Classical theory of resonant transition radiation in multilayer structures

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(Received 15 September 1999; revised manuscript received 22 June 2000; published 27 December 2000)

A rigorous classical electromagnetic theory of the transition radiation in finite and infinite multilayer structures is presented. It makes the standard results of thin-film optics, such as the matrix formalism, accountable; it allows thus an exact treatment of the propagation of the waves induced by the electron. This method is applied to the particular case of the periodic structures to treat the resonant transition radiation (RTR). It is noted that the present theory gives, in the hard x-ray domain, results previously published. The reason for this approach is to make the numerical calculations rigorous and easy. The numerical results of our theory are compared to experimental RTR data obtained recently by Yamada *et al.* [Phys. Rev. A **59**, 3673 (1999)] with a nickel-carbon multilayer structure.

DOI: 10.1103/PhysRevE.63.016613 PACS number(s): 41.60.−m

I. INTRODUCTION

When an electrically charged particle crosses an interface at a constant velocity between two different dielectric media, an electromagnetic radiation is emitted [1]. This radiation is known as transition radiation. It is caused by a collective response of the matter surrounding the particle trajectory to readjust the electromagnetic field of the charged particle. This process has been studied experimentally by means of radiators consisting generally of a small number of thin foils made up with light materials [2–5]. In the pioneering experiments quoted above, no interference effects between the foils have been observed. With the help of more sophisticated radiators, coherent emission resulting from interfoil interferences has been obtained; this emission is called resonant transition radiation (RTR) [6,7]. For a few MeV electrons with the reduced speed β of the order of 0.9995 and xuv emission of wavelength λ of the order of 0.5 nm, the well-known law of resonance $\cos[\alpha]=(1/\beta)-p(\lambda/d)$ (where α is the angle of emission and p an integer corresponding to the order) indicates that the period d of the stack must be in the submicrometer range. In fact, to detect a maximum of intensity, the observation angle α must be of the order of $1/\gamma$ [where $\gamma^2=1/(1-\beta^2)$]; a simple calculation neglecting the refraction effects indicates that to match this condition, the period d must scale as $p\gamma^2\lambda$. In this case, it has been proposed to use radiator multilayer stacks similar to those implemented in soft x-ray optics [8,9]. Recently, Yamada *et al.* [10] have used a Ni/C multilayer target with 15-MeV electrons to produce soft x rays in the range 2–7 keV.

The theoretical studies devoted to the modelization of the RTR are not numerous. Some studies deal with the process of transition radiation at an interface by solving the Maxwell equation, and without taking into account the multiple scattering, they treat the contribution of the different interfaces as a mere interference effect: see Ref. [11] and Eq. (5) in Ref. [12]. A first so-called dynamical approach, which supposes the media to be periodic and infinite, consists of calling upon the Bloch-Floquet theorem for help. Datta and Kaplan [12] have used such an approach in the framework of a quantum description of the phenomenon allowing them to deal with both spontaneous and stimulated radiation. By expanding the classical electromagnetic fields in Bloch waves, Pardo and André [13] have developed a model that accounts globally for the Cherenkov emission, the RTR, and the Bragg resonances. Dubovikov [14] has extended the previous results in a quantum context with emphasis on Bragg resonances.

Recently, with the intent to handle the case of periodic stratified but finite media, we have used a perturbative method in the framework of electromagnetism in continuous media [15]. This work allowed us to compute the yield and spatial distribution of xuv resonant transition radiations.

In the present paper our purpose is, by adopting the point of view of physical optics, to treat the case of finite radiators and to extend it to the infinite ones. The radiator is considered as a stack of homogeneous and absorbing layers through which the light propagates. By applying a classical matrix formalism, the actual propagation, which involves the multiple reflections and the absorption, can be computed rigorously and be compared with experimental data [10] where the stack is made up of a small number of absorbing layers. This method will be tested by considering its limits when the optical indices are small enough (case of hard x rays) so that the kinematical approximations can be applied.

This paper is organized as follows. Section II gives the field produced by an electron crossing the interface between two different homogeneous media. Section III introduces the matrix formalism for the propagation of the field in stack of plane layers. Section IV deals with the propagation in a periodic stack. Section V gives the radiation intensity far from the stack. Section VI tackles the case of hard x rays. Section VII presents numerical applications about the recent experiment of Yamada *et al.* [10] and discusses the results.

