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Ising model on nonorientable surfaces: Exact solution for the Möbius strip and the Klein bottle

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Closed-form expressions are obtained for the partition function of the Ising model on an $\mathcal{M}\times\mathcal{N}$ simple-quartic lattice embedded on a Möbius strip and a Klein bottle. The solutions all lead to the same bulk free energy, but for finite $\mathcal{M}$ and $\mathcal{N}$ the expressions are different depending on whether the strip width $\mathcal{M}$ is odd or even. Finite-size corrections at criticality are analyzed and compared with those under cylindrical and toroidal boundary conditions. Our results are consistent with the conformal field prediction of a central charge $c=1/2$, provided that the twisted Möbius boundary condition is regarded as a free or fixed boundary.

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I. INTRODUCTION

There has been considerable recent interest [1–3] in studying lattice models on nonorientable surfaces, both as new challenging unsolved lattice-statistical problems and as a realization and testing of predictions of the conformal field theory [4]. In a recent paper [1] we presented the solution of dimers on the Möbius strip and Klein bottle, and studied its finite-size corrections. In this paper we consider the Ising model.

The Ising model in two dimensions was first solved in 1944 by Onsager [5], who obtained a closed-form expression of the partition function for a simple-quartic $\mathcal{M}\times\mathcal{N}$ lattice wrapped on a cylinder. The exact solution for an $\mathcal{M}\times\mathcal{N}$ lattice on a torus, namely, with periodic boundary conditions in both directions, was obtained by Kaufman four years later [6]. Onsager and Kaufman used spinor analysis to derive the solutions, and the solution under the cylindrical boundary condition was rederived later by McCoy and Wu [7] using the method of dimers. More recently, we obtained the solution for a finite Ising lattice with a self-dual boundary condition [8]. As far as we know, these are the only known solutions of the two-dimensional Ising model on finite lattices with conventional boundary conditions.

Here, using the method of dimers, we derive exact expressions for the partition function of the Ising model on finite Möbius strips and Klein bottles. As we shall see, as a consequence of the Möbius topology, the solution assumes a form which depends on whether the width of the Möbius strip is even or odd. However, all solutions yield the same bulk free energy. We also present results of finite-size analyses for corrections to the bulk solution, and compare with those deduced under other boundary conditions. Our explicit calculations confirm that the central charge is $c=1/2$, provided that the twisted Möbius boundary condition is regarded as a free or fixed boundary.

The organization of this paper is as follows. The partition function for a $2M\times N$ Möbius strip is evaluated in Sec. II, with details given in Sec. III. The partition function for a $(2M-1)\times N$ Möbius strip is evaluated in Sec. IV, and results for the Klein bottle are given in Sec. V. In Sec. VI we carry out a finite-size analysis for large lattices, and results at criticality are given for $2M\times N$ lattices.

II. $2M\times N$ MÖBIUS STRIP

To begin with, we consider a $2M\times N$ simple-quartic Ising lattice $\mathcal{L}$ embedded on a Möbius strip, where M and N are integers, $2M$ is the width, and N is the length of the strip, which can be either even or odd. The Ising model has anisotropic reduced interactions K_h along the (horizontal) length direction, and K_v along the (vertical) width direction. The example of a 4×5 Möbius strip $\mathcal{L}$ is shown in Fig. 1.

To facilitate considerations, it is convenient to let the row of N vertical edges located in the middle of the strip take on a different interaction K_1, as shown. The desired result is then obtained by setting $K_1=K_v$. In addition, by setting $K_1=0$ the Möbius strip reduces to an $M\times 2N$ strip with a ''cylindrical'' boundary condition, namely, periodic in one direction and free in the other, for which the partition function was evaluated by McCoy and Wu [7]. By setting $K_1=\infty$ the two center rows of spins coalesce into a single row with an (additive) interaction, which in this case is $2K_h$. These are two key elements of our consideration.

Following standard procedures [7] we write the partition function of the Ising model on $\mathcal{L}$ as

$$Z^{\mathrm{Mob}}_{2M,N}(K_h,K_v,K_1)=2^{2MN}(\cosh K_h)^{2MN}(\cosh K_v)^{2(M-1)N}\times(\cosh K_1)^N G(z_h,z_v,z_1), \tag{1}$$

where $z_h=\tanh K_h$, $z_v=\tanh K_v$, and $z_1=\tanh K_1$, and

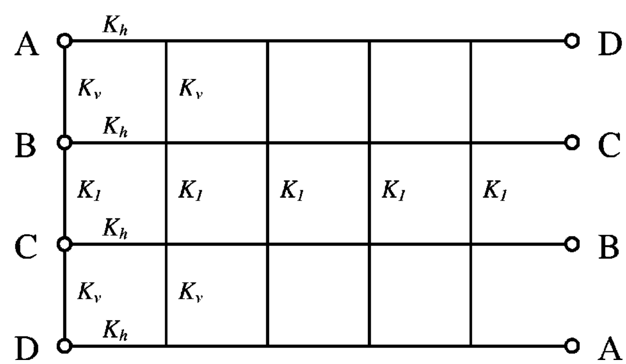

FIG. 1. A 4×5 Möbius strip $\mathcal{L}$. Vertices labeled A,B,C, and D are repeated sites.

