Stimulated transitions between the self-trapped states of the nonlinear Schrödinger equation

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We investigate a particle confined within a double-well potential, its behavior described by a one-dimensional nonlinear Schrödinger equation. Transitions between the two lowest self-trapped states of this system are studied, in the two-mode approximation, under the influence of the external time-dependent perturbation. If the perturbation is harmonic in time, with the frequency ω, then transitions between the states become possible if the amplitude of the perturbation F exceeds some threshold value $F_c(\omega)$; above this threshold motion of the system becomes chaotic. If the perturbation is broadband noise, then transitions between the states are possible at arbitrarily small F and occur in the process of the system's energy diffusion.

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I. INTRODUCTION

The nonlinear Schrödinger equation (NLS),

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta \psi + U(\vec{r})\psi + \lambda |\psi|^2 \psi, \tag{1}$$

serves as a tool for the account of many physical phenomena. The stationary version of this equation was introduced by Deigen and Pekar to describe the self-trapped (autolocalized) states of electrons in deformable crystal lattice ([1], see also [2]). Gross [3] and Pitayevskii [4] derived Eq. (1) as the mean-field approximation for the macroscopic wavefunction $\psi(\vec{r},t)$ of the Bose-Einstein condensate of a nonideal Bose gas at vanishing temperature (see also [5]). The third avatar of the NLS came in the realm of nonlinear optics, where $\psi(\vec{r},t)$ represents the envelope of a quasimonochromatic electromagnetic wave. The exact solution of the one-dimensional homogeneous $(U(\vec{r})=0)$ NLS found by Zakharov and Shabat [6] formed the modern paradigm of soliton theory [7].

Although the solutions of the NLS were extensively studied, it seems that comparatively little is known about their properties in the case of time-dependent potentials. In this paper we address a specific problem of this class. We shall consider the one-dimensional Eq. (1) with a potential that consists of two parts: a permanent potential $U(x)$ that has the form of symmetric double well, and some time-dependent potential $V(x,t)$ that will be called the perturbation. In the absence of a perturbation the properties of the stationary solutions of Eq. (1), that have the form

$$\psi(x,t) = \Phi(x)\exp\left(-i\frac{E}{\hbar}t\right), \tag{2}$$

depend essentially on the nonlinear coefficient λ. At small λ there is an infinite set of modes (2) that have symmetric wave functions—odd or even, $\Phi(x) = \pm\Phi(-x)$. As λ increases above some threshold value λ_c, a pair of stationary solutions (2) with broken symmetry $\Phi_s(x) \neq \pm\Phi_s(-x)$ appear. These solutions describe the states of the particle that are self-trapped in one of the wells of the permanent potential. For $\lambda < 0$ the corresponding energy E_s is lower than that of any symmetric mode [8].

Our main concern will be the following question: if the initial state of the system is one of these self-trapped states, then how will the system evolve under the influence of the nonstationary perturbation? In particular, can the perturbation transfer the system completely into the opposite self-trapped state?

There is a favorable circumstance that allows us to simplify the problem. It happens that at moderate $\lambda > \lambda_c$ even the self-trapped modes of high asymmetry can be accurately described by linear combination of the two lowest symmetric modes, the even $\Phi_0(x)$ and the odd $\Phi_1(x)$ [9]. Therefore in studying the problem we can restrict ourselves by analysis of the evolution of the two-level system. In Sec. II we derive the basic equations for this model. In Sec. III we study the influence of the external perturbation that harmonically depends on time. The evolution of the system under the influence of broadband noise is studied in Sec. IV. Section V contains the concluding discussion.

II. THE BASIC EQUATIONS

For future use we introduce the following quantities related to the (supposedly real) eigenfunctions $\Phi_0(x)$ and $\Phi_1(x)$:

$$J_{00} = \int_{-\infty}^{\infty} \Phi_0^4\, dx, \quad J_{01} = \int_{-\infty}^{\infty} \Phi_0^2 \Phi_1^2\, dx,$$
$$J_{11} = \int_{-\infty}^{\infty} \Phi_1^4\, dx. \tag{3}$$

