Glass transition of a particle in a random potential, front selection in nonlinear renormalization group, and entropic phenomena in Liouville and sinh-Gordon models

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We study via renormalization group (RG), numerics, exact bounds, and qualitative arguments the equilibrium Gibbs measure of a particle in a d-dimensional Gaussian random potential with *translationally invariant logarithmic* spatial correlations. We show that for any $d \geqslant 1$ it exhibits a transition at $T=T_c>0$. The low-temperature glass phase has a nontrivial structure, being dominated by *a few* distant states (with replica symmetry breaking phenomenology). In finite dimension this transition exists only in this ''marginal glass'' case (energy fluctuation exponent $\theta=0$) and disappears if correlations grow faster (single ground-state dominance $\theta>0$) or slower (high-temperature phase). The associated extremal statistics problem for correlated energy landscapes exhibits universal features which we describe using a nonlinear Kolmogorov (KPP) RG equation. These include the tails of the distribution of the minimal energy (or free energy) and the finite-size corrections, which are universal. The glass transition is closely related to Derrida's random energy models. In $d=2$, the connection between this problem and Liouville and sinh-Gordon models is discussed. The glass transition of the particle exhibits interesting similarities with the weak- to strong-coupling transition in Liouville ($c=1$ barrier) and with a transition that we conjecture for the sinh-Gordon model, with correspondence in some exact results and RG analysis. Glassy freezing of the particle is associated with the generation under RG of new local operators and of nonsmooth configurations in Liouville. Applications to Dirac fermions in random magnetic fields at criticality reveal a peculiar ''quasilocalized'' regime (corresponding to the glass phase for the particle), where eigenfunctions are concentrated over *a finite number* of distant regions, and allow us to recover the multifractal spectrum in the delocalized regime. [S1063-651X(00)11510-7]

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I. INTRODUCTION

Despite significant progress in the past two decades, disordered systems continue to pose considerable theoretical challenges. Two important questions still largely open, are, respectively, to which extent the (better understood) mean-field models are relevant to describe low-dimensional physical systems, and, in the special case of two dimensions, to what extent the powerful field-theoretic treatments developed for pure models can be adapted to treat disordered models.

A celebrated controversy is whether the structure found in the solution of mean-field models for spin glasses and other complex disordered systems, both in the statics [1] and in the dynamics [2], has any counterpart in the world of experimentally relevant low-dimensional models. Specifically, it has been vigorously questioned [3] whether the breaking of the phase space in ''many pure states,'' predicted to occur in mean field, may also occur in short-range models, and how it can be properly defined [4,5]. The unusual nature of the technique used to solve the statics, i.e., the replica method with a hierarchical breaking of the permutation symmetry between replicas in the limit $n\to 0$ (RSB), did not contribute to make the physics transparent. A distinct structure, which remarkably parallels the one in the statics, has been found [2] to occur in the nonequilibrium dynamics. The dynamical problem can be studied by *a priori* better defined methods and leads to predictions which are in principle directly testable in experiments, such as a nontrivial generalization of the fluctuation dissipation relations. Even so, it has been emphasized that mean-field models, which usually involve infinite range or an infinite number of component limits, may not capture physical processes important in low dimensions. The alternative ''droplet picture'' in its simplest form [3] postulates the existence of a single ground state with excitations (droplets) of (free) energy ΔE scaling with their size x as $\Delta E \sim x^{\theta}$, $\theta>0$. It provides a more conventional scaling description of the glass physics, as being controlled by zero-temperature RG fixed points where temperature is formally irrelevant (with eigenvalue $-\theta$).

Another important advance was the exact solution of simpler prototype models, such as the random energy model (REM) [6], where one considers simply a collection of independently distributed energy levels, as well as its generalization, the GREM [7], or the directed polymer on the Cayley tree (DPCT) with disorder [8]. These solutions being direct with no use of replica, their results can be fully relied upon. They exhibit a similar physics, with a glass transition, and in the glass phase they exhibit an exponential tail for the distribution of the free energy $P(f) \sim e^{\beta_c f}$ for negative f. This feature is crucial to recover the same physics, and indeed many observables were found to be similar [9]. In fact, the alternative solution of the REM using replicas, given in [6], or that of the DPCT [10] *do* involve RSB. In the REM model the structure of the glass phase is particularly transparent as being dominated by *a few* states [6,9].

It is important to go beyond models defined in mean field or on hierarchical (or ultrametric) structures and to study simple yet nontrivial (and nonartificial) finite d models with full statistical translational invariance. In this paper we study the model of a particle in a Gaussian random potential $V(\mathbf{r})$ with spatial correlations which are *invariant by translation* and which grow as the *logarithm* of the distance. We consider this model in any dimension d, but in $d=2$ it has also been studied recently since it is of direct relevance for several physical systems [11–20]. One example is a spin model with XY symmetry and random gauge quenched disorder, which arises naturally in describing Josephson junction arrays [21] or two-dimensional (2D) crystalline structures with smooth disorder, e.g., flux lattices in superconductors [22], or electrons at the surface of helium [23]. In this model, a single topological defect (an XY vortex) or a single neutral pair sees precisely a random potential with logarithmic correlations [14–19]. Another example arises in a model of localization of Dirac fermions in a random magnetic field, motivated by quantum Hall physics. There, the zero-energy $E=0$ normalized wave function is identical to the Boltzmann weight of the particle studied here [11–13]. This wave function is ''critical'' in a sense discussed below.

Here we study this model using a renormalization-group (RG) approach, bounds, numerical methods, and qualitative arguments. We show that it exhibits a transition at $T=T_c>0$ in any $d\geqslant 1$. We find that in the high-temperature phase the particle is essentially delocalized over the whole system, while in the low-temperature glass phase the Gibbs measure is concentrated in a few minima. The fact that such a simple (finite d) model exhibits a genuine glass transition is already surprising. Indeed, as we argue, this transition exists *only* for such a ''marginal'' type of correlation (which corresponds to $\theta=0$ in the glass scaling mentioned above [24]). It disappears [for Gaussian $V(\mathbf{r})$] if correlations grow faster (with only a low-temperature phase and single ground-state dominance) or slower (with only a high-temperature phase). Logarithmic growth of correlations thus produces exactly the right balance between the depth of the energy wells and their number (entropy). Note that for slower growing correlations one can recover a transition but only by artificially rescaling the disorder variance with the size of the system: in the extreme case of uncorrelated variables, it is the REM model. Here, by contrast, there is a genuine phase transition in the thermodynamic limit, with no need for rescaling. Most interestingly, the glass phase is nontrivial. The Gibbs measure is concentrated in *a few distant minima* which remain in a finite number in the thermodynamic limit. This is because the extrema of random variables with such correlations exhibit an interesting property of ''return near the minimum'': there is, with a finite probability in a sample of size $L\to+\infty$, at least one second minimum far away (at distances of order L), and with a finite energy difference with the absolute minimum. And there are not too many (a thermodynamic number) of these secondary minima, leading to a zero entropy. As in the REM, this property leads here naturally to a nontrivial ground-state structure, reminiscent, as we discuss, of a genuine property of replica symmetry breaking in the replicated theory. The low-temperature limit corresponds to a nontrivial problem of extremal statistics of *correlated* variables, studied here.

Another interesting property of this model is its relation to the Liouville model (LM) and the sinh-Gordon model (SGM) in $d=2$ (and their boundary restriction in $d=1$): $V(\mathbf{r})$ turns out to be the Liouville field while the LM and SGM partition functions arise simply as generating functions of the probability distribution of the partition sum $Z[V]=\int_{\mathbf{r}}e^{-\beta V(\mathbf{r})}$ of a single particle. The high-temperature phase for the particle corresponds to the weak-coupling regime for the LM and SGM, where we find that known exact results compare well with results for the particle. In the SGM we predict here the existence of a transition (more appropriately, a ''change of behavior''). It corresponds to the glass transition for the particle, which also exhibits interesting similarities with the weak- to strong-coupling transition in the Liouville theory (and the so-called $c=1$ barrier). The glassy freezing of the particle is associated, in the LM and SGM, to new local operators and nonsmooth configurations being generated under RG.

To study the model, we will introduce a RG approach based on a Coulomb gas renormalization in the manner of Kosterlitz. It leads to a nonlinear RG equation [of the so-called Kolmogorov-Petrovskii-Piscounov (KPP) type] for the full probability distribution of the ''local disorder.'' Indeed, a separation between the long-range part of the disorder and the local, short-range part arises naturally in our approach. The RG equation has traveling-wave types of solutions. The corresponding well-known problem, in such nonlinear equations, of the selection of the velocity of the traveling front and its freezing for $T\leqslant T_c$ is related to glassy freezing of the particle free energy and, in the LM or SGM, to the ''selection'' of the anomalous dimensions (and at the transition dimension degeneracy it leads to logarithmic operators). When temperature is lowered, the local disorder becomes broadly distributed and the freezing occurs when its tails become relevant. Our RG method indicates that the physics depends only weakly on d. We will take advantage of this fact and check our results using simulations in $d=1$.

It is important to compare the present work to previous studies of the model. The existence of a freezing transition in $d=2$ has been conjectured previously [18,16,17,13]. In Ref. [18], the analysis was based on an explicit approximation which neglects spatial correlations (called here and below the REM approximation). Various efforts to include spatial correlations were made in Refs. [16,17,13]; some are described below. Although very interesting, none of these works fully established the existence of a transition (which is done here in Appendix A), nor developed analytical methods allowing to obtain results beyond the simpler REM approximation [25], or to prove their validity. The problem is thus still largely open and the present work contains some new attempts to go beyond the REM approximation. In particular, one wants to know what is the precise universality class of the model, which we hope can be determined from the RG method introduced here. This RG method yields some remarkable universal features of the probability distribution of the free energy and of its finite-size corrections, different from the REM. It shows that the problem is more closely related to the directed polymer on a Cayley tree. A *qualitative* analogy between the present model and the DPCT was

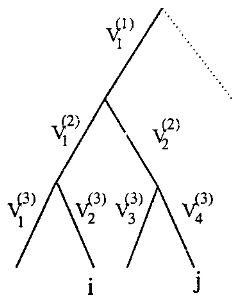

FIG. 1. Directed polymer approximation: sites are the tree end points. The v_q^p are uncorrelated of variance 2σ. The random potential at i is $V_i = v_1^{(1)} + v_1^{(2)} + v_2^{(3)}$ and at j it is $V_j = v_1^{(1)} + v_2^{(2)} + v_4^{(3)}$. Thus $\overline{(V_i - V_j)^2} = 4\sigma d(i,j)$, where $d(i,j) \sim \ln|i-j|$ is the distance (in generations) on the tree.

in fact cleverly guessed recently in Refs. [17,11]. It is based on the observation that the energy of polymer configuration on a tree also scales logarithmically with the overlap distance defined on the tree (see Fig. 1). It is remarkable that this connection naturally emerges here from the Kosterlitz-type RG performed on this problem, via the KPP equation. It is all the more surprising, since the model studied here has statistical translational invariance, while a tree has a hierarchical structure. The solution of Derrida and Spohn [8] (and the mapping onto the DPCT proposed in [17,11]) would be exact for random variables $V(\mathbf{r})$ correlated with a *hierarchical* (i.e., ultrametric) matrix of correlation. Here instead the correlations are translationally invariant and it is thus important to understand the origin of the analogy with the DPCT and to which extent it holds. The RG procedure developed in this paper is an attempt to address these questions. The result is that we can make the mapping precise: at least for the universal observables studied here (e.g., the tails of the free-energy distribution), the mapping is onto a *continuum branching process*, i.e., a continuum limit of a Cayley tree (whereas [17,11] could not be so specific).

The present model has also been studied in the context of random Dirac problems and localization. An early study [26] of the $E=0$ wave function established that it was critical (in the sense of corresponding to a ''delocalized'' wave function, while $E \neq 0$ has finite localization length). However, this study missed the glass transition. Later studies [13] computed the multifractal spectrum based on the REM approximation (in the sense defined above [25]) and noticed the existence of a strong disorder regime. These and other studies [11,12], however, focused on properties of the high-temperature phase: it was conjectured [12] that the (conformal) Liouville field theory (LFT) (i.e., a continuum limit of the LM) was able to describe all spatial correlations of the model in the high-temperature phase. These works call for more investigations. First, the *glass transition* and the peculiar physical properties of the low T (i.e., strong disorder) phase have not been addressed, even at the most qualitative level. We thus find it useful to present the problem from a different perspective by comparing with other types of correlated disorder, or by recasting it as a problem of extremal statistics. Although well known properties [27] of extremal statistics of *uncorrelated* variables were often used to study model disordered systems (see, e.g., [28–30]), a lot remains to be understood about the (more realistic) case of correlated variables. Second, the question of the universality class is in our opinion far from established. Evidence for the LFT description mostly comes from reproducing the multifractal spectrum as given by the REM approximation and one would like to check it against more detailed predictions. The present RG procedure is a step towards clarifying the connection between this model and solvable models such as Derrida's REM and Derrida-Spohn DPCT. In this respect, finite-size corrections are important to understand, as they are found to exhibit universal prefactors allowing to distinguish between various universality classes. In addition, they determine the anomalous dimensions, and thus control the critical behavior, in the full disordered XY model as shown in [19]. Since they are found to be very large, they are also crucial in order to analyze the results of numerical simulations. In particular, although we confirm the result of [11], we also conclude that the sizes used in the numerical study of [11] were in fact vastly insufficient for drawing firm conclusions: we do perform here a more detailed finite-size analysis on much larger samples to confirm analytical predictions.

The model studied here is thus related to a surprising number of interesting problems. Let us mention for completeness that it also has connections to problems such as two-dimensional interfaces, or films, confined between two walls (for $\beta = +\infty$ it is the confinement entropy of a film), wetting transitions [31], extremal statistics of correlated variables useful, e.g., for problems of ''persistence'' in nonequilibrium dynamics [32], and finally, to the clumping transition of a self-gravitating planar gas [33]. We will not explore these connections in detail here.

This paper is organized as follows. In Sec. II A, the single-particle model is defined and in Sec. II B, the random energies approximation (REM) is applied, which amounts to neglecting the spatial correlations of the random potential. The full problem, with correlations taken into account, is related to the description of extremal statistics in Sec. II C, and three different classes of correlations are identified in Sec. II D from qualitative arguments. A new renormalization (RG) technique is applied to this problem in Sec. III. The resulting nonlinear scaling equation for the distribution of the local disorder is studied in Sec. III C, and is found to be related to the Kolmogorov KPP equation, which admits front solutions. This connection between front solutions of nonlinear equations and the renormalization of disordered models is pursued in Sec. III D, where a solution to the REM is found via a similar nonlinear RG (details in Appendix C). The nontrivial nature of the glass phase is discussed in Sec. III E together with its relations to replica symetry breaking. In Sec. IV we present a numerical analysis of the problem of the particle in a random potential in $d=1$. Section V is devoted to the connection between the particle model (and its transition) and entropic phenomena in the Liouville and sinh-Gordon models. A direct RG analysis in Sec. V C allows us to recover the corresponding change of behaviors in these models. Section VI contains the applications to the properties of the critical wave function of a Dirac fermion in a random magnetic field, in particular the multifractal properties and the property of quasilocalization. Appendix A contains an outline of the proof of the existence of a transition, Appendix B is a review of well-known (and not so well-known) results about extremal statistics, and Appendix D contains an extended model which exhibits three phases.

II. MODEL AND QUALITATIVE ANALYSIS

In this section, we define the model of a single particle in a correlated random potential. Then we describe the random energy model (REM) approximation used in previous studies, which consist in neglecting correlations. We then pose the new questions which we want to address here for the true model and present a qualitative analysis showing physically why we expect that logarithmic correlations (as opposed to faster growing or slower growing correlations) are the only case which leads to (i) a glass transition and (ii) a low-temperature phase with a nontrivial structure of quasidegenerate distant minima

A. The model

The equilibrium problem of a single particle in a d-dimensional random potential is defined by the canonical partition function

$$Z[V]=\sum_{\mathbf{r}} e^{-\beta V(\mathbf{r})}, \tag{1}$$

where $\beta=1/T$ is the inverse temperature, in a sample of finite size L (and total number of sites L^d) and for a given configuration of the random variables $V(\mathbf{r})$. The equilibrium Gibbs measure, or probability distribution for the position of the particle, is

$$p(\mathbf{r})=e^{-\beta V(\mathbf{r})}/Z[V]. \tag{2}$$

We are interested here in cases where the random variables $V(\mathbf{r})$ can be correlated. As discussed below, the statics (and dynamics) of this problem in the limit of large sizes depends on the type of correlations, the distribution of the disorder, and the dimensionality of space d. Some of these cases and their dynamical aspects (such as the Sinai model) have been extensively studied, e.g., in the context of diffusion in random media [34]. Even logarithmic correlations in $d=2$ were studied then [35], but it was not realized at that time that a static glass transition could exist in that case.

Correlated random potentials $V(\mathbf{r})$ are most conveniently studied for Gaussian distributions, on which we focus, parametrized by the correlator $\Gamma(\mathbf{r},\mathbf{r}')=\overline{V(\mathbf{r})V(\mathbf{r}')}$ [and we choose $\overline{V(\mathbf{r})}=0$]. Non-Gaussian extensions will be mentioned. Unless specified otherwise, the correlations will be chosen translationally invariant $\Gamma(\mathbf{r},\mathbf{r}')=\Gamma_L(\mathbf{r}-\mathbf{r}')$ with cyclic boundary conditions, or in (discrete) Fourier space $\overline{V(\mathbf{q})V(\mathbf{p})}=\Gamma(\mathbf{q},\mathbf{p})=\Gamma(\mathbf{q})\delta_{\mathbf{p},-\mathbf{q}}$. We will often denote $\overline{[V(\mathbf{r})-V(\mathbf{r}')]^2}=\tilde{\Gamma}(\mathbf{r}-\mathbf{r}')=2\int_{\mathbf{q}}\Gamma(\mathbf{q})(1-\cos[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')])$ [with $\int_{\mathbf{q}}=1/L\Sigma_{\mathbf{q}}\to\int d^d\mathbf{q}/(2\pi)^d$].

One important quantity is the free energy,

$$F[V]=-T\ln Z[V], \tag{3}$$

and, since it fluctuates from configuration to configuration, as $F[V]=\overline{F[V]}+\delta F[V]$ we will be interested in its average $F=\overline{F[V]}$ and in its distribution. From the convexity of the logarithm follows the well-known exact bound for F in terms of the annealed free energy F_A:

$$-T\,\overline{\ln Z}=F\geqslant F_A=-T\ln\overline{Z}, \tag{4}$$

$$F\geqslant-\left(Td\ln L+\frac{1}{2T}\overline{V(\mathbf{r})^2}\right) \tag{5}$$

for the Gaussian case.

In this paper we will mainly focus on the case of correlations growing *logarithmically* with distance:

$$\overline{[V(\mathbf{r})-V(\mathbf{r}')]^2}\sim 4\sigma\ln\frac{|\mathbf{r}-\mathbf{r}'|}{a},\quad a\ll|\mathbf{r}-\mathbf{r}'|\ll L \tag{6}$$

which also requires a small distance ultraviolet (uv) cutoff a (we can set here $a=1$ in accordance with the definition 1 of a discrete model, but in the following sections we will consider a continuum version and vary a). This behavior is achieved in d dimension by choosing a propagator in Fourier space $\Gamma(\mathbf{q})\sim 2\sigma(2\pi)^d/S_d q^d$. The $d=2$ case is also of special interest as the propagator is the usual Coulomb one:

$$\Gamma(\mathbf{q})\sim\frac{4\pi\sigma}{q^2} \tag{7}$$

and boundary conditions must be specified later on. It is important to note that for LR correlations the single site variance $\overline{V(\mathbf{r})^2}=\Gamma_L(0)$ diverges with the system size, e.g., for Eq. (6) one has $\Gamma_L(0)\sim 2\sigma\ln(L/a)$.

For such logarithmic correlations (as well as for weaker correlations [36]) one will find that F scales as $d\ln L$ (consistent with the number of degree of freedom being L^d in this problem). Thus it is natural to define the intensive free energy,

$$f(\beta)=\lim_{L\to+\infty}\frac{F[V]}{d\ln L}, \tag{8}$$

which will be found to be self-averaging. The above bound gives

$$f(\beta)\geqslant-\left(\frac{1}{\beta}+\frac{\sigma}{d}\beta\right). \tag{9}$$

Thus we will find that $F[V]\sim f(\beta)d\ln L$ with subdominant corrections. These corrections have a nonfluctuating universal $O(\ln(\ln L))$ piece as well as an $O(1)$ fluctuating part $\delta F[V]$ both of which we will study.

B. The REM approximation

A useful *approximation* to the problem studied here, which can be called the REM approximation, consists in neglecting all correlations but keeping the on-site variance exact [18,13]:

$$\Gamma_L(\mathbf{r}) \to \Gamma_L^{\mathrm{REM}}(\mathbf{r}) = \Gamma_L(0)\,\delta_{\mathbf{r},\mathbf{r}'} = 2\sigma \ln\left(\frac{L}{a}\right)\delta_{\mathbf{r},\mathbf{r}'} . \quad (10)$$

The corresponding Gaussian REM model can then be solved, being identical to [6], and one finds that it exhibits a transition at $\beta_c = \sqrt{d/\sigma}$ with

$$f(\beta) = -\left(\frac{1}{\beta} + \frac{\sigma}{d}\beta\right), \quad \beta < \beta_c , \quad (11a)$$

$$f(\beta) = -\frac{2}{\beta_c}, \quad \beta > \beta_c . \quad (11b)$$

Most previous studies of the original model (all in $d=2$) amount to studying the REM approximation and argue that it is a good approximation. Indeed, as we will also find here, this REM approximation appears to give the exact result for some observables [e.g., for $f(\beta)$]. In particular, it does seem to give correctly the transition temperature β_c.

C. Beyond the REM approximation: Extremal statistics of correlated variables

Since it is not obvious *a priori* why logarithmic correlations can be considered so weak as to be neglected, one would like to go beyond the REM approximation and describe the effect of the neglected correlations [37]. One would like to understand why this approximation works for some observables (and for which ones) and whether it gives exactly the universality class of the model (i.e., all universal behavior of observables). The answer to the latter is negative: as our analysis will reveal, the correlations do matter for the more detailed behavior and the original models (1) and (6) are *not* in the same universality class as the REM model.

In fact, the problem at hand is related to describing universal features of the extremal value statistics for a set of *correlated* random variables. Indeed, the zero-temperature limit ($T=0$ for fixed L) of the problem defined by Eq. (1) amounts to finding the distribution of the *minimum* $-\lim_{T\to 0} T \ln Z_L = V_{\min} = \min_r(\{V_r\})$ of a set of *correlated* random variables. In the case of uncorrelated (or short-range correlated) variables, a lot is known in probability theory on this problem (see, e.g., [38]), some of which is summarized in Appendix B. For the type of distributions considered here (Gaussian and some extensions), the distribution of the minimum $V_{\min}$ has a strong universality property, being given, up to nontrivial rescaling and shift (see Appendix B and below), by the Gumbell distribution:

$$\mathrm{Prob}(y < x) = \mathcal{G}(x) = \exp(-e^{x}). \quad (12)$$

The Gumbell distribution thus appears as the distribution of the zero-temperature free energy in the REM. For the case of a Gaussian distribution, the standard probability theory results are usually given in terms of a variable X_r such that $\overline{X_r^2} = 1$. One can simply rescale $V_r = \sqrt{2\sigma \ln L}X_r$ from Appendix B and get

$$V_{\min} = -2\sqrt{\sigma d}\ln L + \frac{1}{2}\sqrt{\frac{\sigma}{d}}\ln(4\pi d \ln L) + \sqrt{\frac{\sigma}{d}}y, \quad (13)$$

where y is distributed as in Eq. (12).

