length. Black circles are placed where $s_0=\xi$ [Eq. (29)]; on the left of these circles we have $s_0<\xi$ and Eq. (31) is valid, on their right $s_0>\xi'$ and Eq. (32) is used. For all the equations the parameters values are $\lambda=1$ d^{-1}, $\gamma=1$, $\beta=1.1$ d^{-1}, and $s_b=10$.

The MFPT of the threshold ξ when the initial point is also $s_0=\xi$ is, from Eq. (23),

$$T_\xi(\xi)=\frac{\gamma\beta}{\lambda(\gamma\beta-\lambda)}e^{\xi[\gamma-(\lambda/\beta)]}-\frac{1}{\gamma\beta-\lambda}, \tag{29}$$

the frequency of upcrossings of ξ is [Eq. (24)]

$$\nu(\xi)=Ce^{-\xi[\gamma-(\lambda/\beta)]}, \tag{30}$$

and the MFPT of ξ when $s_0<\xi$ is, from Eq. (25),

$$T_\xi(s_0)=\frac{\gamma\beta}{\gamma\beta-\lambda}\left[\frac{\gamma\beta}{\lambda(\gamma\beta-\lambda)}e^{\xi[\gamma-(\lambda/\beta)]}-\frac{1}{\gamma\beta-\lambda}e^{s_0[\gamma-(\lambda/\beta)]}-\frac{1}{\beta}(\xi-s_0)-\frac{1}{\gamma\beta}\right]. \tag{31}$$

Finally, the MFPT of ξ', with $s_0>\xi'$, reads from Eq. (27)

$$T_{\xi'}(s_0)=\frac{C\gamma\beta-\lambda(\gamma\beta-\lambda)}{C(\gamma\beta-\lambda)^2}[e^{\xi'[\gamma-(\lambda/\beta)]}-e^{s_0[\gamma-(\lambda/\beta)]}]+\frac{\gamma}{\gamma\beta-\lambda}(s_0-\xi'), \tag{32}$$

that in the unbounded case assumes the simple form

$$T_{\xi'}(s_0)=\frac{\gamma}{\gamma\beta-\lambda}(s_0-\xi'),\quad \gamma>\frac{\lambda}{\beta}. \tag{33}$$

Equations (31) and (32) are plotted in Fig. 2 taking the initial condition s_0 as variable. Four curves are traced for different values of the threshold ξ (or ξ') and a black circle is placed on each curve where $s_0=\xi$ [the position of the circles is therefore also described by Eq. (29)]. On the left of these circles we have $s_0<\xi$ and Eq. (31) is valid, while on their right $s_0>\xi'$ and Eq. (32) is used. Equations (31) and (32) derive from the imposition of different boundary conditions for the differential equation (12); this leads to the already mentioned inequality $T_\xi(\xi)\neq T_{\xi'}(\xi'^{+})=0$ and to the discontinuity of each curve at $s_0=\xi$ (ξ'). Also note that, due to the presence of an upper bound at $s_b=10$, the parts of the curves on the rhs are bent downward with respect to the linear expression given by Eq. (32): in fact, as pointed out before, the values of $T_{\xi'}(s_0)$ decrease as a consequence of the restriction imposed to the trajectories by the presence of the bound. This effect is more evident when the starting point s_0 is closer to the bound.

A final comment regards the parameters λ, γ, and β: the unbounded Takács problem is stationary only when $\gamma>\lambda/\beta$ and, also in the bounded case, the sign of $\gamma-\lambda/\beta$ is very important in determining the form of Eqs. (31) and (32). In Fig. 2 we used $\lambda=1$, $\gamma=1$, and $\beta=1.1$ [the choice of the dimension of λ (e.g., d^{-1} or s^{-1}) depends on the process under consideration and determines the dimension of $T_\xi(s_0)$]. If we had taken $\beta<1$ the values of $T_\xi(s_0)$ would have dramatically decreased, while those of $T_{\xi'}(s_0)$ would have increased. In the special case when $\gamma=\lambda/\beta$ the steady-state probability distribution becomes uniform, $p_c(s)=[\lambda/(\beta+\lambda s_b)]$. The MFPT's in this case are obtained either taking the limit as the numerator and denominator in Eqs. (29), (31), and (32) tend to zero or by directly applying the relations (23), (25), and (27) with the above uniform distribution. One obtains