II. EQUATION OF THE ELECTROMAGNETIC FIELD PRODUCED BY AN ELECTRON CROSSING TWO HOMOGENEOUS MEDIA SEPARATED BY A PLANE INTERFACE

We consider the electromagnetic field associated with an electron crossing the interface at a constant speed between two homogeneous media. We use the Gauss unit system. In these conditions the Maxwell equations read after Fourier transform,

$$i\mathbf{k}\times\hat{\mathbf{H}}=-\frac{i\omega}{c}\hat{\mathbf{D}}+\frac{q}{c}\frac{1}{2\pi^2}\mathbf{v}\delta[\omega-\mathbf{k}\cdot\mathbf{v}], \tag{1}$$

$$i\mathbf{k}\cdot\hat{\mathbf{D}}=\frac{1}{2\pi^2}q\delta[\omega-\mathbf{k}\cdot\mathbf{v}], \tag{2}$$

$$\mathbf{k}\times\hat{\mathbf{E}}=\frac{\omega}{c}\hat{\mathbf{B}}, \tag{3}$$

$$\mathbf{k}\cdot\hat{\mathbf{B}}=0, \tag{4}$$

where $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{B}$ are, respectively, the generic symbols for the electric-field intensity, the electric flux density, the magnetic-field intensity, and the magnetic flux density; q is the algebraic charge of electron, $\mathbf{v}$ is the speed vector; c is the velocity of light in vacuum; ω and $\mathbf{k}$ are, respectively, the angular frequency and the wave vector of the field; δ is the Dirac distribution; and the caret stands for the Fourier transform defined by

$$\hat{\mathbf{A}}[\mathbf{k},\omega]=\frac{1}{(2\pi)^4}\int\int\int\int\mathbf{A}[\mathbf{r},t]e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d\mathbf{r}\,dt,$$

so that

$$\mathbf{A}[\mathbf{r},t]=\int\int\int\int\hat{\mathbf{A}}[\mathbf{k},\omega]e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d\mathbf{k}\,d\omega.$$

If $\varepsilon[\omega]$ denotes the dielectric constant and if the magnetic permeability is assumed to be equal to unity then, in a homogeneous layer $\hat{\mathbf{D}}=\varepsilon[\omega]\hat{\mathbf{E}}$, $\hat{\mathbf{B}}=\hat{\mathbf{H}}$, and the Maxwell equations become

$$\mathbf{k}\times\hat{\mathbf{H}}=-\frac{\omega\varepsilon[\omega]}{c}\hat{\mathbf{E}}-\frac{q}{c}\frac{i}{2\pi^2}\delta[\omega-\mathbf{k}\cdot\mathbf{v}]\mathbf{v}, \tag{5}$$

$$\mathbf{k}\cdot\hat{\mathbf{E}}=-\frac{i}{2\pi^2\varepsilon[\omega]}q\delta[\omega-\mathbf{k}\cdot\mathbf{v}], \tag{6}$$

$$\mathbf{k}\times\hat{\mathbf{E}}=\frac{\omega}{c}\hat{\mathbf{H}}, \tag{7}$$

$$\mathbf{k}\cdot\hat{\mathbf{H}}=0. \tag{8}$$

By combining the above equations it follows that

$$\hat{\mathbf{E}}[\mathbf{k},\omega]=\frac{qi}{2\pi^2\varepsilon[\omega]}\frac{\dfrac{\omega\varepsilon[\omega]}{c^2}\mathbf{v}-\mathbf{k}}{k^2-\dfrac{\omega^2\varepsilon[\omega]}{c^2}}\delta[\omega-\mathbf{k}\cdot\mathbf{v}]. \tag{9}$$