Much less is known in the case of variables with stronger correlations studied here, though it is more important in practice. The statistics of $V_{\min}$ in the logarithmically correlated case is thus one of the open issues discussed here. One key question is to determine what is universal in the distribution of the minimum of correlated variables. Here, we can formulate the question as follows: given Gaussian random variables satisfying Eq. (6), what in the distribution of the minimum (i.e., of the ground-state energy for fixed large L) is universal, i.e., depends only on σ and not on the details of the correlator $\Gamma(\mathbf{r})$ at short scale. Writing

$$V_{\min} \sim \overline{V_{\min}} + \delta V_{\min} \quad (14)$$

one finds, for the logarithmic correlator, that the averaged ground-state energy must satisfy

$$\overline{V_{\min}} \geq -2\sqrt{\sigma d}\ln L \quad (15)$$

which follows from the above annealed bound, together with the fact that $\partial f/\partial T = -S \leq 0$. Furthermore, one will find here that $V_{\min} \sim e_{\min} d \ln L$ up to a positive subdominant—universal—piece and that $e_{\min} = -2\sqrt{\sigma/d}$ saturates the bound. In the distribution of $\delta V_{\min} \sim O(1)$ we can clearly expect *less* universality than in the problem of random variables with short-range correlations [39].

D. Qualitative study of a particle in a random potential

Before describing the RG method, which allows us to go beyond the REM approximation, let us give some simple qualitative arguments and numerical results which illustrate the main physics of the thermodynamics of a particle in a correlated random potential. To put things in context, we discuss several types of correlations (short range, long range, and marginal). We focus on $d=1$ for simplicity but the arguments extend to any *finite d*.

Whether there is a single phase or not here comes simply from whether the entropy of typical sites wins or not over the energy of the low-energy sites. When there is a low-temperature phase, to decide its structure one must pay special attention to distant secondary local minima.

Indeed, when there is a low-temperature phase, it is controlled by the regions with most negative potential. To investigate its structure one can start, for a given system of size L, with the $T=0$ state, which is determined by the absolute minimum over the system, denoted $V_{\min}$ and located at $\mathbf{r}_{\min}$. At T very small but strictly positive, each (low-lying) secondary local minimum V will also be occupied with a probability $\sim e^{-(V-V_{\min})/T}$, which is very small except when V

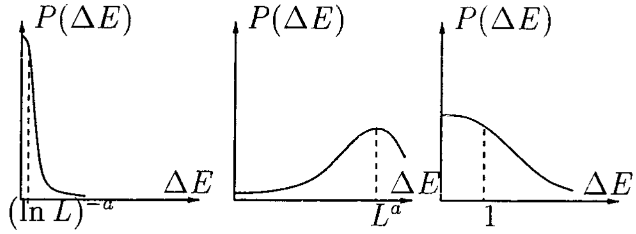

FIG. 2. Three cases for the distribution of energy difference ΔE between absolute and secondary minimum (separated at least by $R \sim L^c$) in a system of size L: (a) short-range correlated potentials, $\Delta E_{\rm typ} \to 0$ logarithmically with size; (b) algebraically growing correlations $\Delta E_{\rm typ} \to +\infty$; (c) logarithmic correlations. $\Delta E_{\rm typ}$ remains constant as the system size increases.

$-V_{\rm min} \sim O(T)$. Thus to characterize the low-temperature phase, we need to know how many of these secondary minima exist and where they are located. For a smooth enough disorder (see, e.g., Fig. 3) there will always be ''trivial'' secondary local minima in the vicinity of $r_{\rm min}$. To eliminate these, we define $V_{\rm min2}(R)$ as the next lowest minimum constrained to be at a distance at least R of the absolute minimum. An interesting quantity to study is then the distribution $P_{R,L}(\Delta E)$ of $\Delta E(R,L) = V_{\rm min2}(R) - V_{\rm min}$ over environments (which *a priori* depends on R and L).

We now distinguish three main cases, according to the behavior of the correlator $\overline{[V(\mathbf{r}) - V(\mathbf{r}')]^2} = \tilde{\Gamma}(\mathbf{r}-\mathbf{r}')$ at large scale (we restrict to Gaussian potentials [40]). In these three cases the distribution $P_{R,L}(\Delta E)$ has markedly different behaviors as illustrated in Fig. 2.

(i) *Short range correlations*. $\tilde{\Gamma}(\mathbf{r}) \to Cst$ at large $\mathbf{r}$, equivalently $\Gamma_L(\mathbf{r}) \to 0$ at large $\mathbf{r}$ (or, e.g., $\Gamma_{\mathbf{q}} \sim q^{-d+\delta}$ with $\delta > 0$). In this case it is clear that *there is only a high-temperature phase* in any finite d and no phase transition. The entropy $Td \ln L$ of typical sites [of energy typically $\sim O(1)$] always wins over the energy of optimal sites ($V_{\rm min} \sim \sqrt{2\sigma d \ln L}$ for Gaussian distributions with on-site variance σ). The optimal energy $V_{\rm min}$ can be estimated using $1/L^d = \int_{-\infty}^{V_{\rm min}} P_1(V) dV$ in terms of the single site distribution $P_1(V)$, which yields the exact leading behavior for uncorrelated disorder [41] (and also for weak enough correlations—see Appendix B). Thus, the particle is delocalized over the system for all $T > 0$. One estimates the number of states within ΔE of the minimum as $N(\Delta E) \sim L^d \int_{V_{\rm min}}^{V_{\rm min}+\Delta E} P_1(V) dV \sim \exp(\Delta E \sqrt{2d \ln L}/\sqrt{\sigma})$ for a Gaussian distribution. Thus there is a large number of sites almost degenerate with the absolute minimum $V_{\rm min}$, separated by finite barriers, and $\Delta E_{\rm typ}$ decays to 0 as a power of $1/\ln L (1/\sqrt{\ln L}$ for a Gaussian) [42]. These minima, however, are irrelevant for the thermodynamics of the system at a fixed finite temperature.

For these minima to play a role and to obtain a transition even for SR disorder, one needs to perform some *artificial rescaling*, as in the REM model [6], either at fixed size to concentrate on the very low T region (e.g., take $\beta \sim \ln L$ in the Gaussian case), or equivalently, to rescale disorder with the system size. By making disorder larger as the system increases, for instance using $P_1(V) \sim e^{-|V/V_{\rm typ}|^\alpha}$ with $V_{\rm typ} \sim (\ln L)^{1-1/\alpha}$, one recovers artificially a transition [30]. For $\alpha = 2$ and uncorrelated $V(\mathbf{r})$ this is exactly the REM studied in [6]. There, the simple argument for the transition is that the averaged density of sites at energy $E = V$ is $\overline{\Omega(E)} = L^d e^{-E^2/(2\sigma_L)}/\sqrt{2\pi\sigma_L}$ [related to the annealed partition sum via $\overline{Z} = \int_E e^{-\beta E} \overline{\Omega(E)}$]. If $\sigma_L = \sigma$ is not rescaled, the average energy is $O(1)$ and the huge entropy of these states always wins. If σ scales with L as $\sigma_L \sim 2\sigma \ln L$, then there is a transition at $\beta_c = \sqrt{d/\sigma}$. Indeed, $\overline{\Omega(E)} \sim \exp(d \ln L[1 - (e/e_{\rm min})^2])$, where $e = E/(d \ln L)$ and $e_{\rm min} = E_{\rm min}/(d \ln L) = -2\sqrt{\sigma/d}$ and there is a saddle point in $\overline{Z}$ at $\langle E \rangle/(d \ln L) = e_{\rm sp} = -\beta e_{\rm min}^2/2$: since $e_{\rm sp}$ must be larger than $e_{\rm min}$ [as $\overline{\Omega(\langle E \rangle)}$ cannot become smaller than 1], the saddle point cannot be valid below $T_c = 1/\beta_c = -e_{\rm min}/2 = \sqrt{\sigma/d}$ and the system freezes in low-lying states. Although this argument implicitly relies on using $\ln \overline{\Omega(E)}$ instead of $\overline{\ln \Omega(E)}$, it does give the correct picture for the REM, as shown in [6].

This picture generalizes to correlated potentials provided $\Gamma_L(r)$ decreases fast enough at large r. The decay must be faster than $1/\ln r$ (which is a rather slow decay) as indicated by the theorems recalled in Appendix B or also by a simple argument given in Appendix B 2 c. Finally, let us point out also that another way to obtain a transition for SR disorder is to take the $d = \infty$ limit before taking the large L limit: there the model (even without rescaling) always exhibits a transition (in the statics and in the dynamics).

(ii) *Long range correlations*. When the typical $V(\mathbf{r}) - V(\mathbf{r}')$ grows with distance as a power law $\tilde{\Gamma}(\mathbf{r}) \sim |\mathbf{r}|^\delta$, *there is only a low-temperature phase* and no transition. The particle is now always localized near the absolute minimum of the potential in the system at $\mathbf{r}_{\rm min}$. The typical minimum energy $V_{\rm min}$ grows as $\sim -L^{\delta/2}$ and thus overcomes the entropy $\sim Td \ln L$ which is never sufficient to delocalize the particle. The structure of this single low-temperature phase is simple: there are no quasidegenerate minima separated by infinite distance (and thus also by infinite barriers) in the thermodynamic limit. As can be seen in Fig. 3, there is typi-

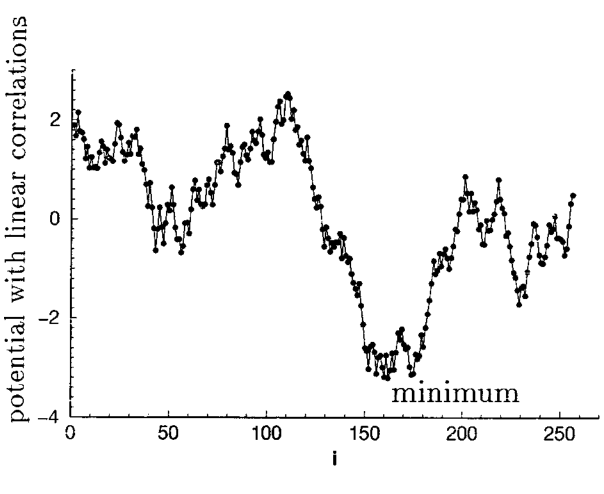

FIG. 3. A typical random potential configuration for algebraically growing correlations.

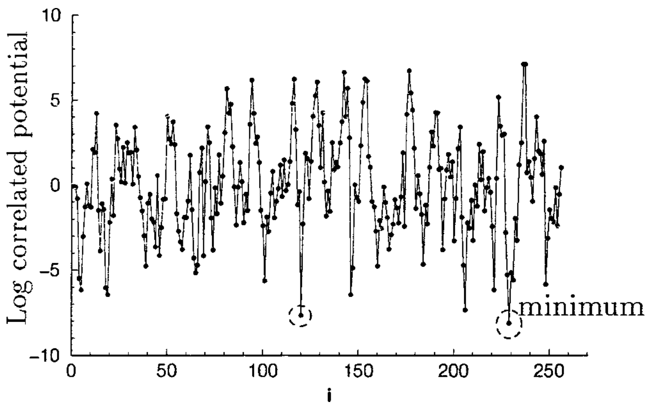

FIG. 4. A typical random potential configuration for logarithmic correlations.

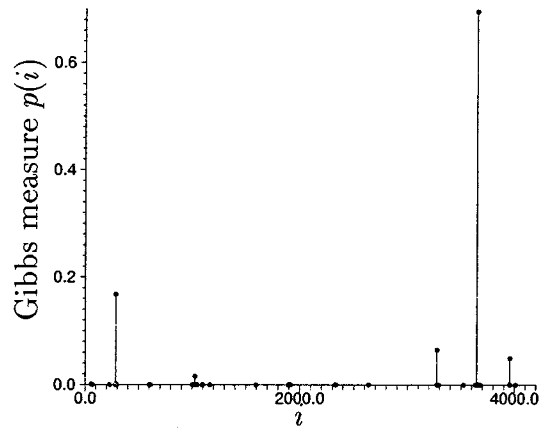

FIG. 6. Gibbs measure in a typical sample in the low-temperature phase ($\beta=3.0>\beta_c=1$), $L=4096$. Only points such that $p_i>10^{-7}$ are indicated.

cally a single minimum, with many secondary ones near it, but none far away. More precisely, as $L\to\infty$, the probability that the lowest-energy excitation $\Delta E(R,L)$ above the ground state (a distance at least $R\sim L^c$ from $\mathbf{r}_{\min}$) will be smaller than a fixed finite (arbitrary) value decays algebraically to 0 with L (and ΔE_{typ} and $\overline{\Delta E}$ increase algebraically with L). This is the familiar scenario from the droplet picture [3], with $\text{Prob}(\Delta E<T)\sim TL^{-\delta/2}$ (i.e., in some configurations which become more and more rare as $L\to+\infty$, there are two far away quasidegenerate ground states). In some cases, e.g., in Sinai's model ($\delta=1$), the distribution of rare events with quasidegenerate minima has been studied extensively [43–45]. For instance, it has been shown [43,44] that there is a well defined limit distribution $Q(R)dR$ (when $L\to+\infty$) to find quasidegenerate minima [46] at a fixed distance between R and $R+dR$, with $Q(R)\sim R^{-3/2}$ at large R.

(iii) *Marginal case, logarithmic correlations*. The most interesting case is when correlations grow as $\tilde{\Gamma}(\mathbf{r})\sim 4\sigma\ln|\mathbf{r}|$. A typical logarithmically correlated landscape is illustrated in Fig. 4. One can already see that, contrary to Fig. 3, it has states with similar energies far away.

Given the growth of correlations, one sees that the *typical* energy differences over a distance L scale as $[V(0)-V(L)]_{\text{typ}}\sim\pm\sqrt{4\sigma\ln L}$. Computing the minimum energy is a harder task here, but if one estimates it as in [18] through the REM approximation $1/L^d=\int_{-\infty}^{V_{\min}}P_1(V)dV$ (which neglects correlations), one finds that it behaves as $V_{\min}\sim-2\sqrt{\sigma d}\ln L$ (for Gaussian disorder). This estimate appears rather uncontrolled here since correlations *grow* with distance, while the theorems for uncorrelated random variables apply *a priori* only for correlations *decaying* slower than $1/\ln r$. In fact, the situation is a bit more complex, and as we will find below from the RG and our numerics, the leading behavior of $V_{\min}$ with $\ln L$ is still correctly given by the REM approximation, although the next subleading—universal—correction is not. Thus the energy of the minimum $-2\sqrt{\sigma d}\ln L$ can now balance the entropy of typical sites $Td\ln L$, which yields the possibility of a transition. The REM approximation of the model indeed yields a transition at $T_c=\sqrt{\sigma/d}$ between a high-temperature phase for $\beta<\beta_c=\sqrt{\sigma/d}$ and a frozen phase $\beta>\beta_c$. This scenario is confirmed by various approaches in the following sections.

An interesting feature of this model is that the low-temperature phase exhibits a nontrivial structure. Unlike

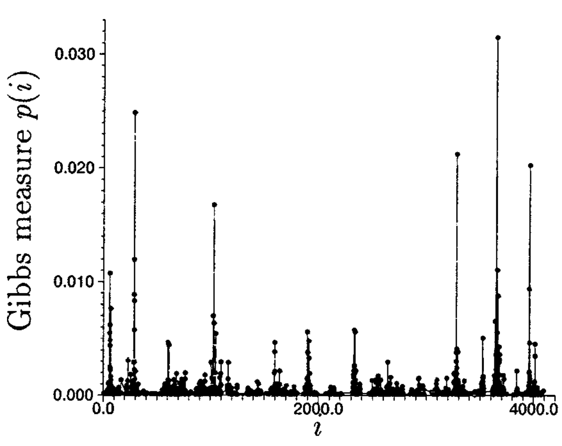

FIG. 5. Gibbs measure in a typical sample in the high-temperature phase ($\beta=0.5<\beta_c=1$), $L=4096$.

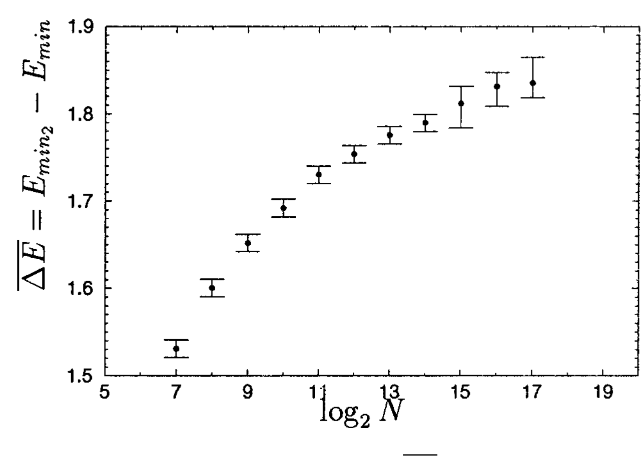

FIG. 7. Averaged energy difference $\overline{\Delta E}$ between the absolute minimum at $r_{\min}$ and the constrained secondary minimum (i.e., the minimum over the set $|r-r_{\min}|>L/3$), as a function of the system size.

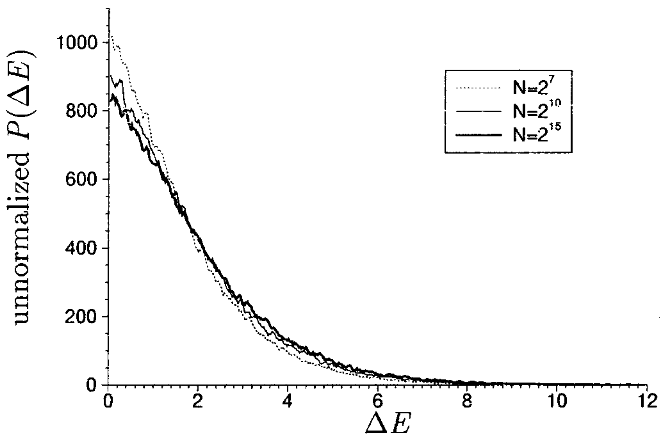

FIG. 8. Distribution of ΔE, the energy difference between the absolute minimum at $r_{\min}$ and the constrained secondary minimum (i.e., the minimum over the set $|r-r_{\min}|>L/3$) for different system sizes.

long-range disorder discussed above, for logarithmic correlations we find that the low-temperature phase is dominated not by one, but by *a few* states in the thermodynamic limit. This is in stark contrast with the standard droplet picture and is reminiscent of the replica symmetry breaking phenomenology, even though we are dealing here with a very simple finite-dimensional system.

One can visualize the transition, and the peculiar nature of the low-temperature phase in Figs. 5 and 6, where a typical Gibbs measure $p(\mathbf{r})$ is shown in both phases, is fairly delocalized at $T>T_c$ (Fig. 5) but peaks around a few states when $T<T_c$ (Fig. 6) separated by a distance of the order of the system size.

This peculiar nature of the frozen phase can be tested by showing that distant secondary local minima with a finite ΔE exist with finite probability in the thermodynamic limit. Thus we have investigated numerically the distribution $P_{R,L}(\Delta E)$ of the lowest excitation. As illustrated in Fig. 2, if the phase is nontrivial, we expect that this distribution has a well-defined limit for, e.g., $R=L/3$ when $L\to\infty$ with a finite typical ΔE. Contrary to the LR disorder, we expect the probability that, e.g., $\Delta E(L/3,L)$ will be smaller than a fixed number to saturate (not to decrease) as $L\to\infty$, i.e., that there is a fixed probability that a second state within ΔE exists far away (as was already apparent in Fig. 4). We show in Fig. 7, Fig. 8, and Fig. 9 numerical evidence that this distribution has a well-defined limit (the details of the simulation are discussed in Sec. IV). Finite-size effects are clearly important in this system, but their magnitude appears compatible with the predictions of our RG approach, as discussed below. Thus we conclude that the numerics are consistent with the existence of such a limit distribution and hence with a frozen phase with a nontrivial structure.

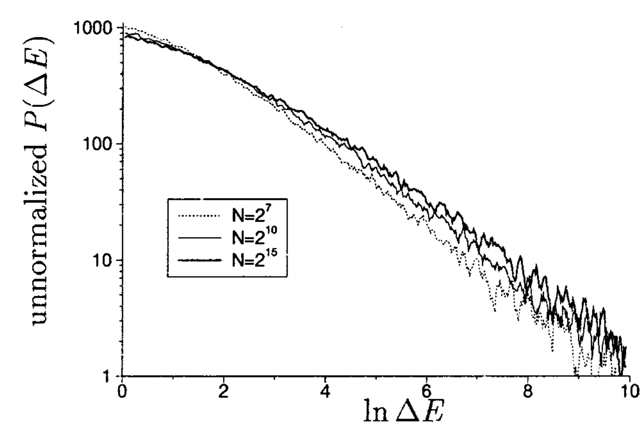

FIG. 9. Distribution of ΔE in log scale.

III. RENORMALIZATION-GROUP APPROACH

A. Idea of the method

We now study the models (1) and (6) using a renormalization approach introduced by us to study $d=2$ disordered XY models [19,20]. There, one is led to study a neutral collection of interacting ± 1 charges (XY vortices) in a random potential $\pm V(\mathbf{r})$ with Eq. (6). The single-particle problem studied here amounts to restricting the Coulomb gas RG of [19,20] to the sector of a single $+1$ charge. Here, however, there is no charge neutrality and one must be careful to study a system of finite size L, as some quantities [such as $\overline{V(\mathbf{r})^2}$] explicitly depend on L, while appropriately defined quantities have a well-defined thermodynamic limit.

The idea is first to formulate the problem in the continuum, with a short distance cutoff a,

$$Z=\int \frac{d^d\mathbf{r}}{a^d}e^{-\beta V(\mathbf{r})}, \tag{16}$$

and an appropriately defined cutoff-dependent distribution for $V(\mathbf{r})$, and second, by coarse graining infinitesimally, to relate the problem defined with a cutoff $a'=ae^{dl}$ to the problem with a cutoff a. In general, this implies being able to follow under this transformation the full probability measure of the potential $V(\mathbf{r})$, which is quite difficult, as complicated correlations can be generated under coarse graining. In some very favorable cases, for instance in the $d=1$ Sinai landscape [where $V(r)$ performs a random walk as a function of the r case $\delta=1$], it is possible to follow analytically an asymptotically *exact* RG transformation (in the statics and in the dynamics [45]). There a very specific real-space decimation procedure is required, which can in principle be extended here, although it may not be tractable beyond numerics. The present case of the logarithmically correlated potential is thus *a priori* less favorable but still, thanks to some known properties of the Coulomb potential, a RG method in the manner of Kosterlitz can be constructed which, we argue, should describe correctly all the universal properties of the model. There are two possible derivations, one which uses replicas and is more precise, and the other one without. We start with the latter, which is physically more transparent.

The key observation is that before (and also after) coarse graining, the logarithmically correlated disorder studied here can naturally be decomposed into two parts as

$$V(\mathbf{r})=V^{>}(\mathbf{r})+v(\mathbf{r}), \tag{17}$$

where $V^{>}(\mathbf{r})$ is a smooth Gaussian disorder with the same LR correlations as the initial $V(\mathbf{r})$ which represents the contribution of the scales larger than the cutoff ae^{l}, and $v(\mathbf{r})$ is a local short-range random potential which represents the contribution of scales smaller than, or of the order of, the cutoff ae^{l}. In the starting model $v(\mathbf{r})$ appears naturally as a Gaussian variable (see below). After coarse graining, $v(\mathbf{r})$ *does not* remain Gaussian, but it *does* remain uncorrelated in space (i.e., correlations of short range a). The decomposition (17) allows us to follow the distribution of the $V(\mathbf{r})$ under coarse graining in a tractable way.