Let us represent the wave function of the system by the superposition of the two lowest symmetric modes,

$$\psi(x,t) = b_0(t)\Phi_0(x)e^{-i\beta_0 t} + b_1(t)\Phi_1(x)e^{-i\beta_1 t}, \tag{4}$$

where $\beta_i = \hbar^{-1}(E_i + \lambda J_{ii})$. By substitution of Eq. (4) in Eq. (1), consequent multiplication by $\Phi_i(x)$, and integration

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over the coordinate x we get the following system of two equations for the complex amplitudes b_i:

$$i\hbar \frac{db_0}{dt} = -\lambda b_0|b_0|^2 J_{00} - 2\lambda b_0|b_1|^2 J_{01} - \lambda b_0^* b_1^2 J_{01} e^{i2(\beta_0-\beta_1)t} + b_0 V_{00}(t) + b_1 V_{01}(t) e^{i(\beta_0-\beta_1)t}, \tag{5}$$

$$i\hbar \frac{db_1}{dt} = -\lambda b_1|b_1|^2 J_{11} - 2\lambda b_1|b_0|^2 J_{01} - \lambda b_0^2 b_1^* J_{01} e^{-i2(\beta_0-\beta_1)t} + b_1 V_{11}(t) + b_0 V_{01}(t) e^{-i(\beta_0-\beta_1)t},$$

where the matrix elements of the perturbation are given by the integrals

$$V_{ij}(t) = \int_{-\infty}^{\infty} \Phi_i(x)\hat{V}(x,t)\Phi_j(x)dx. \tag{6}$$

The system (5) conserves the norm of the state $\psi(x,t)$ (the sum of probabilities $|b_0|^2+|b_1|^2=1$), and the common phase of the wave function (4) is physically irrelevant. Therefore we can describe the evolution of the system by just two real variables. The complex amplitudes could be cast in the form $b_i=\sqrt{n_i}\exp(-i\vartheta_i)$, where n_i and ϑ_i are real time-dependent variables. Let us introduce the population difference $\Delta=n_0-n_1$ and the phase $\Theta=2(\vartheta_0-\vartheta_1)+2(\beta_0-\beta_1)t$. Then the system (5) turns into equations

$$\dot{\Delta} = -B(1-\Delta^2)\sin\Theta + F(t)\frac{\sqrt{1-\Delta^2}}{2}\sin\frac{\Theta}{2}, \tag{7}$$

$$\dot{\Theta} = -\Omega + 2A\Delta + 2B\Delta\cos\Theta - F(t)\frac{\Delta}{\sqrt{1-\Delta^2}}\cos\frac{\Theta}{2} + G(t),$$

where the following notations have been introduced: $\Omega=2(\beta_1-\beta_0)+\lambda\hbar^{-1}(J_{11}-J_{00})$, $A=\lambda\hbar^{-1}(4J_{01}-J_{00}-J_{11})/2$, $B=\lambda\hbar^{-1}J_{01}$, $F(t)=4\hbar^{-1}V_{01}(t)$ and $G(t)=2\hbar^{-1}[V_{00}(t)-V_{11}(t)]$. All these quantities have the same dimensionality, namely that of the frequency. By fixing the unit of the frequency through linking it to one of these parameters (e.g., Ω), in the following we consider them as dimensionless quantities. If the nonlinearity parameter vanishes, $\lambda=0$, then Eqs. (7) become equivalent to the well-known Bloch equations [10].

The nonlinear Bloch equations (7) can be considered as a pair of canonical equations for the conjugated variables Δ,Θ of the nonautonomous system with one degree of freedom with the Hamiltonian function $H=H_0+H_1(t)$, where the unperturbed Hamiltonian is

$$H_0 = \Omega\Delta + A\Delta^2 - B(1-\Delta^2)\cos\Theta, \tag{8}$$

and the perturbation is

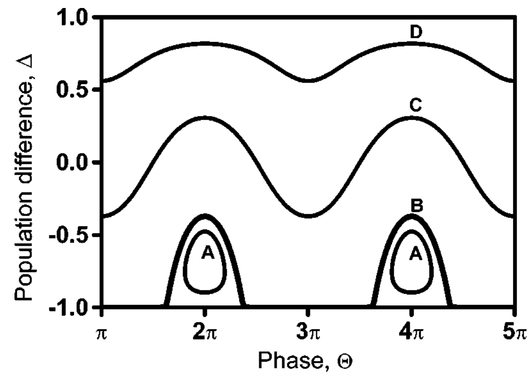

FIG. 1. Phase trajectories of the system (7) in the absence of perturbations on the plane $\Delta-\Theta$ for different values of energy: A-$E=-3.7$, B-$E=E_s=-3.487$, C-$E=0$, D-$E=5$.