The electric field in the space-time domain is obtained by Fourier transforms. By introducing the following quantity embellished by a tilde:

$$\tilde{\mathbf{E}}[\mathbf{k},\mathbf{v}]=\frac{qi}{2\pi^2\varepsilon[\mathbf{k}\cdot\mathbf{v}]}\frac{\dfrac{\varepsilon[\mathbf{k}\cdot\mathbf{v}]\mathbf{k}\cdot\mathbf{v}}{c^2}\mathbf{v}-\mathbf{k}}{k^2-\dfrac{\varepsilon[\mathbf{k}\cdot\mathbf{v}](\mathbf{k}\cdot\mathbf{v})^2}{c^2}}, \tag{10}$$

one obtains this field in cylindrical coordinates which accounts for the symmetry of the problem

$$\mathbf{E}[\rho,z,t]=\int\int\int\tilde{\mathbf{E}}[\mathbf{k},\mathbf{v}]e^{i(\mathbf{k}_\parallel\cdot\boldsymbol{\rho}+k_\perp z-\mathbf{k}\cdot\mathbf{v}t)}d^2\mathbf{k}_\parallel\,dk_\perp\,. \tag{11}$$

The space variable and the wave vector have been split in tangential and normal components, respectively ρ, z and $\mathbf{k}_\parallel$, $k_\perp$.

To use the laws of optics it is convenient to introduce a partial Fourier transform; this quantity, embellished by an overbar is defined by the following expression:

$$\overline{\mathbf{E}}[\mathbf{k}_\parallel,\omega,z]=\frac{1}{(2\pi)^3}\int\int\int\mathbf{E}[\rho,z,t]e^{i(-\mathbf{k}_\parallel\cdot\boldsymbol{\rho}+\omega t)}dt\,d^2\rho. \tag{12}$$

It can be shown (see Appendix A) that the above quantity is given by

$$\overline{\mathbf{E}}[\mathbf{k}_\parallel,\omega,z]=\frac{1}{v}\frac{qi}{2\pi^2\varepsilon[\omega]}\frac{\omega\left(\dfrac{\varepsilon[\omega]}{c^2}-\dfrac{1}{v^2}\right)\mathbf{v}-\mathbf{k}_\parallel}{k_\parallel^2+\left(\dfrac{\omega}{v}\right)^2-\dfrac{\omega^2\varepsilon[\omega]}{c^2}}e^{i(\omega/v)z}. \tag{13}$$

The general solution of the problem is obtained as the sum of the particular solution of the nonhomogeneous Maxwell equations as given by Eq. (7) and the solution of the homogeneous equations. The latter is in terms of electric-field intensity

$$\mathbf{E}_0[\rho,z,t]=\int\int\int\tilde{\mathbf{E}}_0[\mathbf{k}_\parallel,k_\perp]e^{i(\mathbf{k}_\parallel\cdot\boldsymbol{\rho}+k_\perp z-\omega t)}d^2\mathbf{k}_\parallel\,dk_\perp\,, \tag{14}$$

where ω, $\mathbf{k}_\parallel$, and $k_\perp$ are related by the dispersion relation

$$k_\perp^2=\frac{\omega^2}{c^2}\varepsilon[\omega]-\mathbf{k}_\parallel^2. \tag{15}$$

As done previously, one introduces the partial Fourier transform

$$\overline{\mathbf{E}}_0[\mathbf{k}_\parallel,\omega,z]=\frac{1}{(2\pi)^3}\int\int\int \mathbf{E}_0[\rho,z,t]e^{i(-\mathbf{k}_\parallel\cdot\boldsymbol{\rho}+\omega t)}dt\,d^2\rho. \tag{16}$$

Then one obtains readily by calculations similar to the ones of Appendix A:

$$\overline{\mathbf{E}}_0[\mathbf{k}_\parallel,\omega,z]=\int\int\int \tilde{\mathbf{E}}_0[\mathbf{k}']e^{i\mathbf{k}'_\perp z}\delta[\mathbf{k}'_\parallel-\mathbf{k}_\parallel]\times\delta[\omega'-\omega]d^2\mathbf{k}'_\parallel\,dk'_\perp. \tag{17}$$