The precise way of decomposing the disorder in Eq. (17) depends on the details of the cutoff procedure, but should not matter as far as universal properties are concerned. For illustration, let us indicate a simple way to do it; a more detailed discussion is given in [20]. It starts with the well-known continuum approximation in $d=2$ of the lattice Coulomb potential $\tilde{\Gamma}(\mathbf{r}-\mathbf{r}')\approx 4\sigma[\ln(|\mathbf{r}-\mathbf{r}'|/a)+\gamma][1-\delta^{(a)}(\mathbf{r}-\mathbf{r}')]$, where $\delta^{(a)}(\mathbf{r}-\mathbf{r}')=1$ for $|\mathbf{r}-\mathbf{r}'|<a$ and 0 otherwise [$\gamma=\ln(2\sqrt{2}e^{C})$ and $C=0.5772$ is the Euler constant]. This decomposition can be performed more generally, e.g., with other short-distance regularization of the potential $\tilde{\Gamma}(\mathbf{r})$ (which preserves the large distance logarithmic behavior) and in any d, which amounts to modifing the value of γ. Using this approximation, the bare disorder (6) can indeed be rewritten equivalently as a sum (17) of two Gaussian disorders $V^{>}(\mathbf{r})$ and $v(\mathbf{r})$ with no cross correlations and with respective correlators:

$$\overline{[V^{>}(\mathbf{r})-V^{>}(\mathbf{r}')]^2}=4\sigma\ln\frac{|\mathbf{r}-\mathbf{r}'|}{a}[1-\delta^{(a)}(\mathbf{r}-\mathbf{r}')], \tag{18}$$

$$\overline{v(\mathbf{r})v(\mathbf{r}')}=2\sigma\gamma\delta^{(a)}(\mathbf{r}-\mathbf{r}'). \tag{19}$$

With this definition, the problem to be studied is rewritten as

$$Z=\int\frac{d^d\mathbf{r}}{a^d}z(\mathbf{r})e^{-\beta V^{>}(\mathbf{r})},\quad z(\mathbf{r})=e^{-\beta v(\mathbf{r})}. \tag{20}$$

We can now study the behavior of the model under a change of cutoff. There are two main contributions from eliminated short length scale variables. The first one can be seen most simply by rewriting the correlator in Eq. (18),

$$\overline{[V^{>}(\mathbf{r})-V^{>}(\mathbf{r}')]^2}=4\sigma\ln\frac{|\mathbf{r}-\mathbf{r}'|}{a'}[1-\delta^{(a')}(\mathbf{r}-\mathbf{r}')]$$
$$+4\sigma\, dl[1-\delta^{(a')}(\mathbf{r}-\mathbf{r}')], \tag{21}$$

explicitly as the sum of a new LR disorder correlator with cutoff $a'=ae^{dl}$ and a SR disorder correlator [we have discarded terms of order $O(dl^2)$]. Thus the original problem with cutoff a can be rewritten as one with cutoff a' with (i) a new Gaussian LR disorder with identical form of the correlator (18) with a replaced by a' and (ii) a new short-range disorder $v(\mathbf{r})\rightarrow v(\mathbf{r})+dv(\mathbf{r})$ with $\overline{dv(\mathbf{r})dv(\mathbf{r}')}=2\sigma\, dl\;\delta^{(a)}(\mathbf{r}-\mathbf{r}')$ since it is clear from Eq. (21) that when $a\rightarrow ae^{dl}$ the LR disorder produces an additive *Gaussian* contribution dv to the SR disorder.

The second contribution resulting from a change of cutoff is that neighboring regions will merge. Points $\mathbf{r}_1$ and $\mathbf{r}_2$ previously separated as $a<|\mathbf{r}_1-\mathbf{r}_2|<ae^{dl}$ should now be considered as within the same region. The second important observation is that the resulting transformation can only affect the SR part $v(\mathbf{r})$ of the disorder. Indeed, in the region $a<|\mathbf{r}_1-\mathbf{r}_2|<ae^{dl}$ the LR part $V^{>}(\mathbf{r})$ can be considered as constant up to higher-order terms of order dl. One must view this coarse graining as resulting in a "fusion of local environments:" the two local partition sum variables $z(\mathbf{r}_1)$ and $z(\mathbf{r}_2)$ combine into a single one $z(\mathbf{r})$ according to a rule which we will write as $z(\mathbf{r})=z(\mathbf{r}_1)+z(\mathbf{r}_2)$. The exact choice of the form of this fusion rule is again dependent on the cutoff procedure and thus to a large extent arbitrary.

Putting together these two contributions, we obtain the following RG equation for the distribution $P_l(z)$ of the local disorder $z=e^{-\beta v}$ variable (also called "fugacity" in the Coulomb gas context):

$$\partial_l P(z)=\beta^2\sigma(1+z\partial_z)^2P-dP(z)$$
$$+d\int_{z'z''}P(z')P(z'')\delta(z-(z'+z'')). \tag{22}$$

This equation also describes the evolution of the universal part of the total free-energy distribution with the system size. Indeed, the total partition function can be written at any scale as

$$Z(\beta)=\int\frac{d^d\mathbf{r}}{a^d}e^{-\beta V(\mathbf{r})}\approx\int\frac{d^d\mathbf{r}}{(ae^{l})^d}z_l(\mathbf{r})e^{-\beta V_l^{>}(\mathbf{r})}\approx z_{(l*)}, \tag{23}$$

where the $z_l(\mathbf{r})$ are independent variables distributed with $P_l(z)$ and the $V_l^{>}(\mathbf{r})$ are Gaussian distributed as Eq. (18). In the last equality we have coarse grained up to the system size: $L=ae^{l*}$. At this scale, there remains a single site of (random) fugacity z_{l*}. Thus the distribution function of the partition function $Z(\beta)$ can be deduced from the distribution of the random fugacities at scale $l*$. The distribution of the free energy $F=-T\ln Z$ is thus given by $\tilde{P}_{l*}(v=F)$ [where $\tilde{P}(v)dv=P(z)dz$ from the change of variable from z to $v=-T\ln z$]. Note that the $\approx$ in Eq. (23) means that these distributions are the same *a priori* only up to subdominant nonuniversal terms (multiplicative for Z and additive for $\ln Z$).

For a fixed system size L, the above RG equation describes the evolution with the scale l smaller than that $l*$ of the distribution of $z(\mathbf{r})$, which is the local partition sum over scales around $\mathbf{r}$ smaller than or equal to ae^{l} [i.e., of a "local free energy" $-T\ln z_l(\mathbf{r})=v_l(\mathbf{r})$]. The remaining long-wavelength disorder at that scale, $V_l^{>}(\mathbf{r})$, should still be taken into account when computing the total partition sum.

It is striking that Eq. (31) is identical to the RG equation for the partition function of a continuum version of a directed polymer on a Cayley tree (a so-called branching process [8]). We note that it has been derived here for a problem with complete (statistical) translational invariance, with no ad hoc assumption about an underlying tree structure and simply adapting to the present problem the Coulomb gas renormalization in the manner of Kosterlitz. That the correspondence between the two problems naturally appears within the RG with no additional assumptions is even more apparent on the derivation using replicas of the next section. Thus we consider that this establishes on a firm footing the strong connection between the two problems.

Before analyzing the consequences of the above RG equation, let us sketch the more precise derivation using replicas. Other derivations without replicas are also possible and we refer the reader to [20] for more details.

B. Derivation of the RG equation using replicas

Let us consider the whole set of moments $\overline{Z^m}$ which encode for the distribution function $P[Z]$. They can be written as

$$\overline{Z^m}=\int \frac{d^d\mathbf{r}_1}{a^d}\cdots\frac{d^d\mathbf{r}_m}{a^d}e^{(\beta^2/2)\overline{[\Sigma_{i=1,\ldots,m}V(\mathbf{r}_i)]^2}}. \tag{24}$$

This can be rewritten as

$$\overline{Z^m}=\int \frac{d^d\mathbf{r}_1}{a^d}\cdots\frac{d^d\mathbf{r}_m}{a^d}e^{-(\beta^2/4)\Sigma_{i\neq j=1,\ldots,m}\tilde{\Gamma}(\mathbf{r}_i-\mathbf{r}_j)}\times e^{m^2\sigma\beta^2\ln(L/a)}. \tag{25}$$

We have used that $\Gamma(\mathbf{r},\mathbf{r})=\Gamma_L(0)=2\sigma\ln(L/a)$. One can choose a regularisation, e.g., $\Gamma(\mathbf{r}-\mathbf{r}')=\overline{V(\mathbf{r})V(\mathbf{r}')}=-\sigma\ln(|\mathbf{r}-\mathbf{r}'|^2+a^2)/L^2$. Notice that only the large distance behavior of the above correlator is important for the following renormalization.

We now switch to another representation of the replica partition sum. Equation (25) is a partition sum of m particles located at $\mathbf{r}_1,\ldots,\mathbf{r}_m$ corresponding to m replicas. Now instead we will index the configurations using (vector) *columnar replicated charges*. To each point $\mathbf{r}$, within a hard core size a, we associate an m-component vector $\mathbf{n}$ whose components $n^i(\mathbf{r})$ are either 1 or 0 depending on whether the particle corresponding to the ith replica is present within a of $\mathbf{r}$ ($|\mathbf{r}-\mathbf{r}_i|<a$) or not. These charges thus correspond to $\mathbf{n}=(0,1,0,\ldots,0,1,1)$ since several replicas can be present near a given point. Choosing a columnar hard core for the vector charges corresponds to a choice of cutoff, which is arbitrary, but the universal features of the renormalization should not depend on it [47].

The mth moment of $P[Z]$ then reads

$$\overline{Z^m}=\left(\frac{L}{a}\right)^{\beta^2\sigma m^2}\sum_{\{n_\alpha^i\}}{}'\prod_\alpha Y[\mathbf{n}_\alpha]\int_{|\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}|\geq a}\frac{d^d\mathbf{r}_\alpha}{a^d}\times\exp\left[-2\beta^2\sigma\sum_{\alpha<\alpha'}n_\alpha n_{\alpha'}\ln\left(\frac{|\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}|}{a}\right)\right], \tag{26}$$

where the primed sum correspond to a sum over all distinct nonzero configurations of replica charges $\mathbf{n}_\alpha$ at sites $\mathbf{r}_\alpha$. We have defined $n_\alpha=\Sigma_i n_\alpha^i$ as the total number of replicas present in a given charge ($n_\alpha^i=1$). The quantities $Y[\mathbf{n}]$ are functions of the local vector charge and are the so-called vector charge fugacities. In the bare model they appear as soon as the continuum approximation to the lattice Green function is used and read $Y[\mathbf{n}]=e^{-2\sigma\gamma n^2}$. Since we are studying a single-particle problem, there is also an important global constraint on the configuration sum that only one particle in any replica i be present in the system, i.e.,

$$\sum_\alpha n_\alpha^i=1, \tag{27}$$

which is preserved by the RG.

The RG equations for this model read

$$\partial_l Y[\mathbf{n}]=(d+\beta^2\sigma n^2)Y[\mathbf{n}]+\frac{S_{d-1}}{2}\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}}Y[\mathbf{n}']Y[\mathbf{n}''], \tag{28}$$

where the sum is over $\mathbf{n}'$ and $\mathbf{n}''$ nonzero vector charges (also $\mathbf{n}$ is nonzero) and S_{d-1} is the volume of the unit sphere in dimension d. We recall that $n=\Sigma_{i=1}^m n_i$. These equations are obtained by a generalization of the Kosterlitz procedure [48] as follows. The first term comes from an explicit cutoff dependence in Eq. (26). Upon increasing the cutoff infinitesimally $a\to a'=ae^{dl}$, the integration measure and the a dependence in all logarithms combine to give $Y[\mathbf{n}_\alpha]\to Y[\mathbf{n}_\alpha]e^{dl(d+\sigma\beta^2 n_\alpha^2)}$. We have used that $2\Sigma_{\alpha<\alpha'}n_\alpha n_{\alpha'}=m^2-\Sigma_\alpha n_\alpha^2$, which holds due to Eq. (27). The last term in the above equation (28) comes from the fusion of replica charges upon increase of the cutoff. The above RG equations hold for any m.

We should now look for solutions of this set of equations analytically continued to $m\to0$. One way to do that is to find a convenient parametrization for the set of $Y[\mathbf{n}]$. Here we preserve replica permutation symmetry within the RG and we can thus choose $Y[\mathbf{n}]$ to be a function of $n=\Sigma_i n_i$ only. Then we define the parametrization $Y[n]=\int dz\,\Phi_l(z)z^n=\int du\,\tilde{\Phi}_l(v)e^{-\beta n v}$. The different terms in Eq. (28) then translate into

$$n^2Y[\mathbf{n}]=\int dv\,e^{-\beta n v}(\beta^{-1}\partial_v)^2\tilde{\Phi}_l(v), \tag{29}$$

$$\sum_{\mathbf{n}^{(1)}+\mathbf{n}^{(2)}=\mathbf{n}}Y[\mathbf{n}_1]Y[\mathbf{n}_2]=\int_{z',z''}\Phi_l(z')\Phi_l(z'')\delta(z-z'-z'')-2\mathcal{N}\Phi_l(z)+\delta(z)\mathcal{N}^2, \tag{30}$$

where $\mathcal{N}=\int_z\Phi_l(z)$. One then easily converts the equation for $\Phi_l(z)$ into an equation for a normed function $P_l(z)=\Phi_l(z)/\mathcal{N}_>$ defined only for $z>0$, with $\mathcal{N}_>=\int_{z>0}\Phi_l(z)$ (see [20] for details), by noting that $\mathcal{N}_>$ converges quickly to $\mathcal{N}_>=2d/S_{d-1}$. The resulting equation for $P_l(z)$ is exactly the one (22) given above, and its physical interpretation in terms of the probability distribution of the fugacity (i.e., the local partition sum) was given in the preceding section.

What is the small parameter which controls the validity of the above RG equations (with and without replicas)? In a conventional Coulomb gas context, these RG equations are known to become exact in the dilute limit of nonzero (vector) charges [48]. It is easy to see that this corresponds to the tail of the distribution $P(z)$ for large z (or equivalently small v). This is further confirmed, *a posteriori*, by the remarkable universality properties of the resulting nonlinear RG equation (22), analyzed in the following section, which arises precisely in this region of z. So to obtain the universal behavior (e.g., of the distribution of free energy), we are working with sufficient accuracy. On the other hand, the bulk of the distribution $P_l(z)$ seems to be sensitive to details of the cutoff procedure (e.g., details in the fusion rule), and as discussed below it is thus likely (unless proven otherwise) to be nonuniversal.

C. Analysis of RG equation and results

1. KPP front propagation equation and velocity selection

Let us analyze the solutions to the RG equation (22). In terms of the (local free) energy variable $v(\mathbf{r})=-T\ln z(\mathbf{r})$ [from Eq. (20) and its distribution $P_l(v)=P_l(z=e^{-\beta v})\beta e^{-\beta v}$] it has a well-defined zero-temperature limit, since then the fusion rule simply becomes the extremal rule $v'=\min(v_1,v_2)$ leading to

$$\partial_l P(v)=\sigma\partial_v^2 P+dP(v)\left(-1+2\int_v^{+\infty}P(v')dv'\right). \tag{31}$$

To be able to work at all temperatures, it is in fact useful to trade the distributions $P_l(z)$ or $P_l(v)$ for the generating function [8,49]:

$$G_{l;\beta}(x)=\langle e^{-ze^{\beta x}}\rangle_{P_l(z)}=\langle e^{-e^{\beta(x-v)}}\rangle_{P_l(v)}. \tag{32}$$

We will sometimes drop the index β. At zero temperature, the double exponential becomes a θ function and $G_l(x)$ simply identifies with the distribution function:

$$G_{l;\beta=+\infty}(x)=\int_x^{+\infty}P_l(v)dv=\text{Prob}(v>x) \tag{33}$$

and for all β it is a decreasing function of x with $G_l(x\to-\infty)=1$ and $G_l(x\to+\infty)=0$. Note the asymptotic behavior [50] at very large negative x, $1-G_l(x)\sim\langle z\rangle_{P_l}e^{\beta x}$. The temperature appears only via the initial condition [8] and the problem at hand is thus to determine the large l behavior of $G_l(x)$ for a given initial condition.

Equation (31) is easily transformed, at all temperatures, into the Kolmogorov (KPP) nonlinear equation

$$\frac{1}{d}\partial_l G(x)=\frac{\sigma}{d}\partial_x^2 G+F[G], \tag{34}$$

$$F[G]=-G(1-G), \tag{35}$$

which describes the diffusive invasion of a stable state $G=0$ into an unstable one $G=1$. This class of equations admits a family of *traveling wave solutions* $G_l(x)=g[x+m(l)]$ which describe a *front* moving towards negative x and located around $x\sim-m(l)$. This is readily seen by plugging this form in Eq. (34), and assuming that $\partial_l m_\beta(l)\to c$ one obtains the equation for the front shape:

$$\frac{1}{d}cg'(x)=\frac{\sigma}{d}g''(x)+F[g(x)]. \tag{36}$$

The family of such traveling-wave solutions $g_c(x)$ can thus be parametrized by the velocity c. Equation (36) simplifies for large negative x when $g\approx1$. Denoting $\tilde{g}=1-g$ and using that $F[g]\sim-\tilde{g}$ for $g\approx1$, one finds the linearized front equation for $\tilde{g}_c$:

$$\frac{1}{d}c\tilde{g}'=\frac{\sigma}{d}\tilde{g}''+\tilde{g}. \tag{37}$$

This equation allows us to relate the speed of the front c to the asymptotic decay of the front, since if $\tilde{g}(x)\sim e^{\alpha x}$ for large negative x one finds

$$\frac{c}{d}=\frac{\sigma}{d}\alpha+\frac{1}{\alpha}. \tag{38}$$

The problem at hand now is to determine toward which of these front solutions $g_c(x)$ will $G_l(x)$ converge at large l, and thus what will be the asymptotic front velocity. This velocity will determine the intensive free energy of the original problem. Indeed, the convergence at large l of the solutions of nonlinear equations of the type (34) (with a general $F[G]$) towards one of such front solutions, and the corresponding problem of the selection of the front velocity c, is a famous problem, still under current interest in nonlinear physics [51–55].

The simplest argument is to use the fact that for very large negative x, one must have $\tilde{g}(x)\sim e^{\beta x}$ and thus $\alpha=\beta$. This seems to imply that the front velocity is

$$c=c(\beta)=\left(\frac{\sigma}{d}\beta+\frac{1}{\beta}\right)d. \tag{39}$$

This, however, is not always true. First note that the curve $c(\beta)$ has two branches, i.e., in this naive estimate two different β would correspond to the same velocity. The special point $\beta_c=\sqrt{d/\sigma}$ corresponds to $c=c^*=2d\sqrt{\sigma/d}$. For more general nonlinear equations one usually relies on the so-called marginal stability criterion (e.g., which shows that the large β branch is unstable and can be eliminated) [51,8]. Here there are rigorous results available: the Bramson theorem [56] ensures the following results, which are *independent* of the precise form of $F[G]$ (up to some rather weak conditions on $F[G]$ [56]).

(i) At high temperature, $\beta<\beta_c=\sqrt{d/\sigma}$, the asymptotic front is indeed an exponential for large negative x and $G_l(x)$ uniformly converges towards the traveling-wave solution

$g_{c(\beta)}[x+m(l)]$, where the velocity is given by Eq. (39), thus it is continuously dependent on temperature.

(ii) At low temperature, $\beta \geqslant \beta_c$, the velocity *freezes* to the value $c=c^*$ and the front decays as

$$\tilde{g}(x) \sim -x e^{\beta_c x} \tag{40}$$

for large negative x, thus independent of the temperature. The solution $G_l(x)$ uniformly converges towards the traveling-wave solution $g_{c^*}[x+m(l)]$. Thus in that regime, one must then distinguish two regions in $G_l(x)$ at large l, the front region and the region very far ahead of the front $[x+m(l) \gg \sqrt{l}]$ where the decay is again as $G_l(x) \sim \exp(\beta x)$ as it should be: this will be discussed again below.

There are additional rigorous results from [56] and in particular the remarkable fact that not only the velocity but also the *corrections to the velocity* are *universal* (independent of $F[G]$), i.e., one has for the position of the traveling wave $m_\beta(l)$ at ''time'' l,

$$m(l) = \left(\frac{\sigma}{d}\beta + \beta^{-1}\right) dl + Cst, \quad \beta < \beta_c = \sqrt{\frac{d}{\sigma}}, \tag{41a}$$

$$m(l) = \sqrt{\frac{\sigma}{d}}\left(2\ dl - \frac{1}{2}\ln l\right), \quad \beta = \beta_c, \tag{41b}$$

$$m(l) = \sqrt{\frac{\sigma}{d}}\left(2\ dl - \frac{3}{2}\ln l\right), \quad \beta > \beta_c. \tag{41c}$$

2. Results for the fugacity and free-energy distribution and extremal statistics

These results on the KPP equation (34) can now be translated [via Eq. (32)] into results for the fugacity distribution $P_l(z)$ and for the distribution of free energy $\tilde{P}_l(v)$. One finds that $P_l(z)$ and $\tilde{P}_l(v)$ also take the form of a front at large l, e.g.,

$$\tilde{P}_l(v) \to p(v+m(l)) \tag{42}$$

with $p(v')$ related to $g(x)$ by $g(x) = \int_{v'} p(v') e^{-e^{\beta(x-v')}}$. Thus we obtain that the local free energy is

$$-\beta^{-1}\langle \ln z \rangle \sim -m_\beta(l) \tag{43}$$

up to a finite constant, where the position of the front $m_\beta(l)$ is given above in Eq. (41a). Using the result (23), $N = d\ln(L/a) = dl^*$, we obtain using the RG that the free energy $F[V]$ of the system of size L reads

$$F[V] = f_L(\beta) d \ln L + \delta F, \tag{44}$$

where δF is a fluctuating part of $O(1)$ of probability distribution $p(\delta F)$ and the intensive free energy reads

$$f_L(\beta) = -\left(\frac{\beta}{\beta_c^2} + \frac{1}{\beta}\right) + O\left(\frac{1}{\ln L}\right), \quad \beta < \beta_c = \sqrt{\frac{d}{\sigma}}, \tag{45a}$$

$$f_L(\beta) = -\frac{1}{\beta_c}\left(2 - \frac{1}{2}\frac{\ln(\ln L)}{d \ln L}\right) + O\left(\frac{1}{\ln L}\right), \quad \beta = \beta_c, \tag{45b}$$

$$f_L(\beta) = -\frac{1}{\beta_c}\left(2 - \frac{3}{2}\frac{\ln(\ln L)}{d \ln L}\right) + O\left(\frac{1}{\ln L}\right), \quad \beta > \beta_c, \tag{45c}$$

where the factors $\frac{1}{2}$ and $\frac{3}{2}$, which arise in the finite-size corrections, are *universal*.

Thus we have found using our RG method that in any dimension $d \geqslant 1$ the original models (1) and (6) exhibit a phase transition at $\beta = \beta_c(d)$. This transition is very similar to the freezing transition of the continuous version of the random directed polymer on the Cayley tree. Our RG thus confirms that the REM approximation (10) to the model does give the transition at the same β_c, and with same asymptotic intensive free energies (11b) as Eq. (45c). It allows, however, for a more detailed study and shows that the universal finite-size corrections differ in the two models. In the REM, the above formula with the factor $\frac{1}{2}$ holds in all the low-temperature phase, which is not the case for the present model. Thus the present model is in a different universality class than the REM. The physics that we find here is much closer to the one of the directed polymer on the Cayley tree: it remains to be seen whether this can be extended to other observables.