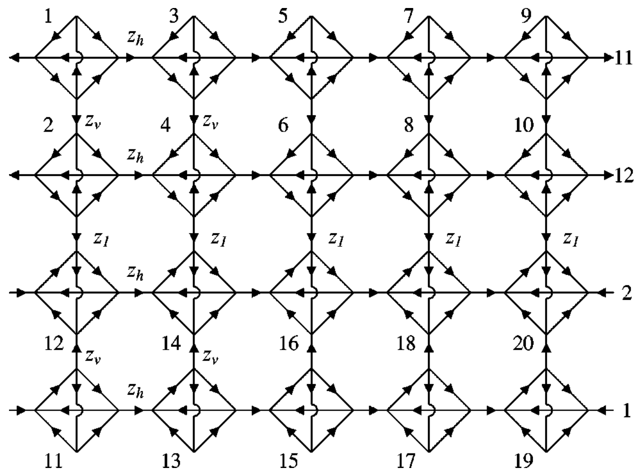

FIG. 2. The dimer lattice $\mathcal{L}_D$ corresponding to the 4×5 Möbius strip. The cities are numbered from 1 to 20, and the six edges within each city all carry the weight 1.

$$G(z_h,z_v,z_1)=\sum_{\text{closed polygons}} z_h^{n_h} z_v^{n_h} z_1^{n_1} \tag{2}$$

is the generating function of all closed polygonal graphs on $\mathcal{L}$ with edge weights z_h, z_v, and z_1. Here, n_h, n_v, and n_1 are the numbers of polygonal edges with weights z_h, z_v, and z_1, respectively.

The generating function $G(z_h,z_v,z_1)$ is a multinomial in z_h, z_v, and z_1 and, due to the Möbius topology, the integer n_1 can take on any value in $\{0,N\}$. Thus we have

$$G(z_h,z_v,z_1)=\sum_{n_1=0}^{N} T_{n_1}(z_h,z_v) z_1^{n_1}, \tag{3}$$

where $T_{n_1}(z_h,z_v)$ are polynomials in z_h and z_v with strictly positive coefficients.

To evaluate $G(z_h,z_v,z_1)$, we again follow the usual procedure of mapping polygonal configurations on $\mathcal{L}$ onto dimer configurations on a dimer lattice $\mathcal{L}_D$ of $8MN$ sites, constructed by expanding each site of $\mathcal{L}$ into a ''city'' of four sites [7–9]. The resulting $\mathcal{L}_D$ for the 4×5 $\mathcal{L}$ is shown in Fig. 2.

Since the deletion of all z_1 edges reduces the lattice to one with a cylindrical boundary condition solved in Ref. [7], we orient all edges with weights z_h, z_v, and 1 as in Ref. [7]. In addition, all z_1 edges are oriented in the direction shown in Fig. 2. Then we have the following theorem:

Theorem: Let A be the $8MN\times 8MN$ antisymmetric determinant defined by the lattice edge orientation shown in Fig. 2, and let

$$\mathrm{Pf}\,A(z_h,z_v,z_1)=\sqrt{\det A(z_h,z_v,z_1)} \tag{4}$$

denote the Pfaffian of A. Then

$$\mathrm{Pf}\,A(z_h,z_v,z_1)=\sum_{n_1=0}^{N} \epsilon_{n_1} T_{n_1}(z_h,z_v) z_1^{n_1}, \tag{5}$$

where $\epsilon_{4m}=\epsilon_{4m+1}=1$ and $\epsilon_{4m+2}=\epsilon_{4m+3}=-1$ for any integer $m\geqslant 0$.

Remark: Define

$$X_p=\sum_{m=0}^{[N/4]} T_{4m+p}(z_h,z_v) z_1^{4m+p}, \quad p=0,1,2, \text{ and } 3, \tag{6}$$

where $[N/4]$ is the integral part of $N/4$, so that $G(z_h,z_v,z_1)=X_0+X_1+X_2+X_3$. It then follows from Eq. (5) that we have

$$\mathrm{Pf}\,A(z_h,z_v,\pm i z_1)=X_0+X_2\pm i(X_1+X_3). \tag{7}$$

As a consequence, we obtain

$$G(z_h,z_v,z_1)=\tfrac{1}{2}[(1-i)\mathrm{Pf}\,A(z_h,z_v,iz_1) + (1+i)\mathrm{Pf}\,A(z_h,z_v,-iz_1)], \tag{8}$$

where, as evaluated in the next section, the Pfaffian is given by

$$\mathrm{Pf}\,A(z_h,z_v,z_1) =[z_v(1-z_h^2)]^{MN} \times\prod_{n=1}^{N}\left[\frac{\sinh(M+1)t(\phi_n)-c(z_1)\sinh Mt(\phi_n)}{\sinh t(\phi_n)}\right], \tag{9}$$

with

$$c(z_1)=\frac{z_v^2(1+z_h^2+2z_h\cos\phi_n)+2(-1)^n z_1 z_h \sin\phi_n}{z_v(1-z_h^2)},$$

$$\cosh t(\phi)=\frac{\cosh 2K_h\cosh 2K_v-\sinh 2K_h\cos\phi}{\sinh 2K_v}, \tag{10}$$

$$\phi_n=(2n-1)\pi/2N.$$

Here we have used the fact that $\Pi_{n=1}^{N}=\Pi_{n=N+1}^{2N}$ in the product in Eq. (9). Substituting these results into Eq. (1), and setting $K_1=K_v$, we are led to the following explicit expression for the partition function:

$$Z_{2M,N}^{\mathrm{Mob}}(K_h,K_v,K_v)=\tfrac{1}{2}(2\sinh 2K_v)^{MN}(\cosh K_v)^{-N} \times[(1-i)F_+ +(1+i)F_-], \tag{11}$$

where

$$F_\pm=\prod_{n=1}^{N}\left[e^{Mt(\phi_n)}\left(\frac{e^{t(\phi_n)}-c(\pm i z_v)}{2\sinh t(\phi_n)}\right) - e^{-Mt(\phi_n)}\left(\frac{e^{-t(\phi_n)}-c(\pm i z_v)}{2\sinh t(\phi_n)}\right)\right]. \tag{12}$$