After some manipulations (see Appendix B) one gets

$$\overline{\mathbf{E}}_0[\mathbf{k}_\parallel,\omega,z]=\overline{\mathbf{E}}'_{0i}[\mathbf{k}_\parallel,\omega]e^{ik_\perp[\mathbf{k}_\parallel,\omega]z}-\overline{\mathbf{E}}'_{0r}[\mathbf{k}_\parallel,\omega]e^{-ik_\perp[\mathbf{k}_\parallel,\omega]z}, \tag{18a}$$

where

$$\overline{\mathbf{E}}'_{0i}[\mathbf{k}_\parallel,\omega]=\frac{1}{2k_\perp[\mathbf{k}_\parallel,\omega]c^2}\left(2\omega\varepsilon[\omega]+\omega^2\frac{d\varepsilon[\omega]}{d\omega}\right)\tilde{\mathbf{E}}_{0i}[\mathbf{k}_\parallel,\omega] \tag{18b}$$

and

$$\overline{\mathbf{E}}'_{0r}[\mathbf{k}_\parallel,\omega]=\frac{1}{2k_\perp[\mathbf{k}_\parallel,\omega]c^2}\left(2\omega\varepsilon[\omega]+\omega^2\frac{d\varepsilon[\omega]}{d\omega}\right)\tilde{\mathbf{E}}_{0r}[\mathbf{k}_\parallel,\omega]. \tag{18c}$$

It is well known that the tangential component of the electric-field intensity and the normal component of the electric flux density must be continuous at the interface.

If one denotes by

$$\overline{\mathbf{E}}_{0\parallel}[\mathbf{k}_\parallel,\omega,z]=\overline{\mathbf{E}}'_{0\parallel i}[\mathbf{k}_\parallel,\omega]e^{ik_\perp[\mathbf{k}_\parallel,\omega]z}-\overline{\mathbf{E}}'_{0\parallel r}[\mathbf{k}_\parallel,\omega]e^{-ik_\perp[\mathbf{k}_\parallel,\omega]z} \tag{19a}$$

and

$$\overline{\mathbf{D}}_{0\perp}[\mathbf{k}_\parallel,\omega,z]=\varepsilon[\omega](\overline{\mathbf{E}}'_{0\perp i}[\mathbf{k}_\parallel,\omega]e^{ik_\perp[\mathbf{k}_\parallel,\omega]z}-\overline{\mathbf{E}}'_{0\perp r}[\mathbf{k}_\parallel,\omega]e^{-ik_\perp[\mathbf{k}_\parallel,\omega]z}), \tag{19b}$$

the tangential component of the electric-field intensity and the normal component of the electric flux density associated with the homogeneous equation, the continuity conditions require that the following quantities:

$$\frac{1}{v}\frac{qi}{2\pi^2\varepsilon[\omega]}\frac{-\mathbf{k}_\parallel}{k_\parallel^2+\left(\frac{\omega}{v}\right)^2-\frac{\omega^2\varepsilon[\omega]}{c^2}}e^{i(\omega/v)z}+\overline{\mathbf{E}}'_{0\parallel i}[\mathbf{k}_\parallel,\omega]e^{ik_\perp[\mathbf{k}_\parallel,\omega]z}-\overline{\mathbf{E}}'_{0\parallel r}[\mathbf{k}_\parallel,\omega]e^{-ik_\perp[\mathbf{k}_\parallel,\omega]z} \tag{20a}$$

and

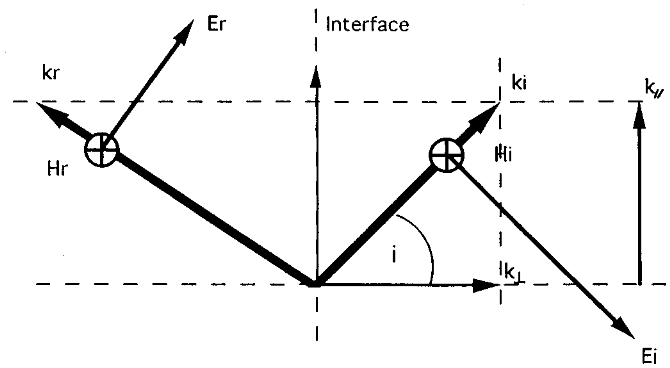

FIG. 1. Distribution of the electromagnetic field at an interface and corresponding geometry.