The RG method also yields the distribution of the $O(1)$ fluctuating part δF of the free energy, and in particular at $T=0$ it gives a result for the extremal statistics of the correlated variables. We must now carefully distinguish between what is clearly universal (and thus for which we can be confident that the RG approach gives the exact result) and what may not be (as it depends on the details of the cutoff procedure, yielding, e.g., a different KPP nonlinearity $F[G]$).

Let us start with $T=0$. We find [cf. Eqs. (44) and (45c)] that the minimum $V_{\min}$ of L^d logarithmically correlated variables behaves as

$$V_{\min} = -2\sqrt{\sigma d}\ln L + \frac{3}{2}\sqrt{\frac{\sigma}{d}}\ln(\ln L) + \delta V \tag{46}$$

and δV is a fluctuating part of order $O(1)$. Since at $T=0$ one has $p(v) = \tilde{g}'(v)$, from the result (40) we get that the tail of the distribution of $u = \delta V - \langle \delta V \rangle$ for $u \to -\infty$ is universal and behaves as

$$p(u) \sim -u e^{\beta_c u} \tag{47}$$

with $\beta_c = \sqrt{d/\sigma}$. Thus we find a distribution *different from the Gumbell distribution*, and thus correlations do matter.

The question of what is universal in this distribution is nontrivial. We find from our method that the full distribution of $P(u)$ depends on the detailed form of the front (and thus on $F[G]$ and *a priori* on the cutoff procedure) and is thus less likely to be universal (although this remains to be investigated). Hence we believe that universal features include *at least* the tail of the distribution (47).

The above result (47) carries through the tail of the distribution of the free energy $u=F-\langle F\rangle$ for $u\to-\infty$ for $T<T_c$ and it was shown in [8] that for $T>T_c$ one has

$$p(u)\sim e^{u\beta_c^2/\beta},\quad \beta<\beta_c. \tag{48}$$

D. More on fronts, REM via nonlinear RG, and extremal statistics

To illustrate how the previous results fit in a broader context, let us show how the simpler properties of extremal statistics of uncorrelated variables and of the random energy model can be recovered within the same RG framework. This provides, in passing, yet another solution of the REM.

1. Uncorrelated variables with fixed distribution: Gumbell via RG

Let us consider $N=e^{ld}=(L/a)^d$ independent random variables $V(r)$ $r=1,\ldots,N$ with a fixed distribution $P(V)$ (d here does not play any role as the true variable is ld but we keep it for the sake of comparison). The generating function of the distribution of the partition function $Z[V]=\Sigma_r e^{-\beta V(r)}$ of model (1) reads

$$G_l(x)=\langle \exp(-Z[V]e^{\beta x})\rangle_{P(V)} = \left(\int dV P(V)\exp(-e^{\beta(x-V)})\right)^{e^{ld}}. \tag{49}$$

It satisfies the equation

$$\frac{1}{d}\partial_l \ln\ln\frac{1}{G}=1. \tag{50}$$

Or, interestingly enough, it obeys a KPP-type equation with no diffusion term:

$$\frac{1}{d}\partial_l G=F[G], \tag{51}$$

$$F[G]=G\ln G. \tag{52}$$

The Gumbell distribution now emerges naturally from the front solutions of this equation. Writing $G_l(x)\sim g(\alpha_l(x+m_l))$ and assuming $\partial_l(\alpha_l m_l)\to c$ yields $cg'=g\ln g$, whose solutions with the above boundary conditions are $g(y)=\exp(-\gamma e^{y/c})$ (γ being a positive constant). We have assumed $\partial_l\alpha_l\to 0$. Since there is some freedom of choice for α_l and m_l, one can always set $c=\gamma=1$. The determination of the rescaling factors α_l and $m(l)$ is performed in Appendix C. At $T=0$ one has $P(V_{\min})=-G'(V_{\min})$ and one recovers the known results from probability theory for the convergence to the Gumbell distribution detailed in Appendix B, but the generating function $G_l(x)$ takes a Gumbell form also at finite T.

2. REM via RG

We now turn to an alternative derivation of the solution to the Gaussian REM model using a RG approach and a traveling-wave analysis. This allows us to make some connections with the correlated case studied previously. Let $l=\ln L$ and $\ln N=ld$.

We want to write a RG equation for

$$G_l(x)=(\langle e^{-e^{\beta(x-V)}}\rangle_{P_l(V)})^{e^{ld}}, \tag{53}$$

where the single site distribution $P_l(V)$ is now scaled with l. We introduce

$$\tilde G_l(x)=\langle e^{-e^{\beta(x-V)}}\rangle_{P_l(V)}=\exp(e^{-ld}\ln G_l(x)). \tag{54}$$

Let us choose the single site distribution $P_l(V)$ which corresponds to the REM approximation (10) of the model studied here [(1) and (6)], i.e., the Gaussian:

$$P_l(V)=\frac{1}{\sqrt{4\pi\sigma l}}e^{-V^2/4\sigma l}. \tag{55}$$

It satisfies

$$\partial_l P_l(V)=\sigma\partial_V^2 P_l(V). \tag{56}$$

One easily checks that it implies that

$$\partial_l\tilde G_l(x)=\sigma\partial_x^2\tilde G_l(x). \tag{57}$$

This leads to the equation for $G_l(x)$:

$$\partial_l G=\sigma\partial_x^2 G+dG\ln G-\sigma(1-e^{-ld})\frac{1}{G}(\partial_x G)^2. \tag{58}$$

Thus the RG equation of the REM, for large l, reads

$$\partial_l G=\sigma\partial_x^2 G+dG\ln G-\sigma\frac{1}{G}(\partial_x G)^2 \tag{59}$$

and is almost a KPP equation, except that it has an additional gradient (KPZ-type) term. This term here plays an important role and yields a different universality class from KPP. We now search for the front solutions.

Let us rewrite the exact equation (58) using the function $h=-\ln G$ (remember that $0<G<1$):

$$\partial_l h=dh+\sigma h''+\sigma e^{-ld}h'^2. \tag{60}$$

For large l we can neglect the decaying nonlinear part, and we now look for a solution of the linear equation. The only front solution of the form $h(x)=\tilde h(x+m(l))$ with $\partial_l m(l)\to c$ which satisfies the boundary conditions $h(-\infty)=0$ and $h(+\infty)=+\infty$ is the exponential

$$h_l(x)=e^{\alpha(x+m(l))}, \tag{61}$$

$$\partial_l m(l)=c=\frac{d}{\alpha}+\sigma\alpha. \tag{62}$$

By using again the $h_l(x)\sim e^{\beta x}$ boundary condition at $x\to-\infty$, we find $\alpha=\beta$ and

$$c(\beta)=\frac{d}{\beta}+\sigma\beta \tag{63}$$

as in Eq. (39). This is correct in the high-T phase and yields the correct REM value for the intensive free energy $f(\beta)=c(\beta)/d+O(1/\ln L)$ as in Eq. (11b) (and also correctly yields the absence of nontrivial finite-size corrections). Thus for the REM in the high-T phase we find

$$G_l(x)\approx\exp(-e^{\beta(x+m(l))}) \tag{64}$$

thus again a Gumbell form, with $\alpha_l=\beta$ and $m(l)=[(d/\beta)+\sigma\beta]l$.

To see the transition to a low-T phase for $\beta\geqslant\beta_c=\sqrt{d/\sigma}$ and the freezing of the velocity at $c=c^*=2\sqrt{d\sigma}$, one needs to carry a slightly more detailed analysis (discarding again the decaying nonlinear part). The general solution of the linear part of Eq. (60) is

$$h_l(x)=\int dx'\frac{1}{\sqrt{4\pi\sigma l}}e^{ld-[(x-x')^2/4\sigma l]}h_0(x'), \tag{65}$$

where $h_0(x')$ can be interpreted as the $h_l(x')$ at earlier time l_0 such that the nonlinear terms can already be neglected and decay as $h_0(x')\sim e^{\beta x'}$ for $x'\to-\infty$.

This formula nicely exhibits the REM transition. In the high-T phase, using the asymptotic form $h_0(x')\sim e^{\beta x'}$ we find that there is a saddle point at $x'=x+2\sigma\beta l$. This gives $h_l(x)\sim e^{\beta(x+c(\beta)l)}$ with $c(\beta)$ given in Eq. (63). The front $h_l(x)$ is centered at $x^*=-c(\beta)l$ and consistency requires that the corresponding saddle point x'^* moves to $-\infty$ so that the asymptotic form of $h_0(x')$ can indeed be used. Hence we have $x'^*\sim[\sigma\beta-(d/\beta)]l$. Thus the saddle point becomes inconsistent and the high-T solution ceases to hold, for $\beta\geqslant\beta_c=\sqrt{d/\sigma}$.

The solution in the low-T phase is easy to find. Setting $x=-m(l)+y$ one finds for large l

$$h_l(y)\sim e^{ld-(1/4\sigma l)m(l)^2-(1/2)\ln(4\pi\sigma l)}e^{(c^*/2\sigma)y}\times\int dx'e^{-(c^*/2\sigma)x'}h_0(x'), \tag{66}$$

where we have denoted $c^*=\lim_{l\to+\infty}m(l)/l$ and neglected the additional factor $e^{-x'^2/(4\sigma l)}$ in the integral. This is correct provided the integral

$$\int dx'e^{-(c^*/2\sigma)x'}h_0(x') \tag{67}$$

is convergent, i.e., $c^*<2\beta\sigma$. The consistent choice for c^* and $m(l)$ must be

$$c^*=2\sqrt{\sigma d},\quad m(l)=\sqrt{\frac{\sigma}{d}}\left(2ld-\frac{1}{2}\ln(4\pi\sigma l)\right)+O(1), \tag{68}$$

which ensures that Eq. (66) has a proper limit $h_l(y)\sim Ae^{(c^*/2\sigma)y}=Ae^{\beta_c y}$, which is again a Gumbell form for $G_l(x)$ but now is temperature-independent. This holds for $\beta\geqslant\beta_c=\sqrt{d/\sigma}$.

From this method of solving the REM we have recovered the result of [6], namely that for $\beta\geqslant\beta_c$ the free energy behaves as

$$f_L(\beta)=-\frac{m(l)}{dl}=-\frac{1}{\beta_c}\left(2-\frac{1}{2}\frac{\ln(\ln L)}{d\ln L}\right)+O\left(\frac{1}{\ln L}\right). \tag{69}$$

In addition, we recover for $T=0$ the result for the minimum $V_{\min}$ in the REM approximation:

$$V_{\min}=-2\sqrt{\sigma d}\ln L+\frac{1}{2}\sqrt{\frac{\sigma}{d}}\ln(\ln L)+\delta V \tag{70}$$

with $u=\delta V-\langle\delta V\rangle$ distributed with a Gumbell distribution:

$$\text{Prob}(u>x)=\exp(-Ae^{\beta_c x}), \tag{71}$$

where A is a constant.

3. Conclusion on RG fronts and extremal statistics

Thus we have seen in two examples that extremal statistics problems (and their $T>0$ thermodynamic model counterpart) can be studied using the nonlinear RG equation with traveling-wave solutions. In one example (uncorrelated rescaled variables, i.e., the REM) the RG equation is exact, while in the second (logarithmically correlated variables) we only know it presumably in the tails. The front position represents the typical value of the minimum $V_{\min}$ as a function of $l=d^{-1}\ln N$ while the shape of the front gives the distribution of the $V_{\min}$ (respectively of the free energy F). This suggests that a broader class of such models can be approached by these methods, and raises the question of universality.

Studies of such nonlinear equations [55] usually distinguish between pushed fronts where the velocity relaxes exponentially in l (velocity selection by nonlinear terms) and pulled fronts (velocity selection by the marginal stability criterion). The extremal statistics (and the glassy phase) correspond to the pulled fronts. There one expects a very broad universality as stressed in [53,54]: not only is the asymptotic front universal, but also the velocity and its corrections. In a nutshell, the argument for the universal $\frac{3}{2}\ln l$ corrections to the front position comes from matching of the universal tail of the front $g(y)\sim(Ay+B)e^{\beta_c y}$ with $y=x+m(l)$ with the far tail region, so far ahead of the front that one can linearize the KPP equation and get

$$1-G_l(x)\approx e^{\beta_c y}\psi(y),\quad \partial_l\psi_l(y)=\sigma\partial_y^2\psi_l(y). \tag{72}$$

The only matching solution is $\psi_l(y)=y/l^{3/2}e^{-y^2/(4\sigma l)}$. Inserting $y=x+m(l)=x+c^*l+C\ln l$ immediately yields $C=\frac{3}{2}$ for proper matching. As discussed in [54], this universality extends for *pulled fronts* in a very broad class of nonlinear (or coupled nonlinear) equations and holds for steep enough initial conditions (i.e., in the glass phase in our language).

This argument fails in some cases, such as at the bifurcation between pushed and pulled fronts (e.g., at the glass transition $\beta=\beta_c$ or equivalently when the initial condition has slow decay $\sim\exp(\beta_c x)$] (see, e.g., the analysis in [52]). Interestingly, it clearly fails also for the nonlinear equation corresponding to the REM model, which is thus in a different universality class (this may be related to the fact that fronts are unbounded here [57]). Presumably what happens there is that the coefficient A vanishes, and the solution is exactly $e^{\beta_c y}$, hence the $\frac{1}{2}\ln l$ [since the above matching function is now $\psi(y)\sim l^{-1/2}e^{-y^2/(4\sigma l)}$].

Next is the question of universality. We will address it only for our model of Gaussian variables with logarithmic correlations. We have recast the RG equation (22) into a KPP equation with a specific nonlinear term $F[G]$. From our RG we have obtained $F[G]=-G(1-G)$. The structure of the RG derivation suggests that we have obtained correctly the two lowest orders of $F[G]$. From the above discussion this is enough for the universality. Thus, and we call it the restricted universality scenario, it is likely that higher-order terms $F[G]=-G(1-G)+O((1-G)^3)$ are nonuniversal and thus that only the tail of the distribution of the minimum of log-correlated variables is universal.

Let us mention, however, that we were not able to rule out another scenario, the broad universality scenario, such that the true distribution of the minimum of log-correlated variables is indeed universal. If this were true, the following conjecture would be tempting: since we know that for *uncorrelated* variables the KPP RG equation is exact with $F[G]=G\ln G$ and $\sigma=0$ (and is asymptotically exact even for weakly correlated ones—see Appendix B), one could conjecture an interpolating KPP equation (34) with $F[G]=G\ln G$ and $\sigma>0$, which would give exactly the distribution of the minimum of log-correlated variables. Unfortunately we have been unable to confirm (but also to strictly rule out) numerically this conjecture, due to the very large finite-size corrections, as discussed in Sec. IV.

E. Structure of low-temperature phase and replica symmetry breaking

Let us now return to the structure of the low-temperature phase for the particle in the d-dimensional random potential with logarithmic correlations. We argue that (i) it has a nontrivial structure, with a few states, and (ii) this structure is reminiscent of the so-called ''replica symmetry breaking'' [5]. This nontrivial structure can be characterized more precisely here as the various states of the model correspond to the different positions of the particle, and have thus a natural meaning in real space. In particular, the minima of the ''energy landscape'' (or metastable states) are nothing but the local minima in the sample of the random potential for our problem. A precise characterization of these ''local minima'' is given below. Also, approximate replica solutions of our model are shown in the following to exhibit RSB at low T.

1. Spatial distribution of secondary minima

Let us start with a simple argument: for a given realization of disorder, we divide our system into two subsystems of size $L^d/2$, and call $V_{\min 1}$ and $V_{\min 2}$ the two corresponding minima in each subsystem.

Within the REM approximation, we know from Eq. (13) that $V_{\min 1}-V_{\min 2}\sim(y_1-y_2)\sqrt{\sigma/d}\sim O(1)$, where y_1 and y_2 have independent Gumbell distributions. Thus clearly in that case there is a nontrivial structure: the secondary minimum (defined as being constrained to lie within the other subsystem) is typically within $\Delta E=O(1)$ in energy of the absolute minimum (and within this approximation the distribution is also easily computed).

The RG analysis performed in this paper indicates that adding correlations will not change this conclusion. Indeed, one first coarse-grains up to scale $l_0=\ln(L)-(1/d)\ln 2$. At this scale, the system can be described by two local energies (one for each half) of minima v_1 and v_2 distributed according to $P_{l_0}(v)$, to which should be added a term δV which correlates the two halves and is Gaussian of variance $\sim(2\sigma/d)\ln 2$. This, however, does not change the fact that the difference $V_{\min 1}-V_{\min 2}\sim O(1)$. Thus one still finds that there exist secondary minima of $O(1)$ in energy from the minimum, and a typical distance L away from the absolute minimum. As discussed in Sec. II D, this property was also confirmed by numerical simulations.

It is natural, in view of the analogy with the directed polymer on the Cayley tree, to introduce the ''overlap'' between two different states (i.e., positions of particles) $\mathbf{r}_1$ and $\mathbf{r}_2$ as

$$q(\mathbf{r}_1,\mathbf{r}_2)=1-\frac{\ln(a+|\mathbf{r}_1-\mathbf{r}_2|)}{\ln L}. \tag{73}$$

We expect it to be non-self-averaging and characterized by the ''overlap distribution:''

$$P_2(q)=\overline{\sum_{\mathbf{r}_1,\mathbf{r}_2} p(\mathbf{r}_1)p(\mathbf{r}_2)\,\delta(q-q(\mathbf{r}_1,\mathbf{r}_2))}. \tag{74}$$

Although we have not attempted to compute this function directly using our RG, it is natural to expect that, as in the REM and the DPCT, it is nontrivial for $T<T_c$ and reads

$$P_2(q)=\frac{T}{T_c}\,\delta(q)+\left(1-\frac{T}{T_c}\right)\delta(1-q). \tag{75}$$

Similarly one expects that in a given disorder environment, the probability of finding an overlap q between two thermal realizations becomes in the large-L limit

$$\tilde{Y}(q)\,dq=(1-Y)\,\delta(q)+Y\,\delta(q-1) \tag{76}$$

with $\overline{Y}=1-T/T_c$ and Y has the same distribution as in the REM. Thus the natural expectation, from the DPCT analogy, is that the overlap in the low-T phase will be either 1 or 0 [i.e., secondary minima—of energy difference of order T—will be either near the absolute one $\ln r_{12}/\ln L\to 0$, or a distance $r_{12}\sim O(L)$ typically a fraction of the system size away]. It would, however, be of interest to investigate further these properties in the present model, in particular to obtain more detailed information at intermediate scales, e.g., correlations probing the whole range $\ln r_{12} \sim (\ln L)^a$ with $0 \leqslant a \leqslant 1$.

2. Approximate replica symmetry breaking solutions of the model

Let us now turn to the replica representation and discuss how the present model exhibits a form of ''replica symmetry breaking.'' The replicated partition sum reads

$$\overline{Z^m} = \int \frac{d^d \mathbf{r}_1}{a^d} \cdots \frac{d^d \mathbf{r}_m}{a^d} e^{-2\sigma\beta^2 \Sigma_{i<j} \ln|\mathbf{r}_i - \mathbf{r}_j|/a} e^{m^2 \sigma \beta^2 \ln(L/a)}. \tag{77}$$

It turns out that various approximations of this partition function (specifically the REM and the DPCT approximations) are dominated, in the limit $m \to 0$, by replica symmetry breaking configurations.

In the context of $2d$ Dirac fermions with random vector potential (see Sec. VI), an estimate of Eq. (77) was given in [26]. For small β it is clear that the exponential containing the logarithmic attraction between replicas does not decay fast enough and thus the integral is dominated by the configurations where the replicas are all $O(L)$ far apart, thus

$$\overline{Z^m} \sim \left(\frac{L}{a}\right)^{\beta^2 \sigma m^2 + dm - \beta^2 \sigma m(m-1)} = \left(\frac{L}{a}\right)^{md(1+(\sigma/d)\beta^2)}. \tag{78}$$

This estimate of Ref. [26] is in fact incorrect as it misses the glass transition. Indeed, one can redo this argument using configurations where m/p packets of p replicas are $O(L)$ far apart [while in each packet the replicas (independent particles) are close to each other]. This estimate was performed in Ref. [59] and gives instead

$$\overline{Z^m} \sim \left(\frac{L}{a}\right)^{\beta^2 \sigma m^2 + d(m/p) - \beta^2 \sigma m(m-p)}. \tag{79}$$

The interaction term is proportional to the number of pairs of replicas in different packets, which is $m(m-p)/2$. In the limit $m \to 0$, one can then optimize over $0 < u = p < 1$, i.e.,

$$\overline{Z^m} \sim \exp\left[d \ln \frac{L}{a} \max_{0<u<1} \left(\frac{1}{u} + \frac{\sigma}{d} \beta^2 u \right) \right]. \tag{80}$$

For $\beta < \beta_c = \sqrt{d/\sigma}$ the saddle is for $p = 1$ and one recovers the above expression. For $\beta > \beta_c = d/\sigma$ one finds that the saddle is for $u = \beta_c/\beta = T/T_c$, which gives

$$\overline{Z^m} \sim e^{d \ln(L/a) 2(\beta/\beta_c)}. \tag{81}$$

Thus this calculation yields a transition. In Ref. [59] it was claimed that it does not correspond to replica symmetry breaking. We believe that this is incorrect and that this *is* a (one-step) RSB estimate of the above partition sum. This is clear since this calculation exactly amounts to the corresponding one for the REM approximation of the model, i.e., replacing in Eq. (78) $\Sigma_{i<j} \ln|\mathbf{r}_i - \mathbf{r}_j|/a$ by $\Sigma_{i<j}(1 - \delta_{r_i, r_j}) \ln(L/a)$. In the REM we know from Ref. [6] that the correct solution for $T < T_c$ can be obtained by performing the analytical continuation to $m \to 0$ on a RSB saddle point (note that the REM finite-size correction $\frac{1}{2} \ln \ln L$ is also obtained from the saddle integration).

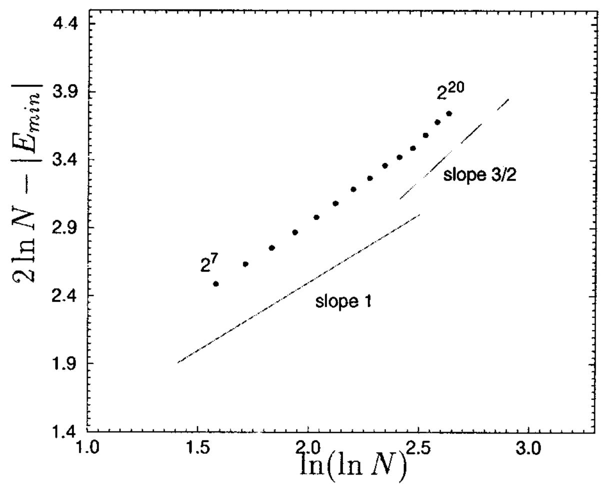

FIG. 10. Zero-temperature limit: finite-size corrections to the minimal energy. Plotted is $2l - |V_{\min}|$ versus $\ln l (l = \ln N)$.

One can go one step further and use an argument based on universality, which puts the present problem in the DPCT universality class (for some observables such as the free-energy distribution). For the DPCT, it was shown in Ref. [10] that one can also recover the correct result for the averaged free energy by considering directed polymer configurations which break replica symmetry as $m \to 0$. It remains to be demonstrated how to obtain other universal quantities, e.g., the $\frac{3}{2} \ln \ln L$ finite-size corrections, via a RSB saddle-point calculation.