$$H_1(t) = G(t)\Delta - F(t)\sqrt{1-\Delta^2}\cos\frac{\Theta}{2}. \tag{9}$$

In what follows we refer to the value of the function H_0 as the (dimensionless) energy E of the system. In the absence of perturbation the system (7) has two trivial stationary solutions, $\Delta=\pm 1$ and Θ arbitrary, that correspond to the symmetric eigenstates Φ_0 and Φ_1 respectively, and two nontrivial fixed points $\Delta_0=-\Omega/2(A+B)$, $\Theta=0$, or $\Theta=2\pi$, that correspond to the pair of self-trapped states that we call the stationary states. These states are divided from the bulk of the phase space by a separatrix (see Fig. 1). The frequency Ω_0 of small oscillations of Δ and Θ around the stationary values is determined by the expression $\Omega_0^2=B[4(A+B)^2-\Omega^2]/2(A+B)$.

For future numerical calculations we need to specify the parameters of the unperturbed Hamiltonian (8). We have chosen the following set of values: $\Omega=5.388$, $A=1.902$, and $B=2.022$. With this choice the stationary states which are located at $\Delta_0=-0.686$ correspond to the minimal energy of the system $E_-=-3.871$, the separatrix coincides with the isoenergetic line $E=E_s=-3.486$, and the maximal energy $E_+=7.290$ corresponds to the line $\Delta=1$.

III. THE HARMONIC PERTURBATION

We shall assume that the even (diagonal) perturbation is absent, $G(t)=0$, and the odd (nondiagonal) has the form

$$F(t) = F\sin(\omega t + \phi). \tag{10}$$

Numerical simulations show, that for given values of the frequency ω and the initial phase ϕ of the perturbation there is a threshold value $F_c(\omega,\phi)$ of its amplitude such that for $F<F_c$ the phase trajectory of the system remains indefinitely within one loop of the separatrix, while for $F>F_c$ the phase trajectory crosses the separatrix many times and may come close to the opposite stable point. The dependence of $F_c(\omega,\phi)$ on the initial phase is weak: the relative variations of the thresholds due to the change of the initial phase have

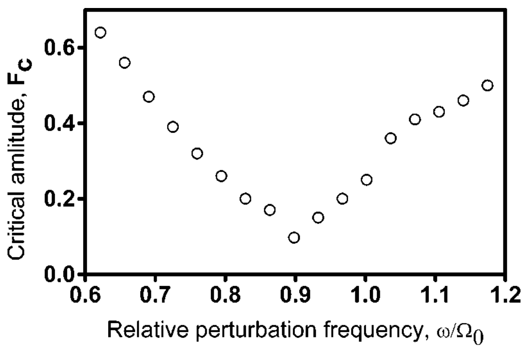

FIG. 2. Dependence of threshold amplitudes F_c of the nondiagonal harmonic perturbation (10) on the relative frequency ω/Ω_0 for the nonlinear Bloch equations (7) with the initial phase $\phi=0$.

the order of a few percent. Hence for the time being we ignore this dependence and shall consider only the dependence $F_c(\omega)$ (see Fig. 2).

The abrupt change in the character of motion at a certain threshold value of the perturbation magnitude strongly indicates the onset of global stochasticity that comes from the overlap of resonances [11,12] and the destruction of the noble tori [12,13]. This is indeed the case: by taking some phase point at the separatrix for the initial conditions, one can see that at the same threshold values $F_c(\omega)$ the stochastic layer around the separatrix explodes and covers the vicinity of the stable states. However, even at rather small excesses of F over the threshold the crossing of the separatrix comes fast, after about ten periods of the field. At these times the chaotic nature of the system's dynamics remains concealed: the motion seems regular and rather simple. Therefore we can try to explain the behavior of the dependence $F_c(\omega)$ in the frame of regular dynamics.