This completes the evaluation of the Ising partition function for the $2M\times N$ Möbius strip. For example, for a 2×5 Möbius strip, this leads to

$$\mathrm{Pf}\,A(z_h,z_v,z_1)=1+z_h^{10}+10z_1z_h^5-5z_1^2z_h^2(1+z_h^2+z_h^4+z_h^6)-20z_1^3z_h^5+5z_1^4z_h^4(1+z_h^2)+2z_1^5z_h^5,$$

$$G(z_h,z_v,z_1)=1+z_h^{10}+10z_1z_h^5+5z_1^2z_h^2(1+z_h^2+z_h^4+z_h^6)+20z_1^3z_h^5+5z_1^4z_h^4(1+z_h^2)+2z_1^5z_h^5, \quad (13)$$

which can be verified by explicit enumerations.

Note that we have $\cosh t(\phi_n)\geqslant 1$, so we can always take $t(\phi_n)\geqslant 0$. For large M, the leading contribution in Eq. (12) is therefore

$$F_\pm \sim \prod_{n=1}^{N} e^{Mt(\phi_n)}, \quad (14)$$

and, hence, from Eq. (11),

$$\frac{1}{2MN}\ln Z_{2M,N}^{\mathrm{Mob}}(K_h,K_v,K_v)\sim\frac{1}{2}\ln(2\sinh 2K_v)+\frac{1}{2N}\sum_{n=1}^{N}t(\phi_n). \quad (15)$$

We now prove the theorem.

Considered as a multinomial in z_h, z_v, and z_1, there exists a one-to-one correspondence between terms in the dimer generating function $G(z_h,z_v,z_1)$ and (linear combinations of) terms in the Pfaffian [Eq. (4)]. However, while all terms in $G(z_h,z_v,z_1)$ are positive, terms in the Pfaffian do not necessarily possess the same sign. The crux of the matter is to find an appropriate linear combination of Pfaffians to yield the desired $G(z_h,z_v,z_1)$. For this purpose it is convenient to compare an arbitrary term C_1 in the Pfaffian with a standard one C_0. We choose C_0 to be a term in which no z_h, z_v, and z_1 dimers are present.

The superposition of two dimer configurations represented by C_0 and C_1 produces superposition polygons. Kasteleyn [10] showed that the two terms will have the same sign if edges of $\mathcal{L}_D$ can be oriented such that all superposition polygons are oriented "clockwise-odd," namely, that there be an odd number of edges oriented in the clockwise direction.

Now since all z_h, z_v, and 1 edges of $\mathcal{L}_D$ are oriented as in Ref. [7], terms in the Pfaffian with no z_1 edges ($n_1=0$) will have the same sign as C_0. To determine the sign of a term when z_1 edges are present, we associate a + sign with each clockwise-odd superposition polygon, and a − sign to each clockwise-even superposition polygon. Then the sign of C_1 relative to C_0 is the product of the signs of all superposition polygons. The following elementary facts can be readily verified.

(i) Deformations of the borders of a superposition polygon always change m_1, the number of *its* z_1 edges, by multiples of 2.

(ii) The sign of a superposition polygon is reversed under border deformations which change m_1 by 2.

(iii) Superposition polygons having 0 or 1 z_1 edges have a sign +.

(iv) There can be at most one superposition polygon having an odd number of z_1 edges, a property unique to nonorientable surfaces.

Let $m_1=4m+p$, where m is an integer and $p=0,1,2$, and 3. Because of point (iv), we need only to consider the presence of at most one polygon having $p=1$ or 3. It now follows from points (i) and (iii) that $\epsilon_{4m}=\epsilon_{4m+1}=+$, and from points (i), (ii), and (iii) that $\epsilon_{4m+2}=\epsilon_{4m+3}=-$. This establishes the theorem.

III. EVALUATION OF THE PFAFFIAN

We now derive expression (9). From the edge orientation of $\mathcal{L}_D$ of Fig. 2, one finds that the $8MN\times 8MN$ antisymmetric matrix A assumes the form

$$A(z_h,z_v,z_1)=A_0(z_v)\otimes I_{2N}+A_+(z_h)\otimes J_{2N}+A_-(z_h)\otimes J_{2N}^T+A_1(z_1)\otimes H_{2N}, \quad (16)$$

where A_0, A_+, A_-, and A_1 are $4M\times 4M$ matrices, I_{2N} is the $2N\times 2N$ identity matrix, and J_{2N} and H_{2N} are $2N\times 2N$ matrices:

$$J_{2N}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix},\quad H_{2N}=\begin{pmatrix} 0 & I_N\\ -I_N & 0\end{pmatrix}. \quad (17)$$

In addition, one has

$$A_0(z_v)=a_{0,0}\otimes I_M+a_{0,1}(z_v)\otimes F_M+a_{0,-1}(z_v)\otimes F_M^T,$$
$$A_\pm(z_h)=a_{\pm1,0}(z_h)\otimes I_M, \quad (18)$$
$$A_1(z_1)=a(z_1)\otimes G_M,$$

where F_M and G_M are $M\times M$ matrices:

$$F_M=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$G_M=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad (19)$$