$$T_\xi(\xi)=\frac{1}{\lambda}+\frac{\xi}{\beta}, \tag{34}$$

$$\nu(\xi)=C=\frac{\lambda\beta}{\beta+\lambda s_b}, \tag{35}$$

$$T_\xi(s_0)=\frac{1}{\lambda}+\frac{\gamma\xi}{\lambda}+\frac{\gamma}{2\beta}(\xi^2-s_0^2), \tag{36}$$

$$T_{\xi'}(s_0)=\frac{1+\gamma s_b}{\beta}(s_0-\xi')-\frac{\gamma}{2\beta}(s_0^2-\xi'^2). \tag{37}$$

In unbounded conditions ($s_b\to\infty$), $T_\xi(\xi)$ and $T_\xi(s_0)$ remain unchanged, the frequency $\nu(\xi)$ tends to zero, and $T_{\xi'}(s_0)$ tends to infinity, due to the nonstationarity of the process in this case.

B. The shot noise process with linear losses

The second application involves a linear loss function of the form $\rho(s)=\beta s$. The decreasing trajectories of s are thus exponential, and the process corresponds to a particular form of shot noise in which the "shots" are exponentially decaying pulses of random heights (e.g., [11]). The steady-state pdf is in this case

$$p(s)=p_c(s)=\frac{C}{\beta}s^{(\lambda/\beta)-1}e^{-\gamma s}, \tag{38}$$

where C is the normalization constant of the pdf,

$$C=\frac{\beta\gamma^{\lambda/\beta}}{\Gamma'\left[\frac{\lambda}{\beta},\gamma s_b\right]},$$

with ($\Gamma'[\cdot,\cdot]=\Gamma[\cdot]-\Gamma[\cdot,\cdot]$, where $\Gamma[\cdot]$ is the Gamma function and $\Gamma[\cdot,\cdot]$ the incomplete Gamma function). When there is no upper bound, $p(s)$ is called a gamma distribution with mean value $\overline{s}=\lambda/\beta\gamma$.

The MFPT of the threshold ξ (ξ') with starting point s_0 can be calculated, using the expression for the steady-state pdf equation (38), from Eqs. (23), (25), and (27). For $s_0=\xi$ one obtains

$$T_\xi(\xi)=\frac{1}{\beta}(\gamma\xi)^{-\lambda/\beta}e^{\gamma\xi}\Gamma'\left[\frac{\lambda}{\beta},\gamma\xi\right]=\frac{1}{\lambda}{}_1F_1\left[1,1+\frac{\lambda}{\beta},\gamma\xi\right], \tag{39}$$

where ${}_1F_1[\cdot,\cdot,\cdot]$ is the confluent hypergeometric function or Kummer function [21]. When $s_0<\xi$ the MFPT of ξ reads

$$T_\xi(s_0)=\frac{1}{\lambda}{}_1F_1\left[1,1+\frac{\lambda}{\beta},\gamma s_0\right]+\frac{\gamma\xi}{\lambda}{}_2F_2\left[1,1;2,1+\frac{\lambda}{\beta};\gamma\xi\right]-\frac{\gamma s_0}{\lambda}{}_2F_2\left[1,1;2,1+\frac{\lambda}{\beta};\gamma s_0\right], \tag{40}$$

where ${}_2F_2[\cdot,\cdot;\cdot,\cdot;\cdot]$ is the generalized hypergeometric function [22]. Finally, the MFPT from s_0 to ξ' is, when $s_0>\xi'$,