Specific features of the perturbing terms in Eqs. (7) create technical complications that are irrelevant to the nature of the phenomenon. The main qualitative features of the separatrix crossing under the influence of the harmonic field could be explained with a toy model—the one-dimensional Duffing oscillator with the equation of motion

$$\ddot{x}+x-x^3=F\sin\omega t \tag{11}$$

and the initial conditions $x(0)=0, \dot{x}(0)=0$. This model also has a stable point, surrounded by a separatrix.

We assume the frequency detuning $\delta=\omega-1$ to be small, $|\delta|\ll 1$. If the perturbation F is weak, then the nonlinearity of the oscillator could be neglected, at least in the lowest approximation. Then the solution of Eq. (11) has the approximate form

$$x(t)\approx-\frac{F}{\delta}\sin\frac{\delta}{2}t\cos\left[\omega t-\frac{\delta}{2}t\right]. \tag{12}$$

From this law of motion, assuming that the oscillator could be linearized in all the range $|x|\leq 1$, we find a crude estimate for the threshold of the separatrix crossing, namely $F_c=|\delta|$.

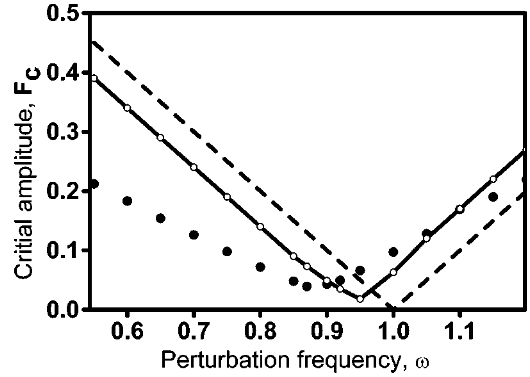

FIG. 3. Dependence of threshold amplitudes F_c of the perturbation on the frequency ω for the Duffing oscillator (11). The dashed line–the zeroth (linear) approximation, ○–estimates found from Eqs. (14) (solid line is an eyeguide); ●–results of the numerical experiment.

Now we improve this estimate by taking into account the nonlinearity. Let us represent the motion of the oscillator in the form $x(t)=A\cos(\omega t+\varphi)$, where A and φ are slowly varying functions of time. The law (12) corresponds to the equations of motion for the slow amplitude and phase,

$$\dot{A}=-\frac{F}{2}\cos\varphi, \quad \dot{\varphi}=-\frac{\delta}{2}, \tag{13}$$

with the initial conditions $A(0)=0$, $\varphi(0)=\pi$. Let us now replace the eigenfrequency of the oscillator $\Omega_0=1$ in the r.h.s. of the second of Eqs. (13) by the eigenfrequency of the nonlinear oscillator $\Omega(A)$ that depends on the amplitude. Then for the model (11), for small A, we have $\Omega(A)=1-3A^2/16$. Consequently the evolution of the system could be described by the system of equations

$$\dot{A}=-\frac{F}{2}\cos\varphi, \quad \dot{\varphi}=-\frac{\delta}{2}-\frac{3}{32}A^2, \tag{14}$$

with the same initial conditions as (13). The threshold of the separatrix crossing could be found from the condition that oscillations may reach the saddle points: $\max A(t)=1$. From the second of Eqs. (14) it is seen, that if $\delta>0$, then the phase shift decreases monotonically, thus decreasing the rate of the amplitude growth. If $\delta<-3/16=-0.188$, then the phase shift increases monotonically while the amplitude stays below its critical value, $A<1$; and again the rate of the amplitude growth decreases with time. But in the band $-3/16=-0.188<\delta<0$ the phase shift at first grows, then reaches a maximum and starts to decrease, passing the zero value at some later time, t_0. Consequently, there are two moments in which the amplitude growth rate is maximal, $t=0$ and $t=t_0$. Thus one may expect that the dependence $F_c(\omega)$ will have a minimum somewhere in the range $13/16=0.812<\omega<1$. The numerical solution of Eqs. (14) shows that this is true: the minimum of $F_c(\omega)$ is reached for $\omega=0.87$, about the middle of this band (see Fig. 3).