F_M^T is the transpose of F_M, and

$$a_{0,0}=\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix},\quad a(z)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$a_{1,0}(z)=\begin{pmatrix} 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad a_{0,1}(z)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

$$a_{-1,0}(z)=-a_{1,0}^T(z),\quad a_{0,-1}(z)=-a_{0,1}^T(z).$$

We use the fact that the determinant in Eq. (4) is equal to the product of the eigenvalues of matrix A. To evaluate the latter, we note that J_{2N}, J_{2N}^T, and H_{2N} mutually commute, so that they can be diagonalized simultaneously. This leads to the respective eigenvalues $e^{i\phi_n}, e^{-i\phi_n}$ and $i(-1)^{n+1}$ and the expression

$$\det A(z_h,z_v,z_1)=\prod_{n=1}^{2N}\det A_M(z_h,z_v,z_1;\phi_n), \tag{21}$$

where

$$A_M(z_h,z_v,z_1;\phi_n)=A_0(z_v)+A_+(z_h)e^{i\phi_n}+A_-(z_h)e^{-i\phi_n}+i(-1)^{n+1}A_1(z_1) \tag{22}$$

is a $4M\times 4M$ matrix. Writing this out explicitly, we have

$$A_M(z_h,z_v,z_1;\phi_n)=\begin{pmatrix} B(z_h) & a_{0,1}(z_v) & 0 & & \\ a_{0,-1}(z_v) & B(z_h) & a_{0,1}(z_v) & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & a_{0,-1}(z_v) & B(z_h) & a_{0,1}(z_v) \\ & & 0 & a_{0,-1}(z_v) & C(z_h,z_1) \end{pmatrix}, \tag{23}$$

where $C(z,z_1)=B(z)+i(-1)^{n+1}a(z_1)$, and

$$B(z)=\begin{pmatrix} 0 & 1+ze^{i\phi_n} & -1 & -1 \\ -(1+ze^{-i\phi_n}) & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}. \tag{24}$$

The evaluation of $\det A_M(z_h,z_v,z_1;\phi_n)$ can be carried out by using a recursive procedure introduced in Ref. [8]. Specifically, let $B_M=B_M(\phi_n)=\det A_M(z_h,z_v,z_1;\phi_n)$, and D_M be the determinant of the matrix $A_M(z_h,z_v,z_1;\phi_n)$ with the fourth row and fourth column removed. Then we have

$$\mathrm{Pf}\,A(z_h,z_v,z_1)=\sqrt{\det A(z_h,z_v,z_1)}=\prod_{n=1}^{2N}\sqrt{B_M(\phi_n)}. \tag{25}$$

Furthermore, by expanding the determinants one finds the recursion relation (which is the same as that in Ref. [8] when $z_h=z_v=z$),

$$\begin{pmatrix} B_M \\ D_M \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} B_{M-1} \\ D_{M-1} \end{pmatrix},\quad M\geqslant 2, \tag{26}$$

with

$$a_{11}=1+z_h^2-2z_h\cos\phi_n,$$

$$a_{12}=-2iz_v^2z_h\sin\phi_n, \tag{27}$$

$$a_{21}=2iz_h\sin\phi_n,$$

$$a_{22}=z_v^2(1+z_h^2+2z_h\cos\phi_n),$$

and the initial conditions (which are different from Ref. [8])

$$B_1\equiv B_1(z_h,z_1)=1+z_h^2-2z_h\cos\phi_n-2(-1)^n z_1z_h\sin\phi_n,$$

$$D_1\equiv D_1(z_h,z_1)=2iz_h\sin\phi_n-i(-1)^n z_1(1+z_h^2+2z_h\cos\phi_n). \tag{28}$$

This leads to the solutions

$$B_M(\phi_n)=B_1\frac{\lambda_+^M-\lambda_-^M}{\lambda_+-\lambda_-}-(a_{22}B_1-a_{12}D_1)\frac{\lambda_+^{M-1}-\lambda_-^{M-1}}{\lambda_+-\lambda_-},$$

$$D_M(\phi_n)=D_1\frac{\lambda_+^M-\lambda_-^M}{\lambda_+-\lambda_-}-(a_{11}D_1-a_{21}B_1)\frac{\lambda_+^{M-1}-\lambda_-^{M-1}}{\lambda_+-\lambda_-}, \tag{29}$$

where $\lambda_\pm=z_v(1-z_h^2)e^{\pm t(\phi_n)}$ are the eigenvalues of the 2×2 matrix in Eq. (26). After some algebraic manipulation, from Eqs. (25) and (29) we obtain the expression [Eq. (9)] quoted in Sec. II.

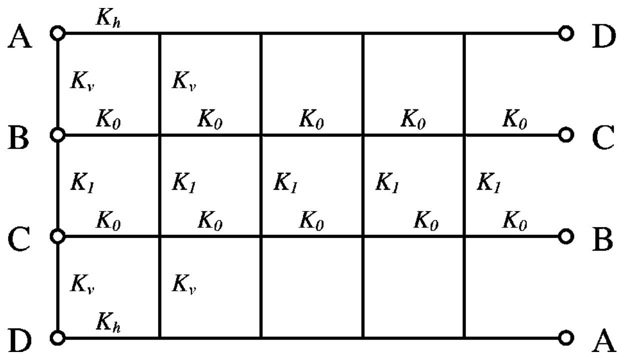

FIG. 3. Labelings of a 4×5 Möbius strip which reduces to a 3×5 Möbius strip upon taking $K_1=\infty$.