$$\begin{aligned}T_{\xi'}(s_0)=&\frac{1}{\beta}(\gamma s_0)^{-\lambda/\beta}e^{\gamma s_0}\left(\Gamma'\left[\frac{\lambda}{\beta},\gamma s_0\right]-\Gamma'\left[\frac{\lambda}{\beta},\gamma s_b\right]\right)\\&-\frac{1}{\beta}(\gamma\xi')^{-\lambda/\beta}e^{\gamma\xi'}\left(\Gamma'\left[\frac{\lambda}{\beta},\gamma\xi'\right]-\Gamma'\left[\frac{\lambda}{\beta},\gamma s_b\right]\right)\\&+\frac{\gamma\xi'}{\lambda}{}_2F_2\left[1,1;2,1+\frac{\lambda}{\beta};\gamma\xi'\right]-\frac{\gamma s_0}{\lambda}{}_2F_2\left[1,1;2,1+\frac{\lambda}{\beta};\gamma s_0\right]+\frac{1}{\beta}(-1)^{\lambda/\beta}\left(\Gamma\left[1-\frac{\lambda}{\beta},-\gamma s_0\right]\right.\\&\left.-\Gamma\left[1-\frac{\lambda}{\beta},-\gamma\xi'\right]\right)\Gamma'\left[\frac{\lambda}{\beta},\gamma s_b\right].\end{aligned} \tag{41}$$

In Fig. 3, $T_\xi(\xi)$ from Eq. (39) and the frequency of crossings, $\nu(\xi)$, are plotted as a function of ξ. The values of λ and γ are kept constant and equal to 1, while β varies from 0.4 to 1.6. Common features for all the curves are the increase of $T_\xi(\xi)$ with ξ and the presence of a maximum of the crossing frequency $\nu(\xi)$. The value ξ_{max} for which $\nu(\xi)$ has a maximum is usually very close to the mean value $\overline{s}$ of the steady-state distribution, because both represent levels of s around which the trajectory preferably evolve. However, only in the unbounded case the two values coincide: in fact, one can set $\rho(s)p_c(s)=\nu(s)$ in Eq. (7), obtaining the equation $\gamma\rho(\xi_{max})-\lambda=0$ for the abscissa of the maximum crossing frequency. When the loss function is linear one obtains $\xi_{max}=\lambda/\gamma\beta$. Such value is in general different from the mean steady-state value which from Eq. (38) results to be

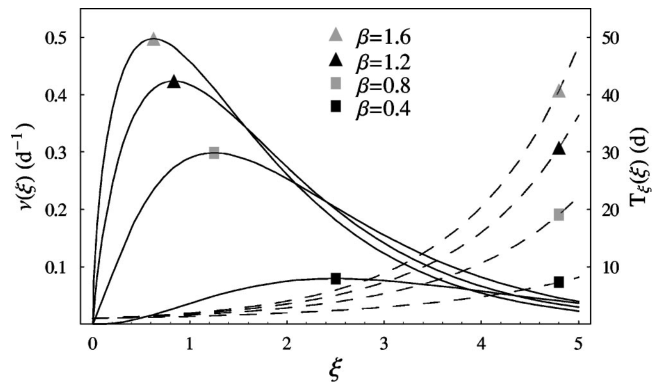

FIG. 3. Mean duration of an excursion below ξ, $T_\xi(\xi)$ (dashed lines), and frequency of upcrossing of ξ, $\nu(\xi)$ (continuous lines), as a function of the threshold value ξ for the unbounded shot noise process with linear losses. The four curves have different values of β; $\lambda=1$ d^{-1} and $\gamma=1$ are kept fixed.

$$\overline{s}=\frac{\lambda}{\gamma\beta}-\frac{(\gamma s_b)^{\lambda/\beta}e^{-\gamma s_b}}{\gamma\Gamma'\left[\frac{\lambda}{\beta},\gamma s_b\right]}$$

and converges to ξ_{max} only in the unbounded case.

C. The hydrologic soil moisture process

Our third example deals with a model with a more complex form of the loss function (see Fig. 4). The special case considered is important to analyze the linkage between climate, soil, and vegetation through the soil moisture dynamics, which represents a problem of fundamental hydrologic interest. This section is presented to show an example of how the previous analytical expressions are applied to more complex dynamics, with results whose interpretation becomes very important for the global understanding of the process.