It is interesting to see how the features associated to RSB arise from the RG developed here, despite the fact that it is *explicitly replica symmetric*. Quite generally, if one can find independent local free-energy variables with an exponential distribution $P(f) \sim e^{\beta_c f}$, one naturally obtains a RSB picture. This is the case here, up to some more detailed universal preexponential structure in $P(f)$. The important feature of our RG is thus that it follows the full distribution $P_l(z)$ of local disorder (i.e., of local Boltzmann weights z) which becomes algebraically broad as $l \to +\infty$. Here this property is sufficient to show that the low-T phase has a structure reminiscent of RSB. Indeed, let us again coarse-grain the system up to an already large scale $L_0 = a e^{l_0}$ but still much smaller than L, the ratio $L/L_0 = e^{l_1} = M$ being *large* but *fixed* as $L \to +\infty$, assuming that L_0 is so large that $P_{l_0}(z)$ has reached its fixed point already (except in a remote tail region corresponding to very rare events). Since one has the decomposition (17), the RG tells us that the sample is divided in M subsystems with free energies $F_i = v_i + V_i^>$, $i = 1, \ldots, M$ where the variables $z_{i=1,\ldots,M} = e^{-\beta v_i}$ are independently drawn from the common distribution $P_{l_0}(z)$ and the $V_i^>$ are still correlated but Gaussian. Neglecting first the $V_i^>$, we are

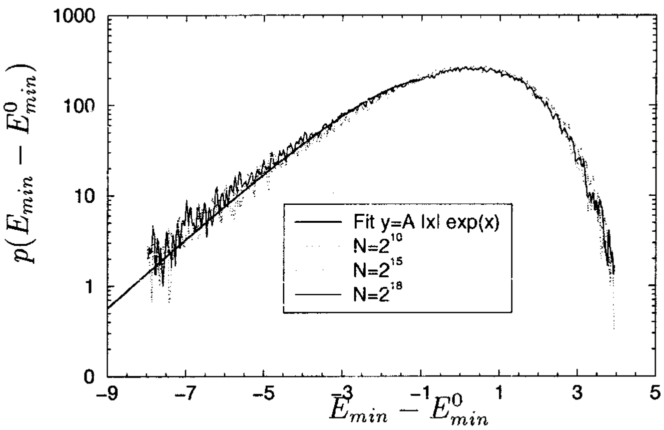

FIG. 11. Distribution of $E_{\rm min}$.

left with a system of M subsystems of Gibbs measure:

$$\frac{z_i}{\sum_j z_j}. \tag{82}$$

Since the z_i are drawn from a distribution with algebraic tails $P(z)\sim 1/z^{1+\mu}$ with $\mu = T/T_c$, one has $\langle z\rangle = +\infty$ for $T<T_c$ and, as is well known, the partition sum (82) is dominated by *a few* of the z_i variables [9,58] (which in essence is the physics associated to RSB). Since the correlated $V^>$ variables are in finite number and with Gaussian tails, they cannot change the exponential tails of the F_i and thus adding them back should not change the above conclusions.

Thus here, although the RG is replica symmetric, since it allows for generation of broad tails, it can capture features usually associated with RSB.

IV. NUMERICAL STUDY

Since we found via the RG and other arguments that there should be a transition in any dimension $d\geqslant 1$, it is particularly convenient to perform numerical simulations in the ''extreme case'' of $d=1$ (i.e., the farther away from mean field). However, even in $d=1$ numerical simulations are delicate because the finite-size corrections are very large (and interesting to study, in order to distinguish various universality classes). Indeed we have found that the main numerical uncertainties come from the finite-size effects and not from the number of averages. In most of the numerical work, averaging over $\sim 10^4$ realizations of disorder was sufficient, while a simulation of a system of size $2^{21}\sim 2\times 10^6$ leads to important corrections to the thermodynamic behavior of the model. In view of this, we believe that the previous numerical investigation [11] was at best approximate.

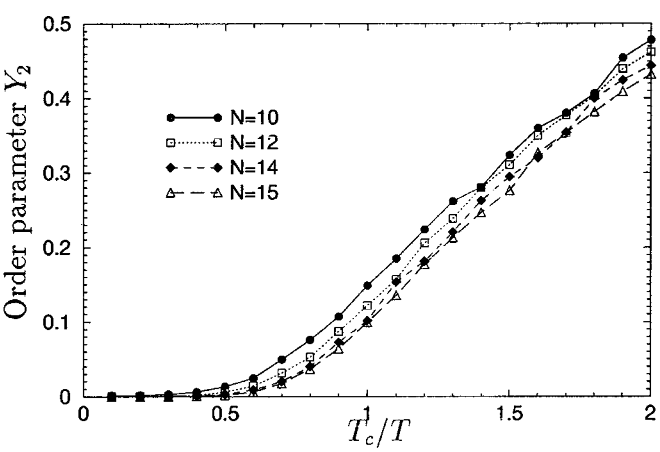

FIG. 12. Plot of $Y_2=\overline{\Sigma_r p_r^2}$ as a function of temperature for different system sizes $L=2^N$.

We have considered a lattice model in $d=1$ with $L=2^n$ sites. The potential $V(r)$ on each site $(r=1,\ldots,L)$ was computed from its Fourier components $V(r)=w_{L/2}(-1)^r + \Sigma_{k=1}^{L/2-1} w_k \cos(2\pi kr/L-\phi_k)$, eliminating the $k=0$ mode, with w_k independent Gaussian variables $\overline{w_k w_{k'}} = \Delta(k)\delta_{k,k'}$ $(k,k'=1,\ldots,L/2)$ and each ϕ_k independently distributed uniformly in $[0,2\pi]$. We choose $\Delta(k)$ such that

$$\Gamma(r-r') = \overline{V(r)V(r')} = \sigma\frac{2\pi}{L}\sum_{k=1}^{L-1}\frac{\cos\left(\frac{2\pi k}{L}(r-r')\right)}{\left|\sin\left(\frac{\pi k}{L}\right)\right|\sqrt{6-2\cos\left(\frac{2\pi k}{L}\right)}} \tag{83}$$

so that $\overline{[V(r)-V(r')]^2} = 4\sigma\ln(r-r')$ for $1 \ll r-r' \ll L/2$. This is the choice which also corresponds to correlations along the axis $y=0$ on a $2d$ square lattice.

The behavior of the model has been studied, without loss of generality, at zero and at finite temperature for a disorder strength $\sigma=1$ (other values of σ can be incorporated in the definition of the temperature scale). We have first computed the average minimum $e_{\rm min} = \overline{V_{\rm min}}/\ln N$ (with $N=L$) for system sizes ranging from $L=2^7=128$ to $L=2^{21}\sim 10^6$ and for each size we have taken the average over 10^4 realizations of disorder. An estimate of the uncertainty on the disorder average was made by measuring the variance of a series of average over 10^4 realizations. This variance was found to be of the order of 10^{-3} for all the value of $\overline{V_{\rm min}}$. The results are plotted in Fig. 10. We recall that the RG prediction reads for $\sigma=1$

$$\frac{1}{\ln N}|\overline{V_{\rm min}}| = 2\ln N - \frac{3}{2}\ln(\ln N) + O(1). \tag{84}$$

We should first note that if one does not assume *anything* about the finite-size corrections, the resulting uncertainty on the ratio $e_{\rm min} = \overline{V_{\rm min}}/\ln N$ is very large even for sizes $L = 2^{21}$ since the ratio $\frac{3}{2}\ln(\ln N)/\ln N \approx 0.3$. Hence with no assumption it is hard to estimate $e_{\rm min}$ to better than 10% accuracy.

However, if one assumes that $e_{\rm min} = -2$, the plot in Fig. 10 shows the existence of the $\ln(\ln N)$ corrections with a slope definitely larger than 1 and consistent with $\frac{3}{2}$ (although the accuracy is not excellent). It is, however, sufficient to rule out a REM-type behavior and is consistent with the RG prediction (84).

Next, we have plotted the distribution of $V_{\min}$ in Fig. 11 and compared with the prediction of the RG for the tails. Here also the agreement is satisfactory.

Finally, we have plotted the ''glass order parameter'' $\overline{Y_2 = \Sigma_r p_r^2}$ which is nonzero when the system is dominated by a few states (see Fig. 12). It is consistent with a very slow convergence towards $Y_2=(1-T/T_c)\theta(T_c-T)$ but clearly other forms cannot be ruled out.

V. RELATIONS WITH LIOUVILLE AND SINH-GORDON MODELS

In this section we describe the relation between the problem of the particle in the log-correlated random potential and the Liouville and sinh-Gordon models. Exact results on the sinh-Gordon model are compatible with (and also point out towards) the existence of the transition at $\beta=\beta_c$.

A. Relations with the sinh-Gordon model in $d=2$ and $d=1$

Let us start with the correspondence with the sinh-Gordon model. Although less direct, it is also simpler to analyze, as the model does not contain subtle boundary condition problems. The interesting thing about the connection is that the sinh-Gordon model is integrable in $d=2$ and $d=1$ (boundary sinh-Gordon) [60–62].

The connection requires introducing a slightly different version of the initial problem, defined by the partition function,

$$Z_{\rm sh}[V]=Z[V]+Z[-V]=\sum_{\mathbf{r}}\ (e^{-\beta V(\mathbf{r})}+e^{\beta V(\mathbf{r})}), \quad (85)$$

which corresponds to a particle in a random potential which can explore both $V(\mathbf{r})$ and $-V(\mathbf{r})$. A physical realization would be a particle with an Ising spin in a random field. As it turns out, the physics of this disordered model is very similar to the original problem. At low temperature, it is now related to the distribution of the minimum of $-|V(\mathbf{r})|$.

We define the generating function of this model $G_{\rm sh}(x)=\langle\exp(-\mu Z_{\rm sh}[V])\rangle$, with $\mu=e^{\beta x}$, which is related to the distribution of the free energy of the particle. In the continuum limit and in $d=2$, it can be rewritten as

$$G_{\rm sh}(x)=H_{\rm sh}[\mu]=\int DVe^{-S_{\rm sh}[V]}, \quad (86)$$

$$S_{\rm sh}[V]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx\ dy\left(\frac{1}{8\pi\sigma}(\nabla V)^2+2\mu\cosh(\beta V(\mathbf{r}))\right),$$

i.e., the partition function of the sinh-Gordon model in $d=2$. Similarly, the $d=1$ version of our model is related to the well-studied boundary sinh-Gordon model [61] defined as

$$G_{{\rm sh}\,B}(x)=H_{{\rm sh}\,B}[\mu]=\int DVe^{-S_{{\rm sh}\,B}[V]}, \quad (87)$$

$$S_{{\rm sh}\,B}[V]=\int_0^{+\infty}dy\int_0^L dx\left(\frac{1}{4\pi\sigma}(\nabla V)^2 + 2\mu\cosh(\beta V(x,0))\right). \quad (88)$$

Indeed one has, as required, that $[V(x,0)-V(x',0)]^2\sim 4\sigma\ln|x-x'|/a$ at large $|x-x'|$, and one only studies (boundary) observables defined at $y=0$.

In the limit of $\beta=+\infty$ one has in both cases

$$G_{\rm sh}(x)={\rm Prob}(x<\min(V_r,-V_r)), \quad (89)$$

$$={\rm Prob}(x<-\max_r|V_r|), \quad (90)$$

and thus the (properly discretized) partition function of the (boundary) sinh-Gordon model becomes related, in that limit, to the distribution function of the maximum of the set of positive random variables $|V(\mathbf{r})|$. The results described in the preceding sections about the statistics of extrema of such variables imply that some transition must occur as a function of β corresponding to a related ''change of behavior'' in the sinh-Gordon and boundary sinh-Gordon models as well. This is a prediction, as we are not aware of such a change of behavior at $\beta=\beta_c$ being mentioned in the literature. As we now discuss, examination of known results is perfectly compatible with the transition at $\beta=\beta_c$.

Let us first describe the known exact results both in $d=2$ and $d=1$. The extensive free energy of the bulk sinh-Gordon model is defined as

$$f_{\rm sh}=\lim_{L\to+\infty}-L^{-2}\ln G_{\rm sh}, \quad (91)$$

where the model defined in Eq. (87) is considered in finite size L. The model is studied usually using the field $\phi=V\sqrt{2/\sigma}$, the nonlinear term being $2\mu\cosh(\beta V)=2\mu\cosh(b\phi)$ and its free energy depends on the single variable $b=\beta\sqrt{\sigma/2}=\beta/\beta_c$, where $\beta_c=\sqrt{d/\sigma}$ is dimension-dependent. Using the variable b, its exact expression, proposed in Ref. [60], reads when explicited [63],

$$f_{\rm sh}(\mu)=C_2(b)\mu^{(1/1+b^2)}, \quad (92a)$$

$$C_2(b)=\frac{2\pi}{\left(\Gamma\left[\frac{1}{2+2b^2}\right]\right)^2\left(\Gamma\left[1+\frac{b^2}{2+2b^2}\right]\right)^2\sin\left(\frac{b^2\pi}{1+b^2}\right)}\times\left(\frac{\pi\Gamma[1+b^2]}{-\Gamma[-b^2]}\right)^{1/(1+b^2)}. \quad (92b)$$

These results are *a priori* only valid for $b<1$ $(|b|<1)$, as they were obtained in [60] from an analytical continuation of the sine-Gordon model (performing $\mu\to-\mu$ and $b^2\to-b^2$, M being the soliton mass). The constant μ was defined in the continuum model by fixing the normalization of the field $\langle \cos(b\phi(\mathbf{r}))\cos(b\phi\mathbf{r}')\rangle = \frac{1}{2}|x-y|^{-4b^2}$ of the sine-Gordon model.

The $d=1$ version corresponds to the boundary sinh-Gordon model usually studied using $\phi = V/\sqrt{\sigma}$ and $2\mu\cosh(\beta V) = 2\mu\cosh(b\phi)$, with again $b=\beta/\beta_c$ ($\beta_c = 1/\sqrt{\sigma}$). The analogous expression for the free energy reads, from [61],

$$f_{\mathrm{sh}\,B}(\mu) = \lim_{L\to+\infty} -L^{-1}\ln G_{\mathrm{sh}\,B} = C_1(b)\mu^{(1/1+b^2)}, \quad (93)$$

$$C_1(b) = \frac{1}{8\pi^{3/2}}\Gamma\left[\frac{1+2b^2}{2+2b^2}\right]\Gamma\left[\frac{-b^2}{2+2b^2}\right]\left(-\frac{2\pi}{\Gamma[-b^2]}\right)^{1/(1+b^2)}.$$

Let us now comment on these results. The power-law dependence in μ of the free energy is just the naive dimensional result $\sim\mu^{1/(1+b^2)}$ in both cases. This result should hold for $\beta<\beta_c$. However, there is clearly, in both $d=2$ and $d=1$ cases, a singularity as $\beta\to\beta_c^-$ as the amplitude $C(b)$ diverges as $b=\beta/\beta_c\to 1^-$. This is thus in perfect agreement with the existence of a phase transition in the particle model. In the sinh-Gordon model itself, we do not expect strictly speaking a phase transition, as the model is massive both below and above $b=1$, however we do expect some ‘‘change of behavior,’’ which may be related to a change of nature of the excitations around the ground state. This is not ruled out by exact results [64] as it clearly comes here from the physical mass acquiring a nontrivial dependence in the bare mass parameter μ (contrary to the sine-Gordon model, for the sinh-Gordon model there is no presently known exact solution of a lattice version).

Let us now interpret these results for our model. They mean that the generating function $G_{\mathrm{sh}}(x)$ of the free-energy distribution, with $\mu=e^{\beta x}$, takes indeed the form of a traveling wave:

$$G_{\mathrm{sh}} \sim \exp\left(-L^2 C_d\left(\frac{\beta}{\beta_c}\right)\mu^{2/[1+(\beta/\beta_c)]^2}\right) = g(x+cl+\gamma) \quad (94)$$

with $l=\ln L$ and a velocity

$$c = \frac{d}{\beta} + \sigma\beta. \quad (95)$$

This is exactly the velocity given by the KPP equation for the particle model, in the high-temperature phase. It also yields a front $g(y)=\exp(-e^{ay})$ with $a=\beta/[1+(\beta/\beta_c)^2]$ and $\gamma=\beta/[1+(\beta/\beta_c)^2]\ln C_d(\beta/\beta_c)$. This form, however, should be taken with caution as strictly speaking formula (94) is valid only in the limit where L goes to infinity first (at fixed $\mu=e^{\beta x}$). It should be compared with the asymptotic behavior of the front in the region of large positive y. We expect universality in the other region of the front (of very negative y, i.e., $x\ll cl$) and exact knowledge about this region would be equivalent to exact knowledge of the sinh-Gordon model at finite size, which is not yet available.

The physics of the problem of the particle in the random potential leads us to conjecture that the $2d$ sinh-Gordon model (as well as the boundary sinh-Gordon model) will exhibit a change of behavior; the algebraic μ dependance of its free energy will freeze for $\beta\geqslant\beta_c$, which corresponds to the low-temperature glassy phase of the particle model. We thus expect

$$f_{\mathrm{sh}}(\mu)\sim\mu^{\alpha}, \quad (96a)$$

$$\alpha = \frac{1}{1+(\beta/\beta_c)^2}, \quad \beta<\beta_c=\sqrt{d/\sigma}, \quad (96b)$$

$$\alpha = \frac{1}{2}, \quad \beta>\beta_c=\sqrt{d/\sigma}, \quad (96c)$$

and presumably log corrections (at least at $\beta=\beta_c$, and maybe for all $\beta>\beta_c$).

This is confirmed by a renormalization-group analysis directly on the sinh-Gordon and Liouville models discussed below.

B. Relation with the Liouville model in $d=2$

The relation between our original model (1) of the particle in the random potential and the Liouville model proceeds via the generating function,

$$G(x) = \langle \exp(-e^{\beta x}Z[V])\rangle_V = \left\langle \exp\left(-\sum_{\mathbf{r}} e^{\beta(x-V(\mathbf{r}))}\right)\right\rangle_V, \quad (97)$$

which encodes the full probability distribution of the free energy of the particle. In the case of the $d=2$ potential with logarithmic correlations it is identical to the partition function of a Liouville model, which one can write either on the original lattice or in the continuum (with uv and ir cutoffs a and L) as ($\mu=e^{\beta x}$):

$$G(x) = H[\mu] = \int DVe^{-S[V]},$$

$$S[V] = \int d^2\mathbf{r}\left(\frac{1}{8\pi\sigma}[\nabla V(\mathbf{r})]^2 + \mu e^{-\beta V(\mathbf{r})}\right), \quad (98)$$

where the functional integral is normalized such that $H[\mu=0]=1$ (equivalently one redefines $H[\mu]\to H[\mu]/H[\mu=0]$). We call it the Liouville model (LM) since it is important to distinguish it from the *continuum* Liouville field theory (LFT) whose (formal) definition is recalled below. A relation also exists between the correlation functions of the Gibbs measure and some correlation functions in the Liouville model:

$$\langle p(\mathbf{r}_1)\cdots p(\mathbf{r}_n)\rangle = \int_{\mu>0}\mu^{n-1}e^{-\beta(V(\mathbf{r}_1)+\cdots V(\mathbf{r}_n))}e^{-S[V]}. \quad (99)$$

Strictly speaking, the model (98) above is not well defined because of the zero mode $V(\mathbf{r})\rightarrow V(\mathbf{r})+w$ and must be complemented with boundary conditions. In the particle problem studied here, we have chosen periodic boundary conditions with the additional constraint $\Sigma_{\mathbf{r}}V(\mathbf{r})=0$ to pin the zero mode.

On the other hand, many results are known for the (related) continuum Liouville field LFT, of great interest in quantum gravity [65–68]. It is usually defined on an arbitrary genus h manifold with background metric g and associated curvature R by the action [70]

$$S_{\mathrm{LFT}}=\int d^2x\left(\frac{1}{4\pi}(\partial_a\phi)^2+\mu e^{2b_L\phi}+\frac{Q}{4\pi}R\sqrt{g}\phi\right) \tag{100}$$

in conventional notations. The (formal) correspondence to the LM notations above is via $\phi=-V/\sqrt{2\sigma}$ and $b_L=b=\beta/\beta_c$ (only for $b<1$, see below). The (standard) choice is $Q=b_L+1/b_L$, for which the theory is critical and has local conformal invariance (with a central charge $c_L=1+6Q^2=25+c$) [70]. It can also be formulated as the theory of (liquid) random surfaces [68,71], e.g., as random triangulations. There one defines the total area $A=\int_{\mathbf{r}}e^{-2b_L\phi(r)}$, which is merely the partition function $A=Z[V]$ of the particle problem, and studies the distribution $\mathcal{Z}(A)\sim e^{-\mu_c A}A^{\gamma_{\mathrm{string}}-3}$, which is merely $P(Z)$.

The particle model allows us to make precise statements on the Liouville model defined above. The LFT allows for exact calculations (e.g., of correlation functions) and in principle one could hope to translate those in the particule model (Gibbs measure correlations). The relation between the two, however, is rather subtle. For instance, the boundary conditions chosen in the particle problem would correspond to Liouville on a torus $h=0$, except that the additional pinning condition spoils it. We will thus not explore here all these intricacies but give a few general remarks, mostly about the behavior of the Liouville model under coarse graining.

First we know that $G_l(x)$ satisfies a RG equation of the KPP type. Thus upon coarse graining [i.e., as a function of $l=\ln(L/a)$, see Sec. III] the Liouville model partition function satisfies a KPP nonlinear RG equation. The corresponding front velocity gives the scaling of the partition function with μ. The glass transition, with freezing of the front velocity, corresponds exactly in the Liouville model to the transition between two regimes (the so-called $c=1$ barrier in the LFT).

(i) *Weak coupling Liouville*: $b_L=b=\beta/\beta_c<1$. In that regime there is no problem to define a continuum limit. The KPP nonlinear RG front solution of velocity $c(\beta)=2[(1/\beta)+(\beta/\beta_c^2)]\equiv c(b)$ yields $G_l(\mu)\sim\mu^{1/(1+b^2)}$ and in the Liouville model the scaling dimensions are the one obtained by naive dimensional counting, e.g.,

$$\mu\int d^2\mathbf{r}\;e^{-\beta V(\mathbf{r})}=\mu e^{\ln Z[V]}\sim\mu L^{\beta c(\beta)}=\mu L^{2(1+b^2)}. \tag{101}$$

Note that while the continuum LFT has a formal duality $b_L\rightarrow 1/b_L$, the discrete LM naturally selects, as $l\rightarrow+\infty$, the branch $b_L=b<1$. This phenomenon is analogous to the selection of one branch of the curve inverse to $c(\beta)$ in the KPP equation, where $b(b_L)$ plays the role of the spatial decay rate of solutions and c the scaling dimension.