To find the condition for the separatrix crossing, Eqs. (14) should be solved on a finite interval of time, while the phase shift reaches the value $\pm\pi/2$. This could be done in a number of ways that will produce analytical estimates for the threshold values. In the exact resonance (at $\delta=0$) in the zeroth approximation the amplitude dependence on time is linear, $A_0=Ft/2$. By substitution of this expression in the r.h.s. of the second of Eqs. (14), we have in the first approximation

$$\varphi_1(t)=\pi-\frac{1}{128}F^2t^3. \tag{15}$$

The time t_m when the amplitude $A(t)$ reaches the maximum is found from the condition $\varphi_1(t_m)=\pi/2$. Hence from the first of Eqs. (14), in the first approximation, we have the threshold value of the perturbation in the exact resonance:

$$F_c(1)=\frac{27}{16}\left[\int_0^{\pi/2}\theta^{-2/3}\cos\theta\,d\theta\right]^{-3}=0.0666. \tag{16}$$

This agrees with the result of the numerical solution of the system (14) with an accuracy of about 6%, but differs from the value obtained in the numerical simulations by a factor about 1.5.

The studied model (11) shares with the system (7) the ''soft'' character of the nonlinearity of oscillations around the stable points: the eigenfrequency of oscillations decreases with the growth of their amplitude. This common feature is responsible for the similar behavior of $F_c(\omega)$ in the two models (compare Figs. 2 and 3)—namely, the presence of a nonzero minimum at a frequency somewhat lower than that of the small oscillations, Ω_0.

Now we return to the case of the chaotic motion of the system above the threshold. For the system with the Hamiltonian $H=H_0(\Delta,\Theta)+V(\Delta,\Theta)\sin\omega t$, with a small perturbation V the energy half-width of the stochastic layer is given by the Melnikov-Arnold integral

$$\Delta E=\int_{-\infty}^{\infty}\left(\frac{\partial V}{\partial\Delta}\dot{\Delta}+\frac{\partial V}{\partial\Theta}\dot{\Theta}\right)\sin\omega t\,dt, \tag{17}$$

where $\Delta(t)$ and $\Theta(t)$ are taken for the unperturbed motion on the separatrix [11]. Thus we can expect that the motion in phase space will persist inside the domain limited by the isoenergetic line $E=E_s+\Delta E$. Since for the Hamiltonian systems the phase volume is conserved, we may expect the invariant density in the phase space to be uniform. This leads to the invariant distribution of the energy values $w(E)$ of the form

$$w(E)=\eta\frac{2}{\Omega(E)}\quad(E_-<E<E_s),$$
$$w(E)=\eta\frac{1}{\Omega(E)}\quad(E_s<E<E_s+\Delta E), \tag{18}$$

where η is the normalization constant, and the factor ''2'' in the first line accounts for the double degeneracy of the energy states. The comparison of this distribution with the one

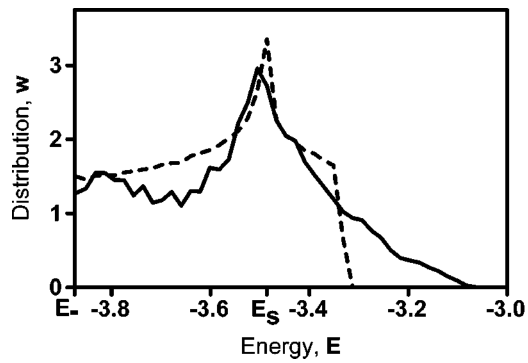

FIG. 4. Distribution w of system's values of energy under the influence of the over-threshold harmonic perturbation with $\omega=\Omega_0=2.896$ and $F=2F_c=0.2$. Dashed line–theoretical distribution Eq. (18), solid line–results of the numerical experiment.

obtained in the numerical simulations is shown in Fig. 4. The general agreement is clearly present, in spite of rather large value of the perturbation magnitude. The discrepancy between the distributions for the energy values around and above $E_s+\Delta E$ is due to the borderline resonances of the stochastic layer and could be anticipated.

If we define the vicinity of the stable state by the condition $E<E_*$, then the average fraction of time spent in this domain is

$$\mu(E_*)=\eta\int_{E_-}^{E_*}\frac{dE}{\Omega(E)}\approx\eta\frac{(E_*-E_-)}{\Omega_0} \tag{19}$$

and the average transition time from vicinity of one of the self-trapped states to vicinity of the other is about

$$T\sim\frac{\tau}{\mu(E_*)}, \tag{20}$$

where τ is the energy relaxation time. For the over-threshold perturbation amplitude the latter has value of about the period of the harmonic field.