When the lateral contributions can be neglected, the soil moisture balance at a point is expressed as [9]

$$nZ_r\frac{ds}{dt}=I[s,t]-E[s]-L[s], \tag{42}$$

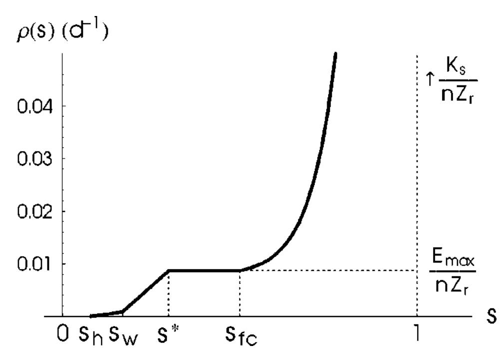

FIG. 4. The loss function $\rho(s)$ for the soil moisture process.

where n is the soil porosity, Z_r is the depth of active soil or root depth, and s is the relative soil moisture content ($0 \leqslant s \leqslant 1$). Infiltration from rainfall, $I[s,t]$, is the stochastic component of the balance and represents the part of rainfall that actually reaches the soil column. $E[s]$ and $L[s]$ are the rates of evapotranspiration and leakage, respectively.

We idealize, at the daily time scale, the occurrence of rainfall as a series of point events in continuous time, arising in a Poisson process of rate λ and each carrying a random amount of rainfall extracted from an exponential distribution [9]. Under this assumption, Eq. (42) is the same as Eq. (1) with $\rho(s) = [E(s)+L(s)]/nZ_r$ and a random driving process represented by Eq. (2). The mean interval between two rainfall events is 1/λ. The value of λ is corrected with the expression $\lambda' = \lambda e^{-\Delta/\alpha}$ to take into account canopy interception (see [9,12] for details), where α is the mean amount of rainfall falling during a precipitation event, and Δ is the maximum depth of rainfall intercepted by the vegetation canopy during a single rain event. Finally, the use of Eq. (3) with $s_b = 1$ and $\gamma = nZ_r/\alpha$ for the jump heights distribution allows to consider the normalization between 0 and 1 of soil moisture and the occurrence of runoff events [9,12].

The piecewise loss function $\rho(s)$ deriving from the evapotranspiration and leakage losses is shown in Fig. 4. There are no losses up to the hygroscopic point s_h (therefore $s_l = s_h$) and linearly increasing evaporation is present from the hygroscopic to the wilting point s_w, which is the soil moisture level below which plants begin to wilt. Evapotranspiration takes place at a linearly increasing rate from the wilting point to s^*, the point that marks the complete stomatal opening, while from s^* on evapotranspiration is at a maximum value E_{max}. From field capacity s_{fc} to soil saturation ($s=1$), the leakage becomes dominant and the losses increase exponentially up to the saturated hydraulic conductivity K_s. A detailed explanation of the rationale behind this form of $\rho(s)$ can be found in [12].

The soil moisture value s^* below which plants begin closing their stomata can be taken as a threshold for the occurrence of vegetation water stress [17,18]. The MFPT of s^*, $T_{s*}(s^*)$, becomes therefore very important for the analysis of plant condition in water-controlled ecosystems. Equation (23) can be used to derive the expression for $T_{s*}(s^*)$ regardless of the piecewise form of the loss function: all the complications arising from this particular form of $\rho(s)$ are in fact embedded in the values of $p(s^*)$ and $P(s^*)$, whose analytical expressions can be found in [12]. In Fig. 5, $T_{s*}(s^*)$ is studied as a function of the frequency λ of the rainfall events and of the mean rainfall depth α, in such a way that the product $\alpha\lambda$ remains constant. This is to compare environments with the same total rainfall $\alpha\lambda T_{seas}$ during a growing season lasting T_{seas}, but with differences in the timing and average amount of the precipitation events. Independently of the differences in the maximum transpiration rates E_{max}, plants experience longer periods of stress either where the rainfall events are very rare but intense or where the events are very frequent and light. From a physical viewpoint, this is due to the relevant losses of transpirable