IV. $(2M-1)\times N$ MÖBIUS STRIP

The $(2M-1)\times N$ Möbius strip is considered in this section. In order to make use of results of the preceding sections, we start from the $2M\times N$ Möbius strip of Sec. II, and let spins in the two center rows of the strip [the Mth and $(M+1)$th rows] having interactions $K_0=K_h/2$. The example of a 4×5 lattice with these interactions is shown in Fig. 3. Then, by taking $K_1=\infty$ $(z_1=1)$, as described in Sec. I, this lattice reduces to the desired $(2M-1)\times N$ Möbius strip of uniform interactions K_h and K_v.

Following this procedure, we have

$$Z^{\text{Mob}}_{2M-1,N}(K_h,K_v)=2^{(2M-1)N}(\cosh K_h\cosh K_v)^{2(M-1)N}\times\cosh^{2N}(K_h/2)G(z_h,z_v,z_0,z_1)|_{z_1=1},\tag{30}$$

where $z_0=\tanh(K_h/2)$, and $G(z_h,z_v,z_0,z_1)$ is the generating function of closed polygons on the $2M\times N$ Möbius net with edge weights as described above.

The generating function $G(z_h,z_v,z_0,z_1)$ can be evaluated as in the previous sections. In place of Eq. (25), we now have

$$\text{Pf}\,A(z_h,z_v,z_0,z_1)=\prod_{n=1}^{2N}\sqrt{\det A_M(z_h,z_v,z_0,z_1;\phi_n)},\tag{31}$$

where

$$A_M(z_h,z_v,z_0,z_1;\phi_n)=\begin{pmatrix} B(z_h) & a_{0,1}(z_v) & 0 & & \\ a_{0,-1}(z_v) & B(z_h) & a_{0,1}(z_v) & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & a_{0,-1}(z_v) & B(z_h) & a_{0,1}(z_v) \\ & & 0 & a_{0,-1}(z_v) & C(z_0,z_1) \end{pmatrix}.\tag{32}$$

Then Eq. (8) becomes

$$G(z_h,z_v,z_0,z_1)=\tfrac{1}{2}[(1-i)\text{Pf}\,A(z_h,z_v,z_0,iz_1)+(1+i)\text{Pf}\,A(z_h,z_v,z_0,-iz_1)].\tag{33}$$

The evaluation of $\det A_M(z_h,z_v,z_0,z_1;\phi_n)$ can again be done recursively. As before, define $B_M=\det A_M(z_h,z_v,z_0,z_1;\phi_n)$, and let D_M be the determinant of A_M with the fourth row and column removed; one again obtains recursion relations (26), and arrives at precisely the same solution [Eq. (29)], but now with different initial conditions

$$B_1=B_1(z_0,z_1),\quad D_1=D_1(z_0,z_1),\tag{34}$$

where the functions B_1 and D_1 are as defined in Eq. (28). After some algebra, this leads to the solution

$$\text{Pf}\,A(z_h,z_v,z_0,z_1)=[z_v(1-z_h^2)]^{(M-1)N}\times\prod_{n=1}^{N}\left[\frac{c_1\sinh Mt(\phi_n)-c_2\sinh(M-1)t(\phi_n)}{\sinh t(\phi_n)}\right],\tag{35}$$

where

$$c_1=\frac{2z_0}{z_h}\{1-z_h[\cos\phi_n+(-1)^n z_1\sin\phi_n]\},$$

$$c_2=2z_0z_v\left\{\frac{1}{z_h}+[\cos\phi_n+(-1)^n z_1\sin\phi_n]\right\}.\tag{36}$$

The substitution of Eq. (35) into Eqs. (33) and (30) now completes the evaluation of the partition function for a $(2M-1)\times N$ Möbius strip.

V. KLEIN BOTTLE

The Ising model on a Klein bottle can be considered similarly. We first consider a $2M\times N$ lattice $\mathcal{L}$, constructed by connecting the upper and lower edges of the Möbius strip of Fig. 1 in a periodic fashion with N extra vertical edges. As in the case of the Möbius strip, it is convenient to let the extra edges have interactions K_2. The desired solution is obtained at the end by setting $K_1=K_2=K_v$.

The Ising partition function for the Klein bottle now assumes the form

$$\begin{aligned}Z^{\mathrm{Kln}}_{2M,N}&(K_h,K_v,K_1,K_2)\\&=2^{2MN}(\cosh K_h)^{2MN}(\cosh K_v)^{2(M-1)N}\\&\quad\times(\cosh K_1\cosh K_2)^N G^{\mathrm{Kln}}(z_h,z_v,z_1,z_2),\end{aligned}\tag{37}$$

where

$$G^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)=\sum_{\text{closed polygons}} z_h^{n_h}z_v^{n_v}z_1^{n_1}z_2^{n_2}\tag{38}$$

generates all closed polygons on the $2M\times N$ lattice $\mathcal{L}$ with edge weights $z_i=\tanh K_i$, $i=h,v,1$, and 2. The desired partition function is then given by

$$\begin{aligned}Z^{\mathrm{Kln}}_{2M,N}&(K_h,K_v,K_v,K_v)\\&=2^{2MN}(\cosh K_h\cosh K_v)^{2MN}G^{\mathrm{Kln}}(z_h,z_v,z_v,z_v).\end{aligned}\tag{39}$$

Again, it is convenient to first write $G^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)$ as a multinomial in z_h,z_v,z_1, and z_2 in the form of

$$G^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)=\sum_{m,n=0}^{N} T_{m,n}(z_h,z_v)z_1^m z_2^n,\tag{40}$$

where $T_{m,n}(z_h,z_v)$ are polynomials in z_h and z_v with strictly positive coefficients.