IV. THE BROADBAND PERTURBATION

The threshold character of the separatrix crossing in the harmonic field stems from the termination of the resonant energy absorption before the value E_s is achieved. If the perturbation is broadband noise, then the energy absorption can go on indefinitely at arbitrarily small field amplitude. The problem of energy absorption from the external noise has been studied extensively as a part of the theory of dissipative systems in contact with the heat bath [14–16]. The averaged evolution of the system can be described as a process of diffusion on the energy axis. The equation that governs this process could be derived from the Fokker-Planck equation [14,15]. It is inconvenient, however, to adjust the known results to our case since our Hamiltonian (7), (8) has a rather unusual structure. Instead we derive the equation for the energy diffusion considering the system *formally* as a quantum one, starting from quantum kinetic equations and going to the classical limit $\hbar \to 0$ to obtain our results (cf. [16,17]).

Let us consider a quantum system with the unperturbed Hamiltonian $\hat{H}_0$ with one degree of freedom and a discrete energy spectrum under the perturbation $\hat{V}\xi(t)$ where $\hat{V}$ depends on the dynamical variables of the system and $\xi(t)$ is a stationary weak broadband noise specified by its spectral density $S(\omega)$. The state of the system could be described by the probabilities ρ_n of finding it in the quantum state $|n\rangle$. The evolution of these probabilities obeys the system of master equations

$$\frac{d\rho_n}{dt} = -\rho_n \sum_{k=-n}^{\infty} \dot{W}_{n,n+k} + \sum_{k=-n}^{\infty} \rho_{n+k} \dot{W}_{n+k,n}. \tag{21}$$

The rates of transitions $\dot{W}_{n+k,n}$ are determined by the perturbation theory formula

$$\dot{W}_{n,n+k} = \frac{2\pi}{\hbar^2} |V_{n,n+k}|^2 S(-\omega_{n,n+k}), \tag{22}$$

where $V_{n,n+k}$ are the matrix elements of the perturbation and $S(-\omega_{n,n+k})$ is the spectral density of noise at the frequency of transition. Let us take the probabilities to be functions not on the level number n, but on its energy: $\rho_n \to \rho(E_n)$. In the quasiclassical case the energy spectrum of the system could be related to the frequency of its classical motion at a given energy $\Omega(E)$:

$$E_{n+k} = E_n \pm \hbar\Omega\left(E_n \pm k\frac{\hbar\Omega}{2}\right), \tag{23}$$

and the matrix elements of the perturbation $V_{n,n+k}$ could be replaced by the Fourier components of the unperturbed motion of the dynamical variable that corresponds to the operator $\hat{V}$: if

$$V(t) = \sum_{k=-\infty}^{\infty} V_k e^{-ik\Omega t}, \tag{24}$$

then

$$V_{n,n+k} \to V_k\left(E_n + k\frac{\hbar\Omega}{2}\right). \tag{25}$$

We assume $\rho(E)$ to be a smooth function, and expand its value to the terms of the second order in $\hbar$ and the values of E_{n+k} and $V_{n,n+k}$ to the first order in $\hbar$. After the substitution of these expansions into Eq. (21) and going to the limit $\hbar \to 0$ we obtain a purely classical equation. At this point we restrict our consideration to the case of white noise with the constant spectral density $S(\omega)=1$. For this case we have

$$\frac{\partial\rho}{\partial t} = \Omega\frac{\partial}{\partial E}\left[D(E)\frac{\partial\rho}{\partial E}\right], \tag{26}$$

where the energy diffusion coefficient $D(E)$ is given by the expression

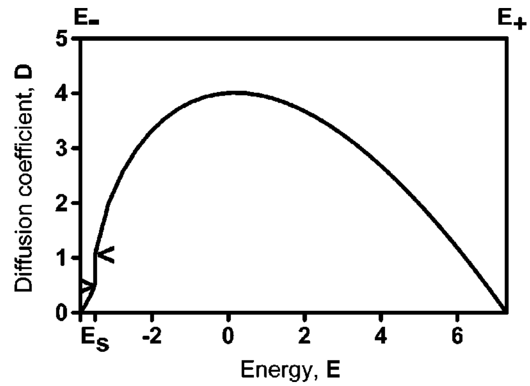

FIG. 5. Dependence of the energy diffusion coefficient D under the influence of the white noise of unit spectral density on the energy E.