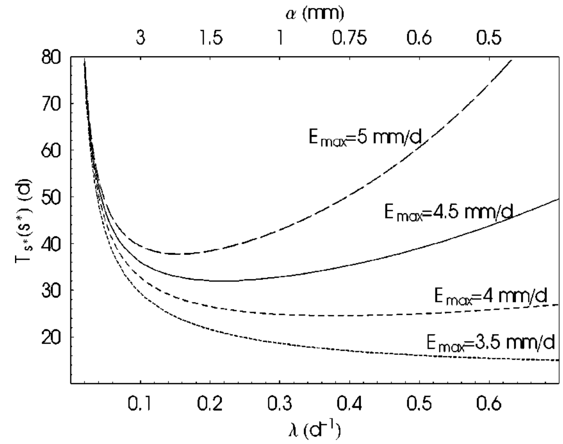

FIG. 5. Mean duration of a plant water stress period, $T_{s*}(s^*)$, as a function of the frequency of the rainfall events λ when the total rainfall during the growing season is kept fixed at 650 mm. The maximum evapotranspiration rate E_{max} is varied between 3.5 and 5 mm/d. The root depth is $Z_r = 60$ cm, the soil is a loam, and $s^* = 0.57$.

water either in leakage (or runoff) or in canopy interception according to the situation, pointing out possible optimal conditions for vegetation.

Figure 6 shows two other important variables for the analysis of plant water status: the mean duration of an excursion from s_w to s^*, which is important for the analysis of the recover of a plant from a period of intense stress [18], and the MFPT of s^* from s_{fc}, $T_{s*}(s_{fc})$, which is useful to analyze the duration of the periods without water stress at the beginning of the growing season in places with a wet winter season [19]. $T_{s*}(s_w)$ is obtained from Eq. (25), while $T_{s*}(s_{fc})$ is calculated from Eq. (27), again without further difficulties for the piecewise form of $\rho(s)$. The variations of $T_{s*}(s_w)$, $T_{s*}(s_{fc})$, and $T_{s*}(s^*)$ with respect to the root depth Z_r are shown in Fig. 6. Since the height of the active soil nZ_r is the ''capacity'' of the system (42), the trajectories of soil moisture become more ''regular'' when deeper soils are considered, so that both $T_{s*}(s_w)$ and $T_{s*}(s_{fc})$ rapidly

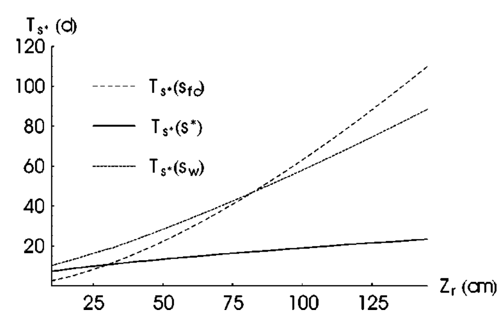

FIG. 6. The effect of variations of the plant root depth on the mean duration of a water stress period $T_{s*}(s^*)$, on the mean duration of a period without water stress at the beginning of the growing season $T_{s*}(s_{fc})$, and on the mean time plants need to recover after a period of intense stress, $T_{s*}(s_w)$. The mean rainfall frequency is $\lambda = 0.2$ d^{-1}, the mean rainfall depth is $\alpha = 2$ cm. The maximum evapotranspiration rate is $E_{max} = 4.5$ mm/d, $s^* = 0.57$, $s_w = 0.24$ and the soil is a loam with field capacity $s_{fc} = 0.7$.

increase with Z_r. A higher value of $T_{s*}(s_{fc})$ is favorable for plants because it implies a longer unstressed period at the beginning of the growing season, while a higher value of $T_{s*}(s_w)$ is problematic for plants which then need a very long time to recover after a period of intense stress. These features lead to important differences in the water use patterns of deep and shallow rooted plants, with advantages and drawbacks in different situations that affect the favorableness of a given environment to different vegetal species.

V. CONCLUSIONS

The mean first passage times of processes driven by white shot noise have been studied in detail for the case of exponentially distributed forcing jumps. The main results of the present work are (i) the simplification of some general expressions for the MFPT's found in the literature [5] [Eqs. (23), (25), and (27)]; (ii) the expression of the linkage of the MFPT's to the steady-state pdf of the process; (iii) the extension of the analysis to processes with piecewise loss function or with an upper bound, which are very important for many geophysical applications like those involving soil moisture dynamics, as explained in Sec. IV; (iv) fully explicit expressions for the MFPT's of the Takács process and the shot noise with linear losses, some of which [Eqs. (31), (32), (40), and (41)] seem not to have been reported before. Special attention has been given to the above-mentioned linkage between the MFPT's and the steady-state pdf of the process, for its importance in the physical interpretation of otherwise more complicated equations.

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