The evaluation of $G^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)$ parallels that of $G(z_h,z_v,z_1)$ for the Möbius strip. One first maps the lattice $\mathcal{L}$ into a dimer lattice $\mathcal{L}_D$ by expanding each site into a city of four sites, as shown in Fig. 2. Orient all k_h, k_v, and k_1 edges of $\mathcal{L}_D$ as shown, and orient all k_2 edges in the same (downward) direction as the k_1 edges. Then this defines an $8MN\times 8MN$ antisymmetric matrix obtained by adding an extra term to $A(z_h,z_v,z_1)$ given by Eq. (16), namely,

$$A^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)=A(z_h,z_v,z_1)+b(z_2)\otimes G'_M\otimes H_{2N}.\tag{41}$$

Here

$$b(z_2)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-z_2\end{pmatrix},$$

$$G'_M=\begin{pmatrix}1&0&\cdots&0&0\\0&0&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&0&0\\0&0&\cdots&0&0\end{pmatrix}.\tag{42}$$

Then, in place of theorem (5), we now have

$$\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)=\sum_{m,n=0}^{N}\epsilon_m\epsilon_n T_{m,n}(z_h,z_v)z_1^m z_2^n,\tag{43}$$

from which, in a similar manner, one obtains the result

$$\begin{aligned}G^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)=\tfrac{1}{2}[&\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,iz_1,-iz_2)\\&+\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,-iz_1,iz_2)\\&-i\,\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,iz_1,iz_2)\\&+i\,\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,-iz_1,-iz_2)].\end{aligned}\tag{44}$$

To evaluate the Pfaffian (43), we note that matrix (41) can again be diagonalized in the $\{2N\}$ subspace, yielding

$$\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)=\prod_{n=1}^{2N}\sqrt{\det A_M^{\mathrm{Kln}}(z_h,z_v,z_1,z_2;\phi_n)},\tag{45}$$

where

$$\begin{aligned}A_M^{\mathrm{Kln}}(z_h,z_v,z_1,z_2;\phi_n)&=A_M(z_h,z_v,z_1;\phi_n)\\&\quad+i(-1)^{n+1}b(z_2)\otimes G'_M.\end{aligned}\tag{46}$$

Now we expand $\det A_M^{\mathrm{Kln}}$ in z_2. Since, upon setting $z_2=0$, the determinant is precisely B_M and the term linear in z_2, the $\{4,4\}$ element of the determinant, is by definition D_M, one obtains

$$\det A_M^{\mathrm{Kln}}(z_h,z_v,z_1,z_2;\phi_n)=B_M+i(-1)^n z_2 D_M,\quad M\geqslant 2,\tag{47}$$

where B_M and D_M were already computed in Eq. (29). This leads to

$$\begin{aligned}\mathrm{Pf}\,A^{\mathrm{Kln}}&(z_h,z_v,z_1,z_2)\\&=\left(1+\frac{z_1z_2}{z_v^2}\right)^N[z_v(1-z_h^2)]^{MN}\times\\&\quad\times\prod_{n=1}^{N}\left[\frac{\sinh(M+1)t-c(z_1,z_2)\sinh Mt}{\sinh t}\right],\end{aligned}\tag{48}$$

where

$$c(z_1,z_2)=\frac{1}{z_v(1-z_h^2)(z_v^2+z_1z_2)}[(1+z_h^2)(z_v^4+z_1z_2)+2z_h(z_v^4-z_1z_2)\cos\phi_n+2(-1)^n\times(z_1+z_2)z_hz_v^2\sin\phi_n]. \quad (49)$$

Setting $z_1=z_2=z_v$ in Eq. (44) and using Eq. (48), after some algebra one obtains

$$G^{\mathrm{Kln}}(z_h,z_v,z_v,z_v)=[z_v(1-z_h^2)]^{MN}\left[\prod_{n=1}^{N}2\cosh Mt(\phi_n)+\mathrm{Im}\prod_{n=1}^{N}\left(\frac{\sinh Mt(\phi_n)}{\sinh t(\phi_n)}D(\phi_n)\right)\right], \quad (50)$$

where

$$D(\phi_n)=\frac{1}{z_v(1-z_h^2)}[(1+z_h^2)(1-z_v^2)-2z_h(1+z_v^2)\cos\phi_n-4i(-1)^nz_hz_v\sin\phi_n], \quad (51)$$

and Im denotes the imaginary part. The substitution of Eq. (50) into Eq. (39) now completes the evaluation of the partition function for a $2M\times N$ Klein bottle.

For a 2×2 Klein bottle, for example, one finds

$$\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)=1+z_h^4+4(z_1+z_2)z_h^2-2(z_1^2+z_2^2)z_h^2+2z_1z_2(1+z_h^2)^2-4z_1z_2(z_1+z_2)z_h^2+z_1^2z_2^2(1+z_h^4),$$

$$G^{\mathrm{Kln}}(z_h,z_v,z_1,z_2)=1+z_h^4+4(z_1+z_2)z_h^2+2(z_1^2+z_2^2)z_h^2+2z_1z_2(1+z_h^2)^2+4z_1z_2(z_1+z_2)z_h^2+z_1^2z_2^2(1+z_h^4), \quad (52)$$

which can be verified by explicit enumerations.