$$D(E) = 2\pi\Omega(E)\sum_{k=1}^{\infty} k^2 V_k^2(E) = \frac{1}{2}\int_0^{2\pi/\Omega}\left(\frac{dV}{dt}\right)^2 dt. \tag{27}$$

The energy dependence of the diffusion coefficient D for the unperturbed system (7) with the perturbation $V(\Delta,\Theta) = \sqrt{1-\Delta^2}\cos(\Theta/2)$ is shown in Fig. 5. We note a discontinuity of $D(E)$ at the separatrix value of energy: $D(E_s+0) = 2D(E_s-0)$. This jump is not a direct consequence of the presence of the separatrix, but reflects both the global structure of the phase space and the behavior of the perturbation V in the neighborhood of the separatrix.

Since the classical distribution in energy $w(E)$ is connected to the probability density $\rho(E)$ by the relation $w(E) = \rho(E)[\hbar\Omega(E)]^{-1}$, we have the equation for the $w(E)$ in the form

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial E}\left(D(E)\frac{\partial}{\partial E}(w\Omega)\right). \tag{28}$$

The stationary solution of Eq. (28) has the form given by Eq. (18), but the second line holds now in all the range $E_s < E < E_+$. If the system was initially in one of the stable states, then in course of the energy diffusion it has the opportunity to migrate to the vicinity of the opposite stable state. The characteristic time of the average transfer is determined by the relaxation time of the distribution to its stationary value; from Eq. (26) it can be estimated as

$$T \sim \frac{(E_+ - E_-)^2}{\langle\Omega D\rangle}, \tag{29}$$

where the angular brackets denote averaging over the energy.

V. CONCLUSIONS

The main question addressed by this paper is: If the system, that is described by the one-dimensional nonlinear Scrödinger equation with the potential of the symmetric double well, is initially in one of the lowest asymmetric (self-trapped) states, then can the time-dependent perturbation transfer the system completely to the opposite asymmetric mode? The answer is yes, but almost completely and only by chance.

For the harmonic perturbation with amplitude F that exceeds the threshold $F_c(\omega)$, the system's motion is captured in the stochastic layer that embraces both domains of vibration around the stable states and a strip around the separatrix. When moving in this domain, the system can come arbitrarily close to the opposite asymmetric state. However, the nature of this process is purely chaotic and, hence, unpredictable. There is no way to create in the nonlinear system the ''π-pulse'' [10] that will transfer the system unambiguously from one of the stable states to another. Finally we note that the threshold magnitude of the perturbation is only numerically small in comparison with the depth of the self-trapping wells: to make the transfer possible, the system must be perturbed strongly.

For the system under the influence of the white noise (or, generally speaking, any sufficiently broadband noise) the process of energy diffusion eventually spreads the probability density over all phase space of the system H_0. In this case the system can occasionally come close to the opposite stable state. We note, however, that the probability of finding the system within one of the self-trapping wells is rather small; with our standard set of parameters it is only about few percent.

The main approximation of our calculations consisted in the truncation of the expansion of the wave function to just two modes [see Eq. (4)]. It was justified by the high quality of this approximation in representing the unperturbed self-trapped states [9]. Whether this accuracy will hold for the seriously perturbed system is quite a different question. For the harmonic perturbation of moderate amplitude the system stays locked within the narrow energy domain (see Fig. 4), and the addition of contributions of modes Φ_i with $i \geqslant 2$, that will lead to the extension of the energy space of the system, will have little influence.

The situation may be different for broadband noise, where the system can reach any point in the phase space. However the main contribution to the energy diffusion coefficient comes from frequencies that are lower than Ω_0: in particular, at the separatrix they contribute about 0.66 of the total value. Thus if the spectrum of noise has high-frequency cutoff just above the Ω_0, then the energy diffusion ceases at the energy $E_h > E_s$ at which $\Omega(E_h) = \Omega_0$ (for our set of parameters $E_h = -0.356$) and the system stays locked within the restricted energy domain. Then, in parallel to the case of the harmonic field, we can conclude the unimportance of the extension of the energy space by addition of higher modes.

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