The regime $b_L=b\leqslant 1$ is the one where the continuum LFT is well defined: there the role of the Q term in Eq. (100) is to shift the conformal dimension of the fields $e^{2\alpha\phi}$ to $\Delta(\alpha)=\alpha(Q-\alpha)$ [while the naive power counting dimensions in the LM are $\Delta(\alpha)=-\alpha^2$, see above] which renders the Liouville term $\int e^{2b\phi}$ *exactly marginal* (and thus the theory critical). It was argued in [12] that the LFT gives the correlations of the Gibbs measure, the operator $p(\mathbf{r})$ [Eq. (2)] corresponding to the Liouville field $e^{2b\phi}$, and is thus of conformal dimension $\Delta(b)=1$ [i.e., $p(\mathbf{r})\sim L^{-2}$]. A hint in favor of this conjecture was that the corresponding LFT conformal dimensions of the composite fields $p(\mathbf{r})^q\sim e^{2qb\phi}$ are simply $\Delta(q)=q(1+b^2-b^2q)$, which correctly reproduces the multifractal spectrum:

$$\int d^2\mathbf{r}\;p(\mathbf{r})^q\sim L^{2(1-\Delta(q))} \tag{102}$$

given in [11] and Sec. VI (in the weak disorder regime $q<q_c$). This is not a very strong test since the same multifractal spectrum can also be obtained within the LM model by considering the dimension of the normalized Gibbs measure (rather than the unnormalized one $e^{-\beta V}$). Indeed the effect of the Q term is to shift

$$e^{2bq\phi}\rightarrow L^{2bqQ}e^{2bq\phi}\sim Z[V]^{-q}e^{2bq\phi}. \tag{103}$$

To convincingly establish the conjecture of [12], the effect of the additional Q term should be checked on the many-point correlations [70], where it is rather more subtle, and further investigation is needed. In particular, the RG described here suggests by extension that the Gibbs measure correlations (or at least some limits of them) should also be computable within the DPCT model. This suggests a direct relation between LFT and DPCT, a check of which would be of great interest. Note also that a critical model, which mimics the effect of the Q term (as adding an average value to ϕ [70]), is studied in Appendix D.

(ii) *Strong coupling Liouville*: $b=\beta/\beta_c>1$. This corresponds to the glass phase for the particle which, interestingly, has a nontrivial structure. As is well known, there are serious difficulties in defining the continuum LFT in that regime. The Liouville parameter b_L is undefined and cannot equal b anymore [69]. Using what we know from the particle problem, we can gain some idea of what happens in the Liouville theory. First let us note that since $G(x)$ is such that for $\beta=+\infty$ and fixed L it is equal to the distribution function of the minimum of the set of $V(\mathbf{r})$,

$$G(x)=\mathrm{Prob}(x<\min_{\mathbf{r}}V(\mathbf{r})). \tag{104}$$

The (infinitely strong coupling) Liouville model can be recast as *an extremal statistics* problem in that limit. The partition sum $Z[V]$ of the particle model being dominated, for

$b>1$, by a few regions of space where $V(\mathbf{r})\sim V_{\min}$ [with little dependence in β, i.e., the Liouville wall becomes a hard wall for all $\beta<\beta_c$ with thickness of order $O(1)$], we expect this spatial heterogeneity to show up in LM as well. From what we have learned in the previous sections, we know that upon coarse-graining the following *effective Liouville model* action S_{eff} is generated:

$$G_l(x)=H_l[\mu]=\int D\tilde{V}\langle e^{-S[\tilde{V},z]}\rangle_{P(z)}, \tag{105}$$

$$S_{\text{eff},l}[V]=\int d^2\mathbf{r}\left(\frac{1}{8\pi\sigma}[\nabla\tilde{V}(\mathbf{r})]^2+z(\mathbf{r})e^{-\beta\tilde{V}(\mathbf{r})}\right), \tag{106}$$

i.e., *a new field* $z(\mathbf{r})$ *is dynamically generated*, and has short-range correlations *but* has a broad power-law distribution,

$$P(z)dz\sim z^{-1+(1/b)}, \tag{107}$$

while $\tilde{V}\equiv V^>$ is the smooth field introduced in Eq. (17). For $b<1$, this dynamically generated local field can be averaged out without changing significantly the action (note that even for $b<1$ it changes properties of operators $e^{-q\beta V}$ for $q>q_c$), while for $b>1$ it changes crucially the physics. One can define the effective Liouville potential $U[V]$ for the smooth field $\tilde{V}$ after averaging over the z field as

$$U_l[\tilde{V}]=-\ln\langle\exp(-ze^{-\beta\tilde{V}})\rangle_{P_l(z)}=-\ln G_l(x=-\tilde{V}), \tag{108}$$

the bare Liouville potential being $U[V]=\mu\exp(-\beta V)$. We can now use the front solution of the KPP equation (i.e., the scaling region in the large L/a limit) described in the previous sections. For $b<1$, since $\langle z\rangle_{P(z)}<+\infty$, we have that for large V

$$U_l[V]\approx c\mu e^{2(1+b^2)l}\exp(-\beta V) \tag{109}$$

and thus the coarse-grained potential is similar to the bare one. However, for $b>1$ one has for large V

$$U_l[V]\approx c\mu e^{4l}V\exp(-\beta V) \tag{110}$$

because of the broad distribution of the z field.

Since the $z(\mathbf{r})$ are highly heterogeneous on short scales, it is not surprising that a continuum limit is hard to obtain for $b>1$. These heterogeneities are linked to the structure of the glass phase reminiscent of replica symmetry breaking. It is tempting to conjecture that it may also be related to the branched polymer structure which appears in LFT for $b>1$, i.e., beyond the $c=1$ barrier [68], or to the spike instability [71] of fluid membranes.

Furthermore, let us notice that the LFT theory at $b=1$ is known to have two marginal operators whose dimensions are degenerate $e^{2b\phi}$ and $\phi e^{2b\phi}$. This is in exact parallel with the behavior of the KPP front solution, which develops at $b=1$ two degenerate linear eigenmodes $\exp(-\beta V)$ and $V\exp(-\beta V)$.

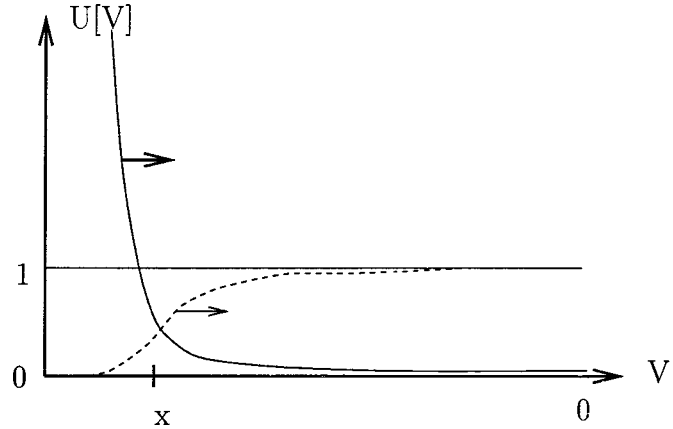

FIG. 13. Liouville wall moving under RG as a traveling wave. Represented is $U[V]$ on the negative V side, of original form $U[V]=e^{-\beta(V-x)}$ and also (dashed line) $G(x)=\exp(-U[V])$. Both move under RG forming a traveling front, whose velocity determines the ''free-energy'' exponent. For sinh-Gordon a second, mirror-image wall is also moving symmetrically towards 0. Freezing in the front velocity occurs at and below the transition at $\beta=\beta_c$.

Thus we have seen that the Coulomb gas RG can be used to understand the behavior of the Liouville model. A scenario is obtained where for $b\geqslant 1$ new short scale degrees of freedom are generated (short scale instability). Averaging over these changes the effective Liouville potential. The parallel with the particle model suggests that the short scale instability in Liouville may be related to the generation of strong inhomogeneities in the Gibbs measure $p(\mathbf{r})$, analogous to structures discussed in the context of replica symmetry breaking. Thus, if the mapping onto the LFT is confirmed, it suggests to also investigate RSB-type effects in strong-coupling LFT.

C. Direct renormalization-group analysis of sinh-Gordon and Liouville models and traveling waves

Let us now illustrate how one can see explicitly the freezing of the free-energy exponent in the strong-coupling phase from renormalization-group approaches *directly* on the sinh-Gordon and Liouville models. Such functional RG methods have been applied to study the analogous problem [31] of the wetting of an interface of height V. Related exact RG methods, with various truncation schemes, have also been applied to the Liouville model, and in the context of quantum gravity to the LFT [72]. In all cases we will illustrate how the main physics lies in the selection mechanism for the traveling-wave solutions of the nonlinear RG equation.

The study proceeds as follows. We consider

$$G(x)=H[\mu=e^{\beta x}]=\int DVe^{-S[V]}$$

$$\text{with}\quad S[V]=\int d^2\mathbf{r}\left(\frac{1}{8\pi\sigma}[\nabla V(\mathbf{r})]^2+U[V]\right). \tag{111}$$

One can perform a Wilson RG analysis (or, if one prefers, a suitably truncated exact RG analysis), and one finds in $d=2$ the flow for the local part $U_l[V]$ as

$$\partial_l U = 2U + \sigma U'' + O(U^2). \tag{112}$$

There may also be corrections to σ to $O(U^2)$ (in the sinh-Gordon model), but we focus for now on the RG to linear order. Let us recall that the initial condition is $U_{l=0}[V]=\mu e^{-\beta V}$ for Liouville and $U_{l=0}[V]=\mu e^{-\beta V}+\mu e^{\beta V}$ for sinh-Gordon, and that we are interested in the small μ limit. In this limit the initial condition corresponds to a very wide well $U[V]$ (e.g., in the sinh-Gordon model) with a very small curvature $U''[0]$. To obtain the free-energy exponent as $\mu\to 0$, one simply iterates the RG until $U_l''[0]\sim O(1)$ at a scale $l^*=\ln(L^*/a_0)$ (more precisely $U_l''[0]\sim 1/(\sigma a_0^2)$, where a_0 is the bare uv cutoff of the model). At this scale, the free energy is $O(1)$, as can be estimated from Gaussian fluctuations (straightforwardly at least in the SG model) and thus the initial free energy is

$$F\sim A_d(\beta)\left(\frac{L}{L^*}\right)^2. \tag{113}$$

Remarkably, it is now possible to use what we learned in the previous sections and demonstrate the ''freezing'' transition at $\beta=\beta_c$ (corresponding to the glass transition for the particle) simply from the RG to this order. Indeed the solution of the truncated equation is

$$U_l[V]=e^{2l}\frac{1}{\sqrt{4\pi\sigma l}}\int dV' \exp\left(-\frac{(V-V')^2}{4\sigma l}\right)U_{l=0}[V]. \tag{114}$$

A straightforward conclusion would then be that the exact solution corresponding to Liouville is

$$U_l[V]=\mu e^{(2+\sigma\beta^2)l}e^{-\beta V} \tag{115}$$

and similarly for the sinh-Gordon

$$U_l[V]=2\mu e^{(2+\sigma\beta^2)l}(e^{-\beta V}+e^{\beta V}) \tag{116}$$

since $\exp(\pm\beta V)$ are exact eigenvectors of the linear RG equation for any β. From Eq. (113) this immediately yields the ''naive dimensional'' result for the free energy,

$$F\sim L^2\mu^{1/[1+(\beta/\beta_c)^2]} \tag{117}$$

with $\beta_c=\sqrt{2/\sigma}$. As we know from the above exact result, this is correct for $\beta<\beta_c$. Note how the potential $U_l[V]$ evolves. Using the notation $\mu=e^{\beta x}$ (natural from our extremal statistics interpretation) it forms a ''Liouville wall,'' which can be seen as a ''front solution'' moving as $\exp(-\beta(V-x-cl))$ towards $U=0$ (and in the sinh-Gordon model there are two symmetric walls moving towards $U=0$ and reaching it at $l=l^*$) (see Fig. 13). The Liouville front velocity is

$$c=\frac{2}{\beta}+\sigma\beta, \tag{118}$$

which, plotted as a function of β, is the famous parabola, such that two values of β correspond to the same c, which is also a well-known property of Liouville theory.

As we now show, Eq. (115) is *incorrect* for $\beta\geqslant\beta_c$. This is so for a subtle reason, as apparently the statement that $\exp(-\beta V)$ is an exact eigenvector of the linear RG [and of Eq. (114)] cannot fail. However, by now we are well used to fronts: in fact we have encountered exactly the same equation in our previous solution of the REM model via RG [$h_l(x)$ in Eq. (65) is identical to $U_l(V)$ in Eq. (114)]. To describe correctly the bare Liouville (or equivalently the sinh-Gordon) model, one should generalize the initial condition $U_{l=0}[V]$, still assuming that $U_{l=0}[V]\sim\exp(-\beta(V-x))$ for $V\gg x$ (x here is very negative corresponding to a small μ). Then one can use the saddle-point method to estimate Eq. (114) as was done in Eq. (65) to evaluate $h_l(x)$ and one discovers that for $\beta>\beta_c$ the velocity freezes into

$$c=2\sqrt{2\sigma} \tag{119}$$

which yields a free energy

$$F\sim L^2\mu^{1/2} \tag{120}$$

instead of the naive dimensional estimate, thus in agreement with our expectation for the SG model (96c). In addition, we find that the decay of the renormalized potential $U_l[V]\sim e^{-\alpha V}$ is frozen at $\alpha=\beta_c$ for all $\beta>\beta_c$ consistent with Eq. (110).

What has happened is that although $U_{l=0}[V]=\exp(-\beta(V-x))$ is indeed formally an exact eigenvector, it is *dynamically unstable*, i.e., if one chooses another function with the same large positive $V-x$ behavior, one gets a different velocity (which is not the case for $\beta<\beta_c$). It is easy to see that the choice $U_{l=0}[V]=\exp(-\beta(V-x))$ exactly for all V does not make sense for $V\to-\infty$. Indeed it is immediately spoiled by the slightest amount of coarse graining (as would appear also by considering the nonlinearities in the RG equation). The simplest way to see it is to notice that the coarse-grained potential

$$\tilde{U}[V]=-\ln\left[\int dv \exp\left([-\mu e^{-\beta(V+v)}]-\frac{v^2}{2s}\right)\right] \tag{121}$$

does not grow as $\sim\exp(-\beta V)$ for large negative V but much slower as $\sim V^2$. To illustrate the point further, let us consider the initial condition,

$$U_{l=0}[V]=\frac{e^{-\beta(V-x)}}{1+e^{-\beta(V-x)}}. \tag{122}$$

It behaves as $e^{-\beta(V-x)}$ for large positive $V-x$ (and thus corresponds to the Liouville model) but goes to 1 on the other side. For $\beta=+\infty$ it is easy to compute $U_l''[V=0]$ from Eq. (114) since $U_{l=0}[V]=\theta(x-V)$. One finds

$$U_l''[V=0] \sim e^{2l-(x^2/4\sigma l)} \tag{123}$$

and thus one has that l^* defined above is such that

$$cl^* = x, \quad c = 2\sqrt{2\sigma}. \tag{124}$$

This is in fact valid for all $\beta > \beta_c$, as was shown in detail in the previous sections.

Thus the freezing transition can be obtained from the linearized (i.e., lowest-order) RG equations, using only elementary insight from coarse graining or the existence of higher-order nonlinear terms. It provides an interesting example where the naive dimensions hold in some regime but are modified in another. Of course, as we have seen in Sec. III D from the study of fronts, to really establish the transition and determine the universality class, one needs to consider higher-order nonlinearities in Eq. (112), which goes beyond the scope of this paper. For the LFT in quantum gravity, the reader can find some exact functional RG studies in Ref. [72]. Although not discussed in this reference, the nonlinear RG there seems to also exhibit traveling front solutions, whose physics may be important in understanding the problem of the $c=1$ barrier.

VI. CRITICAL DIRAC FERMIONS IN A RANDOM GAUGE FIELD

In this section we relate our RG study of the preceding section to the study of the critical wave functions of 2D Dirac fermions in a random magnetic field. We first confirm the results of [13] for the multifractal spectrum, and obtain their finite-size corrections. Then we study the transition from the weak disorder to the strong disorder phase, related to the glass transition for the particle, and find that the strong disorder phase has a new and nontrivial structure, leading to what we call *quasilocalized* eigenstates.

A. Critical wave function of 2D random Dirac

Let us first recall the problem of a massless two-dimensional Dirac fermion in a static random magnetic field [11,13,34]. This model, and its non-Abelian generalizations, has received a lot of attention in connection with the integer quantum Hall effect transitions with disorder. As discussed in [26], two-dimensional Dirac fermions can experience three generic types of disorder: random gauge, random mass, and random potential. Random gauge disorder is believed to be a line of fixed points in this general model and is still not yet fully understood. Here we address only the random gauge disorder model of a Hamiltonian:

$$H = \sigma_\mu (i v_F \partial_\mu - A_\mu(\mathbf{r})), \tag{125}$$

where the $\sigma_{1,2}$ are the 2×2 Pauli matrices and $\mu=1,2$ (we set the Fermi velocity $v_F=1$ from now on). The random magnetic field $\mathbf{B}$ corresponding to the gauge potential $\mathbf{A}$ is chosen to be Gaussian with mean value $\overline{\mathbf{B}(\mathbf{r})}=0$. The type of correlations studied here correspond to the most interesting case where the gauge potential has short-range correlations. In the Coulomb gauge, we can introduce the scalar potential ϕ such that $A_\mu = \epsilon_{\mu\nu}\partial_\nu \phi, B(\mathbf{r}) = -\partial_\mu^2 \phi(\mathbf{r})$. The Gaussian distribution of $\phi(\mathbf{r})$ is thus given by

$$P[\phi] = cte \times e^{-(1/4\pi g)\int_{\mathbf{r}}(\partial_\mu \phi(\mathbf{r}))^2}, \tag{126}$$

where g parametrizes the strength of the random magnetic field B. The correlator of the function $\phi(\mathbf{r})$ is thus

$$\overline{[\phi(\mathbf{r})-\phi(\mathbf{r}')]^2} \sim 2g \ln \frac{|\mathbf{r}-\mathbf{r}'|}{a}. \tag{127}$$

In this model, the wave functions at energy E are localized for all energies other than the critical energy $E=0$. We restrict our study to the $E=0$ critical eigenstate, which satisfy

$$H\Psi_0(\mathbf{r}) = 0. \tag{128}$$

For a system of finite size L with appropriate boundary conditions, there are two independent *normalized* solutions of Eq. (128): the first one can be written $\Psi_{0,1}(\mathbf{r}) = (\Psi_0(\mathbf{r}),0)$ with

$$\Psi_0(\mathbf{r})^2 = \frac{e^{-2\phi(\mathbf{r})}}{\sum_{\mathbf{r}} e^{-2\phi(\mathbf{r}')}}, \tag{129}$$

the second one being $\Psi_{0,2}(\mathbf{r}) = (0,\tilde{\Psi}_0(\mathbf{r}))$, where $\tilde{\Psi}_0(\mathbf{r})$ is given by Eq. (129) changing $\phi(\mathbf{r}) \to -\phi(\mathbf{r})$. We denote $\Sigma_{\mathbf{r}}$ having in mind either a discrete problem or a continuous problem with some short scale cutoff a.

B. Participation ratios and multifractal spectrum

Thus in a given configuration of disorder $\phi(\mathbf{r})$ the quantum probability $|\Psi_0(\mathbf{r})|^2$ is *identical* to the Gibbs probability $p(\mathbf{r})$ defined in Eq. (2) for the particle in the logarithmically correlated random potential $V(\mathbf{r})$ with the correspondence

$$|\Psi_0(\mathbf{r})|^2 = p(\mathbf{r}), \tag{130}$$

$$2\phi(\mathbf{r}) = \beta V(\mathbf{r}), \tag{131}$$

and thus the model depends on a single parameter $g = \frac{1}{2}\beta^2\sigma$. As we have discussed in the previous sections, the particle in the logarithmically correlated random potential undergoes a transition at $\beta_c = \sqrt{2/\sigma}$ at which its Gibbs measure changes from being dominated by many sites (high-T phase) to being dominated to a few sites (low-T phase). Thus in the quantum problem we expect a transition at

$$g = g_c = 1 \tag{132}$$

with a weak disorder phase for $g<1$ and a strong disorder phase for $g>1$. In the weak disorder phase the quantum probability (and thus observables such as the mean-squared position fluctuations $\langle r^2\rangle - \langle r\rangle^2$) is delocalized ($\langle \cdots \rangle$ means averages over Ψ_0). In the strong disorder phase, the quantum probability is more concentrated, but it cannot be called lo-

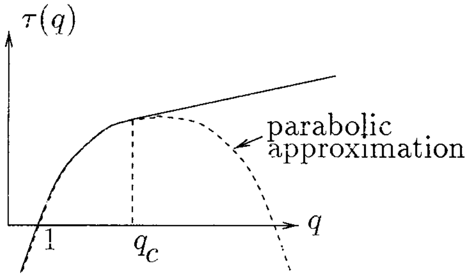

FIG. 14. Multifractal spectrum in the weak disorder phase.

calized in the usual sense (of an exponential decay around a single center) and in fact both phases have rather peculiar properties.

Properties of wave functions can be described by the inverse participation ratios defined from the normalized wave function $\Psi_0(\mathbf{r})$ in a system of size L by

$$R_q(L)=\int d^2\mathbf{r}|\Psi_0(\mathbf{r})|^{2q}=\int d^2\mathbf{r}(p(\mathbf{r}))^q. \tag{133}$$

At a very qualitative level, the nature of the eigenfunction can be inferred from the scaling behavior of the inverse participation ratio with the system size L: for an exponentially localized state $R_q(L)$ scales [73] as $R_q(L)\sim$const for all $q>0$, while for a plane-wave delocalized state we get $R_q(L)\sim L^{-2(q-1)}$. In addition to the localized and delocalized states, there exist states such that $\tau(q)=-\ln R_q(L)/\ln L$ is a nonlinear function of q: they correspond to multifractal wave functions whose moments cannot be described by a single length as usual but rather by a spectrum of exponents. Here, as in [13], we also find intermediate multifractal behavior.

To compute the finite-size inverse participation ratios, we can use the information of the previous sections since

$$s_q(L)=-\ln R_q(L)=-\ln Z_{q^2g}+q\ln Z_g, \tag{134}$$

where we have defined $Z_g=Z(\beta=\sqrt{2g/\sigma})$, where $Z(\beta)$ is the partition function for the particle at inverse temperature β. In particular, we will be interested in the multifractal asymptotic scaling exponent $\tau(q)$ defined by

$$\tau(q)=\lim_{L\to\infty}\frac{s_q(L)}{\ln L}. \tag{135}$$

These exponents were computed previously in [13] using the REM approximation [25]. Here we use our RG results and also obtain finite-size corrections. Note that these correspond to properties of Ψ_0 defined above and could be changed if other boundary conditions were used.

From the previous sections we obtain

$$\ln Z_g=2(1+g)\ln L+\Delta_g,\quad g<1, \tag{136a}$$

$$\ln Z_g=\sqrt{g}[4\ln L-\tfrac{1}{2}\ln(\ln L)]+\Delta_g,\quad g=1, \tag{136b}$$

$$\ln Z_g=\sqrt{g}[4\ln L-\tfrac{3}{2}\ln(\ln L)]+\Delta_g,\quad g>1, \tag{136c}$$

where Δ_g is a sample-dependent variable of order $O(1)$ with a g-dependent distribution (whose tails we have characterized previously). From there we obtain $s_q(L)$, which have different behaviors in the two phases.

(i) Weak disorder phase. For $g<1$ we find, denoting $q_c=1/\sqrt{g}$,

$$s_q(L)=2(q-1)(1-gq)\ln L+A_{q,g}\quad\text{for}\quad |q|<q_c, \tag{137a}$$

$$s_q(L)=\frac{2}{\sqrt{g}}[1-\mathrm{sgn}(q)\sqrt{g}]^2\ln L+\frac{1}{2}\ln\ln L+A_{q,g}\quad\text{for}\quad |q|=q_c, \tag{137b}$$

$$s_q(L)=2q[1-\mathrm{sgn}(q)\sqrt{g}]^2\ln L+\frac{3}{2}|q|\sqrt{g}\ln\ln L+A_{q,g}\quad\text{for}\quad |q|>q_c, \tag{137c}$$

where $A_{q,g}$ is a fluctuating part of order $O(1)$.