For a $(2M-1)\times N$ Klein bottle we can proceed as above by first considering a $2M\times N$ Klein bottle with interactions K_h,K_v,K_1, and K_2 and, within the center two rows, interactions $K_0=K_h/2$, as shown in Fig. 3. This is followed by taking $K_1\to\infty$ and $K_2=K_v$. Thus, in place of Eq. (39), we have

$$Z_{2M-1,N}^{\mathrm{Kln}}(K_h,K_v)=2^{(2M-1)N}(\cosh K_h)^{2MN}\times(\cosh K_v)^{(2M-3)N}\cosh^{2N}(K_h/2)\times G^{\mathrm{Kln}}(z_h,z_v,z_0,1,z_v), \quad (53)$$

where $z_0=\tanh(K_h/2)$, and $G^{\mathrm{Kln}}(z_h,z_v,z_0,z_1,z_2)$ generates polygonal configurations on the $2M\times N$ lattice with weights as shown. Then, as in the above, we find

$$G^{\mathrm{Kln}}(z_h,z_v,z_0,z_1,z_2)=\tfrac{1}{2}[\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,z_0,iz_1,-iz_2)+\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,z_0,-iz_1,iz_2)-i\,\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,z_0,iz_1,iz_2)+i\,\mathrm{Pf}\,A^{\mathrm{Kln}}(z_h,z_v,z_0,-iz_1,-iz_2)], \quad (54)$$

where $\mathrm{Pf}A^{\mathrm{Kln}}(z_h,z_v,z_0,z_1,z_2)$ is found to be given by the right-hand side of Eq. (35), but now with

$$c_1=(1+z_0^2)(1-z_1z_2)-2z_0(1+z_1z_2)\times\cos\phi_n-2(-1)^n(z_1+z_2)z_0\sin\phi_n,$$

$$c_2=\frac{1}{z_v(1-z_h^2)}\{(z_v^2+z_1z_2)[(1-z_hz_0)^2+(z_h-z_0)^2]+2(z_h-z_0)(1-z_hz_0)[(z_v^2-z_1z_2)\cos\phi_n+(-1)^n(z_v^2z_1+z_2)\sin\phi_n]\}, \quad (55)$$

expressions which are valid for arbitrary z_h,z_v,z_0,z_1, and z_2. For $z_0=\tanh(K_h/2)$, the case we are considering, Eq. (55) reduces to

$$c_1=\frac{2z_0}{z_h}[1-z_1z_2-z_h(1+z_1z_2)\cos\phi_n-(-1)^nz_h(z_1+z_2)\sin\phi_n],$$

$$c_2=\frac{2z_0}{z_hz_v}[z_v^2+z_1z_2+z_h(z_v^2-z_1z_2)\cos\phi_n+(-1)^nz_h(z_v^2z_1+z_2)\sin\phi_n] \quad (56)$$

[which reduces further to Eq. (36) after setting $z_2=0$]. The explicit expression for the partition function is now obtained by substituting Eq. (54) into Eq. (53).

VI. BULK LIMIT AND FINITE-SIZE CORRECTIONS

In the thermodynamic limit, our solutions of the Ising partition function give rise to a bulk ‘‘free energy’’

$$f_{\mathrm{bulk}}(K_h,K_v)=\lim_{M,N\to\infty}\frac{1}{2MN}\ln Z(K_h,K_v). \quad (57)$$

Here, $Z(K_h,K_v)$ is any one of the four partition functions. For example, using the solution $Z_{2M,N}^{\mathrm{Mob}}$ given by Eq. (15) for the $2M\times N$ Möbius strip, one obtains

$$f_{\mathrm{bulk}}(K_h,K_v)=\frac{1}{2}\ln(2\sinh2K_v)+\frac{1}{2\pi}\int_0^{\pi}d\phi\,t(\phi), \quad (58)$$

where $t(\phi)$ is given by Eq. (10). This leads to the Onsager solution

$$f_{\rm bulk}(K_h,K_v)=\ln 2+\frac{1}{2\pi^2}\int_0^\pi d\phi\int_0^\pi d\theta \times\ln[\cosh 2K_h\cosh 2K_v-\sinh 2K_h\cos\theta -\sinh 2K_v\cos\phi)]. \quad (59)$$

Steps leading from Eq. (58) to Eq. (59) can be found, for example, in Ref. [11]. The bulk free energy $f_{\rm bulk}(K_h,K_v)$ is nonanalytic at the critical point $\sinh 2K_h\sinh 2K_v=1$.

For large M and N, one can use the Euler-MacLaurin summation formula to evaluate corrections to the bulk free energy. For the purpose of comparing with the conformal field predictions [4], it is of particular interest to analyze corrections at the critical point. We have carried out such an analysis for $2M\times N$ lattices with isotropic interactions $K_h=K_v=K$. In this case the critical point is $\sinh 2K_c=1$ or, equivalently, $2K_c=\ln(\sqrt{2}+1)$ at which we expect to have the expansion

$$\ln Z_{2M,N}(K_c)=2MNf_{\rm bulk}(K_c)+Nc_1(\xi,K_c)+2Mc_2(\xi,K_c)+c_3(\xi,K_c)+\cdots, \quad (60)$$

where $\xi=N/2M$ is the aspect ratio of the lattice.

The evaluation of terms in Eq. (60) was first carried out by Ferdinand and Fisher [12] for toroidal boundary conditions. Following Ref. [12], as well as similar analyses for dimer systems [1,13], we have evaluated Eq. (60) for other boundary conditions. For the $2M\times N$ Möbius strip, for example, one starts with an explicit expression [Eq. (11)] for the partition function, and uses the Euler-MacLaurin formula to evaluate the summations. The analysis is lengthy, even at the critical point. We shall give details elsewhere [14], and quote only the results, here

$$c_1(\xi,K_c)\equiv c_1^{\rm Mob}=I-K_c=-0.087\,618\ldots,$$

$$c_2(\xi,K_c)=0, \quad (61)$$

$$c_3(\xi,K_c)=-\frac{1}{2}\ln 2+\frac{1}{12}\ln\left[\frac{2\vartheta_3^2(0|i\xi)}{\vartheta_2(0|i\xi)\vartheta_4(0|i\xi)}\right]+\frac{1}{2}\ln\left[1+\frac{\vartheta_3(0|i\xi/2)-\vartheta_4(0|i\xi/2)}{2\vartheta_3(0|i\xi)}\right]$$

where

$$I=\frac{1}{2\pi}\int_0^\pi \ln(\sqrt{2}\sin\phi+\sqrt{1+\sin^2\phi})d\phi=0.353\,068\ldots,$$

and $\vartheta_i(u|\tau), i=2,3,$ and 4, are the Jacobi theta functions [15]

$$\vartheta_2(u|\tau)=2\sum_{n=1}^\infty q^{[n-(1/2)]^2}\cos(2n-1)u,$$

$$\vartheta_3(u|\tau)=1+2\sum_{n=1}^\infty q^{n^2}\cos 2nu, \quad (62)$$

$$\vartheta_4(u|\tau)=1+2\sum_{n=1}^\infty (-1)^n q^{n^2}\cos 2nu,$$

with $q=e^{i\pi\tau}$. For the $2M\times N$ Klein bottle, we find, similarly,

$$c_1(\xi,K_c)=0,$$

$$c_2(\xi,K_c)=0, \quad (63)$$

$$c_3(\xi,K_c)=\frac{1}{6}\ln\left[\frac{2\vartheta_3^2(0|2i\xi)}{\vartheta_2(0|2i\xi)\vartheta_4(0|2i\xi)}\right]+\ln\left[1+\sqrt{\frac{\vartheta_2(0|2i\xi)}{2\vartheta_3(0|2i\xi)}}\right].$$

If one takes the limit of $N\to\infty$ ($M\to\infty$) first in Eq. (60), while keeping M (N) finite, one obtains

$$\lim_{N\to\infty}\frac{1}{N}\ln Z_{2M,N}(K_c)=2Mf_{\rm bulk}(K_c)+c_1+\Delta_1/2M+O(1/M^2),$$

$$\lim_{M\to\infty}\frac{1}{2M}\ln Z_{2M,N}(K_c)=Nf_{\rm bulk}(K_c)+c_2+\Delta_2/N+O(1/N^2), \quad (64)$$

where c_1, c_2, Δ_1, and Δ_2 are constants. Results of our findings are listed in Table I above. Also listed are values for toroidal boundary conditions taken from Ref. [12], and values for the cylindrical boundary conditions computed by us using the solution given in Ref. [7].

TABLE I. Results of our findings for different configurations.

	Cylindrical	Toroidal	Möbius	Klein
c_1	$c_1^{\rm Mob}$	0	$c_1^{\rm Mob}$	0
c_2	0	0	0	0
Δ_1	$\pi/48$	$\pi/12$	$\pi/48$	$\pi/12$
Δ_2	$\pi/12$	$\pi/12$	$\pi/48$	$\pi/48$

For a lattice strip of infinite length and finite width, whose free energy is of the form of Eq. (64), the conformal field theory [4] predicts $\{\Delta_1,\Delta_2\}=\pi c/24$ (or $\pi c/6$), where c is the central charge, when the boundary condition in the finite width direction is free or fixed (or periodic). Thus numbers in the first two columns of Table I give rise to the value of

$$c=1/2. \quad (65)$$

Likewise, numbers in the last two columns also yield $c=1/2$, provided that the (twisted) Möbius boundary condition is regarded as a free boundary.

VII. SUMMARY

We have solved and obtained closed-form expressions for the partition function of an Ising model on finite Möbius strips and Klein bottles. The solution assumes different forms depending on whether the width of the lattice is even or odd. For a $2M\times N$ Möbius strip, where $2M$ is its width, the partition function $Z^{\rm Mob}_{2M,N}$ is given by Eq. (11), with $F_\pm$ given by Eq. (12). For a $(2M-1)\times N$ Möbius strip, we employ a trick by first considering a $2M\times N$ lattice and then ''fusing'' it into the desired lattice by coalescing two rows of spins. The resulting partition function $Z^{\rm Mob}_{2M-1,N}$ is given by Eq. (30), in which the generating function $G(z_h,z_v,z_0,z_1)$ is Eq. (33) with the Pfaffians given by Eq. (35).

For a $2M\times N$ Klein bottle the partition function $Z^{\rm K\,ln}_{2M,N}$ is given by Eq. (39) in which the generating function $G^{\rm K\,ln}(z_h,z_v,z_v,z_v)$ is given by Eq. (50). For a $(2M-1)\times N$ Klein bottle the partition function $Z^{\rm K\,ln}_{2M-1,N}$ is given by Eq. (53) in which the generating function $G^{\rm K\,ln}(z_h,z_v,z_0,1,z_v)$, $z_0=\tanh(K_h/2)$, is computed using Eq. (54). All solutions yield the same Onsager bulk free energy [Eq. (59)].

We have also carried out finite-size analyses of all solutions including that of the Ising model under cylindrical boundary conditions at criticality. The analyses yield a central charge $c=1/2$, in agreement with the conformal field prediction [4], provided that the (twisted) Möbius boundary condition is regarded as a free or fixed boundary.

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