(ii) Strong disorder phase. For $g>1$ we find

$$s_q(L)=-2(q\sqrt{g}-1)^2\ln L-\frac{3}{2}q\sqrt{g}\ln\ln L+A_{q,g}\quad\text{for}\quad |q|<q_c, \tag{138a}$$

$$s_q(L)=-\frac{1}{2}\ln\ln L+A_{q,g}\quad\text{for}\quad q=q_c, \tag{138b}$$

$$s_q(L)=A_{q,g}\quad\text{for}\quad q>q_c, \tag{138c}$$

$$s_q(L)=-2|q|\sqrt{g}\left(4\ln L-\frac{3}{2}\ln\ln L\right)+A_{q,g}\quad\text{for } q<-q_c, \tag{138d}$$

$$s_q(L)=-2\left(4\ln L-\frac{1}{2}\ln\ln L\right)+A_{q,g}\quad\text{for}\quad q=-q_c, \tag{138e}$$

where $A_{q,g}$ is a fluctuating part of order $O(1)$.

The corresponding scaling exponents $\tau(q)$ are thus identical to the one found in [13] and in addition we have obtained their finite-size corrections as well as the order-of-magnitude estimate of their fluctuations. In the weak disorder phase for $q>0$,

$$\tau(q)=\begin{cases}2(q-1)\left(1-\dfrac{q}{q_c^2}\right) & \text{for}\quad q\leqslant q_c=\sqrt{\dfrac{1}{g}},\\[2ex] 2q\left(1-\dfrac{1}{q_c}\right)^2 & \text{for}\quad q\geqslant q_c,\end{cases} \tag{139}$$

which means a parabolic form with a termination point at $q=q_c$ as represented in Fig. 14.

In the strong disorder phase $g>1$, i.e., when $q_c\leqslant 1$, the above expression becomes (for $q>0$)

$$\tau(q)=\begin{cases}-2\left(1-\frac{q}{q_c}\right)^2 & \text{for} \quad q\leqslant q_c=\sqrt{\frac{1}{g}}, \\ 0 & \text{for} \quad q\geqslant q_c.\end{cases} \tag{140}$$

Since the inverse participation ratio does not scale with the system size L for each integer q, one could naively conclude that it is the sign of a localized state (see, however, below).

As was discussed in [13], these results can be translated into the spectrum for exponent α. If one assumes that $p(\mathbf{r})$ is of order $L^{-\alpha}$ in a number $L^{f(\alpha)}$ sites, then the above spectrum is recovered if

$$f(\alpha)=8\frac{(d_+-\alpha)(\alpha-d_-)}{(d_+-d_-)^2} \tag{141}$$

with $d_\pm=2(1\pm\sqrt{g})^2$ for $g>1$ and $d_+=8\sqrt{g}$, $d_-=0$ for $g>1$. It is easy to see that

$$\langle(r-\langle r\rangle)^{2k}\rangle\geqslant L^{\max_\alpha((k+1)f(\alpha)-\alpha)} \tag{142}$$

showing that the eigenstate is never localized in the usual sense (exponential decay around a single center) since the exponent is always positive for large enough k. Since $\lim_{q\to\pm\infty}s_q(L)/q=\ln p_{\max,\min}$, one obtains that the maximum of the Gibbs measure $p_{\max}=\max_{\mathbf{r}}p(\mathbf{r})$ and the minimum behave for large L as

$$p_{\max}\sim L^{-2(1-\sqrt{g})}(\ln L)^{-(3/2)\sqrt{g}}, \tag{143}$$

$$p_{\min}\sim L^{-2(1+\sqrt{g})}(\ln L)^{+(3/2)\sqrt{g}}, \tag{144}$$

in the weak disorder phase.

C. Nature of the strong disorder phase: Quasilocalization

Let us now concentrate on the case $g>1$. There, we know from the previous sections that the Gibbs measure of the particle is concentrated in *a few* sites. Thus from Eq. (131) the quantum probability $|\Phi_0(\mathbf{r})|^2$ is also concentrated in *a few* sites, analogous to the RSB picture. This is a very peculiar type of eigenstate. Indeed if one computes the quantum average $\langle r^2\rangle-\langle r^2\rangle$ in a given sample, it has a finite probability to be of order $O(L^2)$. Thus the eigenstate cannot be considered as localized in the usual sense. Since it is peaked around a few sites, we call it ''quasilocalized.'' Around these centers, the wave function decays fast enough to be normalizable. It would be interesting to investigate further the typical spatial decay of such eigenstates around their (multiple) centers, which we expect to be slower than exponential.

VII. CONCLUSION

In this paper we have studied the equilibrium problem of a particle in a random potential with logarithmic correlations, through exact bounds, numerical simulation, qualitative arguments, and a renormalization-group method that we have developed specifically for this problem. We have shown that it exhibits a glass transition at finite temperature $T_c=1/\beta_c>0$ in any dimension. This confirms earlier conjectures and allows for a more detailed study of the problem. The RG method allowed us to obtain the universal features of the free-energy distribution at low temperature. The relation to the problem of extremal statistics of correlated variables was investigated. It has been found that it exhibits universal finite-size corrections, consistent with our numerical calculations.

Most interestingly, we found that this logarithmic model provides a particularly simple example (maybe the simplest) of a finite-dimensional model, i.e., with *translationally invariant disorder correlations*, such that the low-temperature phase is nontrivial. It is nontrivial in the sense that in the thermodynamic limit $L\to+\infty$, there are, with a finite probability, several low-lying states (i.e., possible positions of the particle) with energy differences of order 1, and separated in space by distances of order L. Thus the Gibbs measure at low temperature is dominated by ''a few'' spatially well-separated states. Interestingly, this transition and this type of glass phase occurs only for logarithmically growing correlations, faster growth (e.g., as in Sinai model) yielding only a glass phase with single ground-state dominance, while slower growth yields only a high-temperature phase.

Although oversimplified in some respect (it has no internal space), it does provide one example of a model where the usual droplet picture (which assumes dominance of a single ground state, or several related by a symmetry) does not apply. Rather, it provides one example where some features of the physics usually associated to RSB, namely dominance by a few states with exponential free-energy distributions, can be explicitly exhibited. In fact, due to the finite-dimensional correlations, there are some departures from the behavior observed in the simplest prototype mean-field models (such as the REM), as can be seen, for instance, from the free-energy distribution, which has more structure than a simple exponential. It would of course be interesting to explore further the additional features specific to finite dimensions.

Although the present model is already of obvious physical interest (in 2D it describes, e.g., a single vortex in a random gauge XY model), its nontrivial properties provide a motivation to search for models with more degrees of freedom and with similar features. One way to proceed would be to search for interface models via an internal dimensional expansion around the present model. The key feature, however, appears to be the *marginality* of the model, i.e., the logarithmic growth of typical energy fluctuations. This corresponds to a fluctuation energy exponent $\theta=0$, i.e., the situation where the temperature (i.e., the entropy) is *marginal* in the RG sense. The droplet arguments indeed assume that $\theta>0$, consistent with the single ground-state dominance (and activated behavior typical of a zero-temperature fixed point where T is formally irrelevant). In the situation $\theta=0$, one does expect more generally power laws with T-dependent exponents, reminiscent of mean field. It would thus be of great interest to similarly exhibit other nontrivial marginal models (e.g., spin models with $\theta=0$) with similar features [74]. Spin models where (domain wall) excitations (in root mean square and in average) also scale logarithmically (as vortices in the random gauge XY model) are presumably good candidates.

On the one hand we have developed a specific (Coulomb gas) renormalization-group (RG) approach to describe the model. From the study of the resulting nonlinear (KPP) RG equation, we found explicitly that a freezing phenomenon occurs at the glass transition temperature, and that in the glass phase a broad (power-law) distribution of fugacities develops—or equivalently an exponential distribution of local free energy. It is different from more conventional perturbative RG (e.g., the one which was used to study the dynamics of this model) in the sense that the full distribution of probability is followed. This turns out to be crucial to describe the low-temperature phase.

On the other hand, as we have discussed, two *approximations* of the present model, the REM approximation and the DPCT hierarchical version, can both be solved using replica and do require considering the analytical continuation to $m \to 0$ of contributions of replica symmetry breaking saddle points [10].

This shows that a RG approach which is explicitly replica symmetric but allows to treat broad disorder distributions can be consistent with (approximate) approaches based on RSB saddle points [75]. We have illustrated this on the REM, which can be recast in terms of nonlinear RG equations, with a freezing transition. In fact, one of the striking properties of the model is that the RG equations derived here are similar—to the order we have been working—to the one which holds for a continuous version of the DPCT problem, the branching process. In particular, it indicates that both problems share the same universal finite-size corrections.

We have also analyzed some connections in 2D (and via boundaries in 1D) between the model of the particle and the Liouville and sinh-Gordon models. The intensive free energy of the particle corresponds to the scaling dimension in these models with $b=\beta/\beta_c$. The glass transition corresponds to the weak- to strong-coupling transition at $b=1$. Beyond, corresponding to the glass phase, the scaling dimension freezes, as we have also shown via a direct RG approach on these models. We have seen that under coarse graining, an additional local field appears in the LM and SGM, with broad distribution, and corresponds to inhomogeneous configurations being generated (and broad fluctuations of the local area, since the local partition function corresponds to local area).

The present study raises interesting issues to be explored concerning the relations with the continuum Liouville field theory (LFT). An outstanding question is whether the conjecture of [12] is correct for the correlations. Since we have obtained another result linking the problem to the DPCT, the direct comparison of the LFT and the DPCT remains to be studied. If it holds, it means that the conformally invariant many-point correlations can be related in some limits (large separations with fixed ratios $\ln r_{ij}/\ln r_{kl}$) to the results from the tree problem. It would also raise interesting issues about the continuation of the LFT beyond $b=1$ and its relation with the nontrivial structure of the glass phase (with RSB features) in the equivalent particle model.

We have also extracted from our approach some consequences for the problem of the $E=0$ critical eigenstate of 2D Dirac fermions in a random magnetic field. We have confirmed, via our RG method, previous results concerning the multifractal spectrum and extracted their finite-size corrections. We have found that the nontrivial low T phase of the particle translates into peculiar quasilocalized eigenstates for the quantum problem, peaked around a few distant centers. It raises the question of whether this property can be present in other quantum systems.

Another interesting question is whether the transition studied here has a signature in the dynamics as well. Note that a similar nontrivial structure at low temperature is also present in the the Sinai model with a bias, which renormalizes onto a random walk with algebraic waiting times distribution [58]. However, this is a driven system and it would be interesting to see whether nondriven systems in low dimension can exhibit similar features. Since the barriers grow logarithmically, it is natural to expect an anomalous diffusion exponent $x \sim t^{\nu}$ with $\nu < \frac{1}{2}$, as was found in similar situations [79]. What happens to this exponent ν at T_g will be investigated in the very near future.

Finally, an outstanding question is how the present model can be studied using 2D conformal field theory (CFT). In particular, one wonders what is the signature in this context of the physics that was unveiled here, reminiscent of RSB, using RG with broad distributions. The freezing phenomenon within the nonlinear RG, which transforms the naive scaling dimensions into nontrivial ones, should correspond to a similar mechanism in CFT. Recent progress on CFT classification of disordered models where supersymmetry can be used allows us to hope that such progress is within sight. We hope that the present RG method will apply to study other two-dimensional models with similar features and shed light on the more formal field-theoretic methods.

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APPENDIX A: EXISTENCE OF A TRANSITION

We use the same method as Derrida and Cook [76] for the directed polymer problem [77]. It is easy to compute the first two moments of $P[Z]$, using translational invariance and periodic boundary conditions:

$$\overline{Z} = L^{d} e^{(\beta^2/2)\Gamma_L(0)} \sim L^{d+\beta^2\sigma} \tag{A1}$$

and

$$\overline{Z^2} = L^{d} e^{(\beta^2/2)4\Gamma_L(0)} \sum_{\mathbf{r}} e^{-(\beta^2/2)\tilde{\Gamma}_L(\mathbf{r})} \tag{A2}$$

$$\sim BL^{2d+2\beta^2\sigma}, \quad \beta<\beta_2, \tag{A3}$$

the last estimate being valid as long as the sum over $\mathbf{r}$ is divergent, i.e., $\beta<\beta_2=\sqrt{d/(2\sigma)}$. The constant $B>1$ depends on the details of the model, e.g., for $d=1$ one can write $B=\lim_{L\to+\infty}\int_0^1 dy\ \exp(-\beta^2/2[\tilde{\Gamma}_L(Ly)-4\sigma\ln L])$. Thus for $\beta<\beta_2$ the ratio

$$\frac{\overline{Z^2}}{\overline{Z}^2}\to B \tag{A4}$$

as $L\to+\infty$. In [76] it is shown that the property (A4) implies that

$$\mathrm{Prob}\left(\frac{1}{d\ln L}\ln Z=\frac{1}{d\ln L}\ln \overline{Z}\right)\geqslant 1/B \tag{A5}$$

as $L\to+\infty$. If we take for granted that the free energy is self-averaging, it implies that for $T>T_2=\sqrt{2\sigma/d}$, the quenched and annealed (intensive) free energies coincide exactly, $f(T)=f_A(T)$. Thus for $T>T_2$, the (intensive) entropy is $s(T)=s_A(T)=-\partial_T f_A(T)$ and thus one has

$$s(T)=1-\frac{\sigma}{dT^2},\quad T>T_2. \tag{A6}$$

Since $s_A(T)$ becomes negative below $T=T_g=\sqrt{d/\sigma}$, it implies that there must be a temperature $T_c<T_2$ at which Eq. (A6) breaks down and thus a phase transition occurs. Although this is harder to prove, it seems that here Eq. (A6) holds down to $T_c=T_g$.

Awaiting a rigorous mathematical proof, we have not attempted to prove the self-averaging of f. Not only is it highly reasonable in view of our other results, but in fact if it were not the case, the above argument would imply a rather curious—and unphysical—distribution for f (with a δ peak of nonzero weight smaller than 1). In addition, as noted in [76], by adjusting the small-scale details of the model, the constant B can be chosen as close to 1 as wanted.

APPENDIX B: EXTREMAL STATISTICS OF CORRELATED VARIABLES

In this section we summarize some results on the extremal statistics of a set of random variables. We selected the ones which are useful in putting the problem studied here in a broader context. We recall some of the classic results from probability theory and we have chosen to illustrate them by adding a few simple arguments which emphasize the importance of some of these results to the physics of disordered systems. We denote the N random variables either X_r, $r=1,\ldots,N$ when they are normalized in a particular way, or V_r when they can be readily interpreted as the random potential variables studied here [the two differing by a trivial uniform rescaling $V(r)\equiv V_r\propto X_r$]. They apply directly to describe $d=1$ $(N=L)$ and can be usually extended to $d>1$ $[V(\mathbf{r})$ and $N=L^d]$.

1. Uncorrelated variables

It is natural to start with the case of N uncorrelated variables of identical probability distribution $P(V)$. The distribution $P(V)$ can belong to three classes of extremal statistics, but we will recall only the Gumbell class. Schematically for this class, a well-known theorem [27] states that there exist constants a_N and b_N such that for a fixed $\tilde{y}$,

$$\mathrm{Prob}(V_{\min}>b_N\tilde{y}-a_N)\to\exp(-e^{\tilde{y}}). \tag{B1}$$

The constants a_N and b_N are determined as

$$\ln\int_{-\infty}^{-a_N}dV P(V)=\frac{1}{N}, \tag{B2}$$

$$b_N=N\int_{-\infty}^{-a_N}dy\int_{-\infty}^{y}dV P(V). \tag{B3}$$

For variables X_r chosen from a centered Gaussian of unit variance $P(X)=(1/\sqrt{2\pi})e^{-X^2/2}$, one can choose a_N and b_N as

$$b_N=\frac{1}{\sqrt{2\ln N}}, \tag{B4}$$

$$a_N=\sqrt{2\ln N}-\frac{1}{\sqrt{2\ln N}}\frac{1}{2}\ln(4\pi\ln N), \tag{B5}$$

and thus one can write schematically that

$$X_{\min,N}\approx-\sqrt{2\ln N}+\frac{1}{\sqrt{2\ln N}}\left(\frac{1}{2}\ln(4\pi\ln N)+\tilde{y}\right), \tag{B6}$$

where $\tilde{y}$ is distributed with the Gumbell distribution $p(\tilde{y})=e^{\tilde{y}}\exp(-e^{\tilde{y}})$.

It is useful to note the property of reparametrization associated to a monotonous function $\psi(V)$. If one has Eq. (B1) for the minimum $V_{\min}$ of the variables V_r with the constants a_N and b_N, one also has (under some weak conditions) Eq. (B1) for the minimum $\psi(V_{\min})$ of the variables $\psi(V_r)$ with the constants $a'_N=-\psi(-a_N)$ and $b'_N=b_N/\psi'(-a_N)$. Note also that we have illustrated how to show convergence to Gumbell (and generalized it to finite temperature) in the text.

For completeness, we recall the necessary conditions for the convergence to Gumbell [i.e., $P(V)$ belonging to the Gumbell class]. First $P(V)$ must decay fast enough at $V\to-\infty$ so that there exists y_0 such that

$$\int_{-\infty}^{y_0}dy\int_{-\infty}^{y}P(V)dV<+\infty, \tag{B7}$$

and second, defining

$$R(t)=\frac{1}{\int_{-\infty}^{t}P(V')dV'}\int_{-\infty}^{t}dy\int_{-\infty}^{y}P(V)dV \tag{B8}$$

one must have for all $x<y_0$,

$$\lim_{t\to-\infty} \frac{\int_{-\infty}^{t+xR(t)} P(V)dV}{\int_{-\infty}^{t} P(V')dV'} = e^x. \tag{B9}$$

These conditions are in fact rather broad. Finally, note also the very powerful theorem 2.10.1 of [27] for the rate of convergence to the Gumbell fixed point.

2. Correlated variables

a. General lower bound

We now consider correlated variables with distribution $P(V_1, \dots, V_N)$. Let us start with a simple but very general bound and extract the consequences. One has

$$G(x) = \text{Prob}(V_{\min} < x) \leq \sum_{r=1,N} \text{Prob}(V_r < x) \tag{B10}$$

since the reunion of all events $V_r < x$ implies the event $V_{\min} < x$ and that $\text{Prob}(AUB) \leq \text{Prob}(A) + \text{Prob}(B)$ (the bound is exactly saturated, e.g., when there are strong correlations such that $V_r - V_{r'} > x$ for all $r \neq r'$). For variables which have identical one-particle distribution $P(V_r) = \int \Pi_{r' \neq r} dV_{r'} P(V_1, \dots, V_N)$ one has

$$G(x) \leq N \int_{-\infty}^{x} P(V) dV. \tag{B11}$$

Let us illustrate the consequences for correlated variables $X_1, \dots, X_N$ such that the one-particle distribution is a unit centered Gaussian. Then it implies for $x \to -\infty$

$$G(x) \leq \frac{N}{\sqrt{2\pi} x} e^{-x^2/2}, \tag{B12}$$

from which one immediately sees that it implies

$$\text{Prob}\left(X_{\min} < x_N = -\sqrt{2 \ln N} + \alpha \frac{\ln(4\pi \ln N)}{\sqrt{2 \ln N}} \right) \leq \frac{1}{(4\pi \ln N)^{[1/2-\alpha]}} \underset{N\to+\infty}{\to} 0 \tag{B13}$$

by choosing $x = x_N$ for any $\alpha < \frac{1}{2}$. Thus one has a general lower bound for the minimum of correlated variables. In particular, for Gaussian variables such that $\overline{V_r^2} = 2\sigma \ln N = 2d\sigma \ln L$ one gets

$$V_{\min} > -2d\sqrt{\sigma} \ln L + \sqrt{\sigma}\alpha \ln(4\pi d \ln L) \tag{B14}$$

with probability 1 in the large L limit for any $\alpha < \frac{1}{2}$. Moreover, choosing $\alpha = \frac{1}{2}$ and writing

$$V_{\min} = -2d\sqrt{\sigma} \ln L + \sqrt{\sigma}[\tfrac{1}{2} \ln(4\pi d \ln L) + \tilde{y}] \tag{B15}$$

one finds that

$$\text{Prob}(\tilde{y} < y) \leq e^y. \tag{B16}$$

This yields a lower bound which can be compared with the REM approximation defined in the text. Note that the above upper bound is the exact behavior of the Gumbell distribution at large negative y, so in a sense the REM approximation saturates the bound in the tails. Consequently, to allow for a larger tail (such as ye^{-y}) one needs at least a coefficient of $\ln \ln N$ strictly larger than $\frac{1}{2}$).

b. Short-range correlations and convergence to Gumbell

Let us now consider N centered Gaussian variables X_r with a fixed correlation matrix $\Gamma_{rr'} = \overline{X_r X_{r'}}$, normalized so that $\Gamma_{rr} = 1$. A powerful bound, which refines Eq. (B10) above, allows us to easily demonstrate convergence to the Gumbell distribution for a large class of ''short enough range'' correlations. It compares two arbitrary correlators $\Gamma_{rr'}^{(1)}$ and $\Gamma_{rr'}^{(2)}$ with $\Gamma_{rr}^{(1)} = \Gamma_{rr}^{(2)} = 1$. Their associated $G(x)$ functions satisfy [27]

$$|G_1(x) - G_2(x)| \leq \sum_{r \neq r'} \frac{|\Gamma_{rr'}^{(1)} - \Gamma_{rr'}^{(2)}|}{2\pi(1 - m_{rr'}^2)^{1/2}} e^{-x^2/(1+m_{rr'})} \tag{B17}$$

with $m_{rr'} = \max(|\Gamma_{rr'}^{(1)}|, |\Gamma_{rr'}^{(2)}|)$. It is obtained by bounding $\partial G(x)/\partial \Gamma_{rr'}$ and integrating between Γ_1 and Γ_2. It will be used to compare $\Gamma_{rr'}^{(1)} = \Gamma_{rr'}$ with the uncorrelated case $\Gamma_{rr'}^{(2)} = \delta_{rr'}$.

To address the question of the universality of the Gumbell distribution, let us now consider a $(d=1)$ translationally invariant correlator $\Gamma_{rr'} = \Gamma(r - r')$ with $\Gamma(0) = 1$, where $\Gamma(r)$ is an N-independent function which decays to zero as $r - r' \to +\infty$.

Inserting $x = a_N \tilde{y} - b_N$ of Eqs. (B1) and (B5) into Eq. (B17) one easily gets that if $\Gamma(r)$ decreases fast enough, one has $G(x = a_N \tilde{y} - b_N) = \exp(-e^{\tilde{y}})$ at large N, i.e., one has convergence to the Gumbell distribution with exactly the same coefficients a_N and b_N as in the uncorrelated case, so that Eq. (B6) still holds. As one sees by studying the bound, this result holds as long as $\Gamma(r)$ decreases faster than $1/\ln(r)$ (this is theorem 3.8.2. of [27]). The limiting case (which does not satisfy Gumbell, as discussed below) is $\Gamma(r) \sim \tau/\ln(r)$ at large r.

Let us give a simple self-consistency argument, more enlightening than the bounds, which explains why $\Gamma(r) \sim 1/\ln r$ should be the limiting case between the short-range (Gumbell) universality class and other behaviors. Let us split a set of $2N$ correlated variables $X_1, \dots, X_{2N}$ into subsystem 1, $X_1, \dots, X_N$, and subsystem 2, $X_{N+1}, \dots, X_{2N}$. If correlations are very short ranged (e.g. exponentially decaying), it seems reasonable to first neglect correlations between 1 and 2 and find the minimum in each subsystem, which read, respectively,

$$\tilde{X}_{\min,i} \approx \sqrt{2 \ln N} - \frac{1}{2} \frac{\ln(4\pi \ln N)}{\sqrt{2 \ln N}} + \frac{x_i}{\sqrt{2 \ln N}} \tag{B18}$$

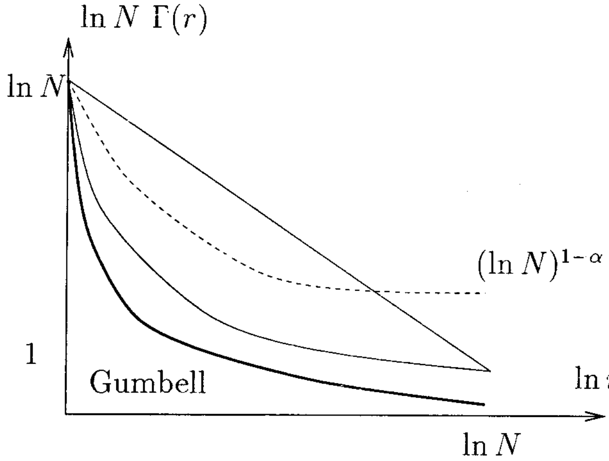

FIG. 15. Correlations as a function of $\ln r$. The straight line corresponds to the log-correlated variables studied here. The thick line corresponds to the limit where the short-range Gumbell (and REM) behavior holds, with $\Gamma(r)\sim 1/(\ln r)^{\alpha}$ and $\alpha<1$; the curved solid line corresponds to the case where a convolution of Gumbell and Gaussian holds (marginal case $\alpha=1$); and the dotted line corresponds to $\alpha>1$ when the mode $q=0$ dominates the behavior.

with $i=1,2$ and where x_1,x_2 are independently distributed with the Gumbell distribution. The symbol $\tilde{V}$ indicates that the minimum (in each subsystem) is with respect to a slightly different distribution from the original one, since all cross correlations between the two different subsystems have been set to zero. The second stage is to add the correlations between the two subsystems. Typically, the minima 1 and 2 will be a distance $\sim N$ apart and thus their original cross correlation is $\sim\Gamma(N)$, and thus, for short-range correlations, very small compared to the fluctuating part $x_i/\sqrt{2\ln N}$. Thus the distribution of the minimum $X_{\min}^{(2N)}$ of the original $2N$ variables should be given with better and better accuracy at large N, as $X_{\min}^{(2N)}=\min(\tilde{X}_{\min,1},\tilde{X}_{\min,2})$ [which is automatically satisfied by the approximation (B18)]. The corrections are irrelevant at large scale provided the typical root mean cross correlation between the subsystems remains smaller than the typical fluctuations of the minimum in each subsystem, a condition which reads

$$\sqrt{\Gamma(N)}\ll 1/\sqrt{\ln N}, \tag{B19}$$

which indeed gives correctly the basin of attraction of the Gumbell distribution. Furthermore, in the limiting case $\Gamma(r)\sim\tau/\ln r$ the above argument shows that the distribution of the x_i should be changed, which is also the case, as we now examine.

So, to summarize, if correlations are short ranged with $\Gamma(r)$ decreasing faster than $1/\ln(r)$, this is the ''SR universality class.'' It includes the REM, and one can check that the finite size corrections in [6] are reproduced (at $T=0$).

c. Long-range correlations and absence of convergence to Gumbell

There is a simple but instructive model of correlated variables which can be easily solved and that illustrates cases where Gumbell does not hold. If one takes

$$V_r'=V_r+U \tag{B20}$$

with V_r a set of uncorrelated Gaussian variables and U a Gaussian variable uncorrelated with the V_r, then clearly, if one chooses the variance of U big enough Eq. (B6) cannot hold. To keep using normalized variables ($\Gamma_{rr}=1$), one defines

$$X_r'=\frac{1}{\sqrt{1+w_N}}X_r+u\sqrt{\frac{w_N}{1+w_N}}, \tag{B21}$$

where u is a centered Gaussian random variable with unit variance. The correlation matrix is then $\Gamma_{rr}'=1/(1+w_N)(\delta_{rr'}+w_N)$. Clearly one has

$$X_{\min}'=\frac{1}{\sqrt{1+w_N}}X_{\min}+u\sqrt{\frac{w_N}{1+w_N}}. \tag{B22}$$

Using the expression (B6) for $X_{\min}$, one sees that for deviations from Gumbell to arise one needs that $w_N\sim\tau/\ln N$. In that case one gets from Eq. (B6) that

$$X_{\min}'\approx\sqrt{2\ln N}-\frac{1}{2}\frac{\ln(4\pi\ln N)}{\sqrt{2\ln N}}+\frac{x_i+\sqrt{2\tau}u+\tau}{\sqrt{2\ln N}}. \tag{B23}$$

These simple considerations thus allow us to understand simply the limiting case, i.e., that if $\Gamma(r)$ decreases as $\tau/\ln(r)$, one has that Eq. (B6) still holds (with the same constants) but the distribution of $\tilde{y}-\tau$ now converges instead to the convolution of the Gumbell distribution and the Gaussian of variance 2τ (see, e.g., theorem 3.8.2. of [27]).

Increasing the range of correlations even further, one gets into a regime where the fluctuating part (in the X variables) is larger than $1/\sqrt{\ln N}$ (and thus in the $V\sim X\sqrt{\ln N}$ variables the dominant finite-size corrections are non-self-averaging). The fluctuations become then entirely Gaussian, being controlled by the U part, i.e., the $q=0$ mode. For instance, if $\Gamma(r)$ decreases as $1/[\ln(r)]^{\alpha}$ with $\frac{1}{3}<\alpha<1$, then (theorem 3.8.4. of [27]) one has

$$P\{V_{\min}>-\Gamma(N)^{1/2}x-[1-\Gamma(N)]^{1/2}\sqrt{2\ln N}$$
$$[2\ln N-\tfrac{1}{2}\ln(4\pi\ln N)]\}\to\int_{-\infty}^{x}2\pi^{-1/2}e^{-x^2/2}. \tag{B24}$$

As illustrated below, this behavior (entirely controlled by the $q=0$ mode) is in a sense more long range, and further away from Gumbell than the problem of log-correlated variables that we are interested in and that we now discuss.

d. Log-correlated variables

The case of log-correlated variables is difficult and little is known. We just make a few comments.

Let us first discuss the form of the correlator. The correlator (for the normalized variables $X_r=V_r/\sqrt{2\sigma\ln N}$ in $d=1$) is of the form

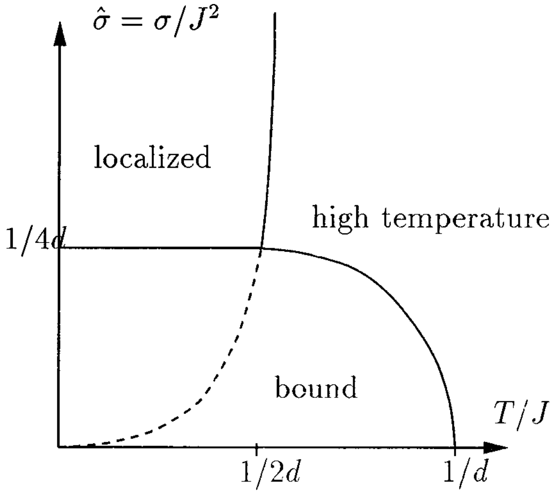

FIG. 16. Phase diagram in the presence of both disorder and external potential. The freezing of the KPP velocity still occurs at $\beta=\beta_c$ and is represented by the dashed line and its solid prolongation: it remains a transition line for $\sigma>4dJ^2$ and becomes a crossover line for $4d\sigma<J^2$.

$$\Gamma(r)=1-\frac{\ln r}{\ln N} \tag{B25}$$

for $1\gg r\gg N$. One must then distinguish the two other regions. For small r, the precise form could vary by adding a short-range correlated noise. This is what we term short scale details, and an important question is the extent of universality of the results (scaling of minima, distribution) with respect to the small-r form. For $r\sim L$, the behavior depends on boundary conditions, which may also be important (see below). For the periodic system in the simulation, $\Gamma(r)=\Gamma(N-r)$ and $\Gamma(r)$ actually becomes negative at $r=N/2$ and of order $c/\ln N$ (see Sec. IV). Adding a small uniform U noise, as described above in Appendix B 2 c, could make $\Gamma(N/2)=0$, so generally speaking one can discuss forms such that $\Gamma(N/2)=0$. Seen as a scaling function of $z=\ln r/\ln N$, $\Gamma(r)$ then converges for large N towards Eq. (B25), but it does have boundary layers at $z=0$ and $z=1$.

It is useful to plot on the same graph the various cases studied in this section. This is represented in Fig. 15. We have represented schematically $\Gamma(r)\ln N$ versus $\ln r$, for the log-correlated form (B25) above, and for the various cases $\Gamma(r)\sim\ln N/(\ln r)^{\alpha}$ with $\alpha>1$ (Gumbell behavior), $\alpha=1$, and $\alpha<1$.

As discussed above, in the log-correlated case the behavior of $\Gamma(r)$ near $\ln r=\ln N$ can be considered as uncertain to order $1/\ln N$. This can be seen either from the $q=0$ mode, which, depending on boundary conditions, one may adjust by this amount, as discussed above, or even looking at the first nontrivial mode, $q=2\pi/L$, which has a contribution of the same order. We know from the preceding paragraph that these contributions can shift the x variable by a Gaussian, so it makes it unlikely that the Gumbell distribution would hold in that case.

To conclude, we have given the various behaviors as a function of the range of correlations. The presence of the $\ln\ln N$ corrections seems to be more robust than the distribution of x. For the marginal case with $q=0$ LR disorder, the same $\frac{1}{2}\ln\ln N$ corrections hold as for the REM, while the distribution is changed. On the other hand, for log-correlated variables, we expect a different coefficient $\frac{3}{2}\ln\ln N$ as discussed in the text (and we do not expect the Gumbell distribution to hold).

APPENDIX C: GUMBELL VIA RG

In a more detailed analysis, Eq. (50) yields $\ln\ln(1/G_l(x))=l+\ln\ln(1/G_0(x))$ which can be rewritten in a frontlike form:

$$G_l(x)=\exp(-e^{l+\phi(x)}), \tag{C1}$$

where $\phi(x)=\ln\ln(1/G_0(x))$. In this appendix we set $dl\to l$. The center of the front is at the $x=-m(l)$ solution of $\phi(-m(l))=-l$. One can Taylor expand $\phi(x)=-l+y+\frac{1}{2}\delta_l y^2+\cdots$ with $y=\alpha_l(x+m(l))$, $\alpha_l=\phi'(-m(l))$, and $\delta_l=\phi''(-m(l))/\phi'(-m(l))^2$. Thus in the variable y, G_l converges to a Gumbell limit distribution $\mathcal{G}(y)=\exp(-e^{y})$. It holds provided higher terms in the Taylor expansion are irrelevant (a necessary, and in the simplest cases sufficient, condition being that the second one $\delta_l\to 0$).

If no rescaling of disorder is performed, in the relevant large negative x region one has $G_0(x)\approx 1$ and thus $\phi(x)\approx\ln[1-G_0(x)]$. Two cases must be distinguished because the limit $T\to 0$ and $N\to+\infty$ do not commute.

(i) *Finite fixed temperature* $T>0$. For $x\to-\infty$ one has $1-G_0(x)\sim C_1(\beta)e^{\beta x}[1+O(e^{\beta x})]$ with $C_k=\int_V P(V)e^{-k\beta V}$ and we assume that $C_1,C_2<+\infty$ exists (distributions falling faster than exponentials). Then the situation is simple as $\phi(x)=\beta x+\ln C_1(\beta)+O(e^{\beta x})$, $m(l)\sim l/\beta+1/\beta\ln C_1(\beta)$, $\alpha_l=\beta$, and $\phi''(x)/\phi'(x)^2\to 0$ exponentially fast. For a Gaussian distribution,

$$m(l)\sim\frac{1}{\beta}l+\frac{1}{2}\sigma\beta. \tag{C2}$$

There is *no transition* to a glass phase.

(ii) *Zero temperature*. It is an extremal statistics problem. Then clearly $1-G_0(x)$ does not decay as an exponential. Let us consider a class of distributions such that $1-G_0(x)\sim(A|x|)^{-\gamma}\exp(-(B|x|)^{\alpha})$ with $\alpha>1$ (plus exponentially small corrections). This contains the Gaussian (of variance σ) of most interest here, for $\alpha=2$, $B=1/\sqrt{2\sigma}$, $\gamma=1$, and $A=\sqrt{2\pi/\sigma}$. Then one easily finds from above that

$$m(l)\approx\frac{1}{B}\left(l-\frac{\gamma}{\alpha}\ln l-\gamma\ln(A/B)\right)^{1/\alpha}, \tag{C3}$$

$$\alpha_l\approx B\alpha\left(l-\frac{\gamma}{\alpha}\ln l-\gamma\ln(A/B)\right)^{1-(1/\alpha)}, \tag{C4}$$

and that $\phi''/\phi'^2 \sim 1/|x|^\alpha$, thus the convergence to the Gumbell front holds. Note that the quantity $\alpha_l m(l) \sim \alpha l - \gamma \ln l + O(1)$ exhibits some universality.

One thus recovers the standard theorems for extremal value statistics reviewed in Appendix B, and the relation to the normalizing constants defined there as

$$m(l) = a_N, \quad \alpha_l = \frac{1}{b_N}, \quad l = \ln N. \tag{C5}$$

In the Gaussian case, using the values given above, one finds that Eq. (C4) indeed yields Eq. (B5) in Appendix B (up to subdominant terms). Although the distribution is universal, the normalizing constants obviously depend on the details of the tail of the distribution. Note in all cases the presence of finite-size corrections involving a logarithm.

There is a very small temperature ($\beta_L \sim \sqrt{\ln L}$ for Gaussian) where the behavior of $G_0(x)$ changes from (i) to (ii). It can be seen by rescaling temperature or equivalently disorder, with system size as in the REM.

Let us examine the case where the constants A_l and B_l are rescaled and now l-dependent (see also, e.g., [30]). One can still use formula (C4). Let us choose $B_l = b l^{-1+1/\alpha}$ and $A_l/B_l = cst$ (which includes the Gaussian REM). One finds at $T=0$ that $m(l) \sim 1/b[l - (\gamma/\alpha^2)\ln l - (\gamma/\alpha)\ln(A/B)]$ and $\alpha_l \sim b\alpha$. In the Gaussian case $\sigma_l = 2\sigma l$ one recovers the REM result,

$$m(l) \approx \sqrt{\sigma}[2l - \tfrac{1}{2}\ln(4\pi l)], \tag{C6}$$

$$\alpha_l \to \frac{1}{\sqrt{\sigma}}, \tag{C7}$$

at $T=0$ [i.e., Eq. (41a) setting $l \to dl$ and $\sigma \to \sigma/d$]. The analysis can be performed at any T and now yields a transition temperature when the behavior of $G_{0,l}(x)$ at large x changes.

APPENDIX D: AN EXTENDED MODEL

A richer phase diagram can be obtained by adding a logarithmic background potential [78] $V_0(\mathbf{r}) = J \ln(|\mathbf{r}|/a)$ to the previous random potential $\overline{[V_d(\mathbf{r}) - V_d(\mathbf{r}')]^2} \sim 4\sigma \ln|\mathbf{r} - \mathbf{r}'|/a$ for $a \ll |\mathbf{r} - \mathbf{r}'| \ll L$ and $\overline{V_d(r)} = 0$ [i.e., writing $V(\mathbf{r}) = V_0(\mathbf{r}) + V_d(\mathbf{r})$] in Eq. (1). The choice of the origin breaks translational invariance. The competition between the disorder and the binding background potential (which if strong enough tends to favor localizing the particle in wells far from $\mathbf{r}=0$) yields the phase diagram of Fig. 16. Another closely related model (model II) which preserves statistical translational invariance and has the same phase diagram is

$$Z_v[V] = 1 + \left(\frac{L}{a}\right)^{-\beta J} \sum_{\mathbf{r}} e^{-\beta V_d(\mathbf{r})}, \tag{D1}$$

which describes a problem with either zero or one particle (vortex) present, the energy cost of the vortex being $J\ln(L/a)$. It is thus a one-vortex toy model of the recently studied XY model with random phase shifts [19,20].

In the absence of disorder, the model with a background potential (model I) trivially exhibits a transition at $\beta = \beta^* = d/J$. At low temperature $\beta > \beta^*$ the Gibbs measure is $p(\mathbf{r}) \sim C(a/r)^{\beta J}$ with $C = Z_{L=\infty}$ a finite constant and the particle is bound to $\mathbf{r}=0$ (it has a finite probability to be within a fixed distance of $\mathbf{r}=0$). At high temperature $\beta < \beta^*$ the Gibbs measure becomes $p(\mathbf{r}) \sim (a/L)^{d-\beta J}(a/r)^{\beta J}$ and the particle is delocalized. This transition can be seen in the free-energy density $f = F/\ln L = -T\ln Z/\ln L$ since

$$f = 0, \quad \beta > \beta^*, \tag{D2}$$

$$f = -(J\beta^* - \beta), \quad \beta < \beta^* \tag{D3}$$

for $\beta < \beta^*$. This first-order transition occurs as f reaches its bound (since $Z > 1$ due to the lattice cutoff, one has that $f \le 0$). The model II has the same f and a similar transition with either one vortex present, $\beta > \beta^*$, or zero, $\beta < \beta^*$.

In the presence of disorder, the RG developed in this paper can be extended straightforwardly and leads to

$$\frac{1}{d}\partial_l G(x) = \frac{J}{d}\partial_x G + \frac{\sigma}{d}\partial_x^2 G + F[G], \tag{D4}$$

$$F[G] = -G(1-G). \tag{D5}$$

The additional term thus results in a simple shift in the front velocity. The position of the front $m(l)$ thus leads to the free energy $f = m(l)/(dl)$, which can have three distinct analytical forms:

$$\beta m(l)/l \sim d\beta\, f(\beta)$$

$$= -(d + \sigma\beta^2 - J\beta) \quad \text{high-}T \text{ phase I}, \tag{D6}$$

$$-\left(2d\frac{\beta}{\beta_c} - J\beta\right) \quad \text{localized phase II}, \tag{D7}$$

$$0 \quad \text{bound phase III}. \tag{D8}$$

The phase diagram is represented in Fig. 16 using the reduced temperature T/J and the dimensionless disorder parameter $\hat{\sigma} = \sigma/J^2$. For $4d\sigma < J^2$ one defines $\beta^*(\sigma) = (1/2\sigma)(J - \sqrt{J^2 - 4d\sigma})$. The RG analysis yields three phases. In the model with the background potential (model I) they are as follows.

The high temperature phase [for $\beta < \beta_c$ when $4d\sigma > J^2$ and for $\beta < \beta^*(\sigma)$ for $4d\sigma < J^2$]: Entropy wins and the particle is delocalized over the system.

The localized phase [for $\hat{\sigma} > \hat{\sigma}_c = 1/(4d)$ and $\beta > \beta_c = \sqrt{d/\sigma}$]: The KPP velocity is frozen. The disorder wins and the particle freezes in wells away from the origin.

The bound-phase [for $\hat{\sigma} < \hat{\sigma}_c = 1/(4d)$ and $\beta < \beta^*(\sigma)$]: The particle is bound to the origin. Within this phase near the phase boundaries (where the bound-state length is large), a crossover can be distinguished as a remnant of the freezing transition. The bound phase arises because of the bound $f \leqslant 0$ (or equivalently the velocity of the KPP equation must remain positive).

In model II, the bound phase corresponds to no vortex present. When one vortex is present, it can be either localized in a few wells or in a high-T phase (as studied in the text of this paper).

Both transitions away from the bound phase are first order, while the transition between the high-temperature phase and the localized phase is continuous. An interesting feature is the multicritical point where the transition becomes continuous.

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[24] The exponent of the growth of correlations $V \sim r^{\alpha}$ would correspond in, e.g., the problem of a manifold in a random potential to $\alpha = \theta/\zeta$, where $r \sim x^{\zeta}$ and θ is the energy fluctuation exponent such that $\delta E \sim x^{\theta}$. Thus $\alpha = 0$ corresponds to $\theta = 0$.

[25] For instance, in [13] a microcanonical approach was studied. The averaged $\overline{\Omega(E)}$ was computed. This calculation [Eq. (3.1) of [13]] is by definition identical to the REM approximation $\overline{\Omega(E)} \equiv \overline{\Omega(E)}_{\text{REM}}$. All further analytical results of [13] are based on this calculation, and hence are in effect obtained using the REM approximation. To fully establish these results within this microcanonical approach, one needs to show the crucial property that $\ln \Omega(E)/\ln L$ in (almost) any sample converges to the annealed average $\ln(\overline{\Omega(E)})/\ln L$. This, however, is not done in [13]. In particular, the result of [13] that the quenched entropy is bounded by the annealed one is not sufficient to establish these results nor the existence of a transition. Although qualitative arguments are given in Appendix B, there is no quantitative analysis of deviations from the REM behavior, especially in *typical* quantities, which are not necessarily related to higher integer moments, as explained in [20].

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[36] But not necessarily for stronger correlations.

[37] The need to understand the problem beyond the REM approximation can be illustrated as follows. Suppose that one simply shifts all variables $V(\mathbf{r})$ by a uniform but random Gaussian variable of variance $\sim \ln L$. This clearly does not change the true transition temperature while it does change the REM approximation transition temperature [it adds a constant to $\Gamma_L(0)$ without changing $\tilde{\Gamma}_L(\mathbf{r})$].

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[39] The problem is subtle, as can be seen by the failure of the following simple argument. Since there are, as will be discussed, only a few low-lying states adding a small Gaussian noise with only short-range correlations (or uncorrelated) $V(\mathbf{r}) \rightarrow V(\mathbf{r}) + w(\mathbf{r})$, one could conclude that the distribution of $\delta V_{\min}$ is changed [since typically $V_{\min} \rightarrow V_{\min} + w(\mathbf{r}_{\min})$]. This argument is insufficient, however, as it would yield the

same conclusion for the REM approximation while there we know (see below) that the distribution does not change: there the probability that a secondary minima becomes, after the shift, the absolute one compensates exactly the additional Gaussian fluctuation.

[40] These results are probably more general than strictly for Gaussian distributions, provided the single site distribution $P(V_r)$ decays faster than exponentially. Clearly, though, one should be careful in making general statements about correlated variables: one can always tailor artificial correlations to produce more or less pathological exceptions, e.g., one can consider a generalized LR Sinai potential $V(r)$ which performs a one-sided random walk, and which clearly has infinitely many exactly degenerate minima, separated by arbitrarily large barriers.

[41] We assume here that the tail of $P_1(V)$ is faster than an exponential.

[42] For example, scaling R algebraically with L.

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[46] Note that, although the Gibbs probability is typically concentrated around a single minimum, some quantities are very sensitive to the rare configurations with several almost degenerate low-lying minima. For instance, the thermal width of a packet $\overline{\langle x^2\rangle-\langle x\rangle^2}$ is easily computed (via methods as in [45]) and shown to grow as $TL^{3/2}$ (since with probability $1/L^{1/2}$ there are two degenerate minima separated typically by L).

[47] As detailed in [20], it arises naturally if one consider first $\overline{Z^m}$ on a lattice, which can be written *exactly* as a vector CG on the lattice, and then takes the continuum approximation of this vector CG model.

[48] B. Nienhuis, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Leibovitz (Academic Press, London, 1987), Vol. 11.

[49] The convention is different from [20] with $x\to -x$ and $G\to -G$.

[50] It is correct if $\langle z\rangle_{P_l}$ remains finite, which is the case if one starts from a well-behaved initial condition. However, below the glass transition, $\langle z\rangle_{P_l}$ grows very fast with l (being dominated by very rare events) and the intermediate asymptotic behavior becomes then different from $1-G_l(x)\sim e^{\beta x}$ (only at much larger x is this behavior recovered).

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