$$\frac{1}{v}\frac{qi}{2\pi^2}\frac{\omega\left(\frac{\varepsilon[\omega]}{c^2}-\frac{1}{v^2}\right)\mathbf{v}}{k_\parallel^2+\left(\frac{\omega}{v}\right)^2-\frac{\omega^2\varepsilon[\omega]}{c^2}}e^{i(\omega/v)z}+\varepsilon[\omega](\overline{\mathbf{E}}'_{0\perp i}[\mathbf{k}_\parallel,\omega]e^{ik_\perp[\mathbf{k}_\parallel,\omega]z}-\overline{\mathbf{E}}'_{0\perp r}[\mathbf{k}_\parallel,\omega]e^{-ik_\perp[\mathbf{k}_\parallel,\omega]z}) \tag{20b}$$

are conserved at the interfaces.

Let us show that the field due to the electron is transverse magnetic, i.e., the Fourier transform of the magnetic field is perpendicular both to the trajectory of the electron and the **k** vector. As a matter of fact, taking into account Eq. (9) and Eq. (7), one gets

$$\hat{\mathbf{H}}[\mathbf{k},\omega]=\frac{qi}{2\pi^2c}\frac{-\mathbf{v}\wedge\mathbf{k}}{k^2-\frac{\omega^2\varepsilon[\omega]}{c^2}}\delta[\omega-\mathbf{k}\cdot\mathbf{v}]. \tag{21}$$

Since this field is the cause of the total field, the latter must have the same symmetry and consequently be transverse magnetic, too.

III. MATRIX FORMALISM FOR THE PROPAGATION OF THE FIELD IN A STACK OF PLANE LAYERS

The stack consists of an arrangement of plane layers characterized by their dielectric constants ε_i and their thicknesses d_i. Introducing the angle i (see Fig. 1) and taking into account the fact that the field is transverse magnetic, it is possible to express the components of the electric field as follows:

$$\begin{aligned}
\overline{E}'_{0\parallel i}[\mathbf{k}_\parallel,\omega]&=-T[\mathbf{k}_\parallel,\omega]\cos[i],\\
\overline{E}'_{0\perp i}[\mathbf{k}_\parallel,\omega]&=+T[\mathbf{k}_\parallel,\omega]\sin[i],\\
\overline{E}'_{0\parallel r}[\mathbf{k}_\parallel,\omega]&=+R[\mathbf{k}_\parallel,\omega]\cos[i].\\
\overline{E}'_{0\perp r}[\mathbf{k}_\parallel,\omega]&=+R[\mathbf{k}_\parallel,\omega]\sin[i].
\end{aligned} \tag{22}$$

With these notations one can rewrite quantities given by Eqs. (20), which remain continuous at the interface, in the following matrix form:

$$M[z]\begin{pmatrix} T \\ R \end{pmatrix} + \begin{pmatrix} S_\perp \\ S_\parallel \end{pmatrix} e^{+ik_z z}, \tag{23a}$$

where

$$M[z] = \begin{pmatrix} \varepsilon[\omega]\sin[i]e^{+ik_\perp[\mathbf{k}_\parallel,\omega]z} & \varepsilon[\omega]\sin[i]e^{-ik_\perp[\mathbf{k}_\parallel,\omega]z} \\ -\cos[i]e^{+ik_\perp[\mathbf{k}_\parallel,\omega]z} & \cos[i]e^{-ik_\perp[\mathbf{k}_\parallel,\omega]z} \end{pmatrix}, \tag{23b}$$

$$\begin{pmatrix} S_\perp \\ S_\parallel \end{pmatrix} = \begin{pmatrix} \dfrac{qi}{2\pi^2} \dfrac{\omega\left(\dfrac{\varepsilon[\omega]}{c^2} - \dfrac{1}{v^2}\right)}{k_\parallel^2 + \left(\dfrac{\omega}{v}\right)^2 - \dfrac{\omega^2\varepsilon[\omega]}{c^2}} \\ \dfrac{qi}{2\pi^2\varepsilon[\omega]} \dfrac{1}{v} \dfrac{-k_\parallel}{k_\parallel^2 + \left(\dfrac{\omega}{v}\right)^2 - \dfrac{\omega^2\varepsilon[\omega]}{c^2}} \end{pmatrix}, \tag{23c}$$

and

$$k_z = \frac{\omega}{v}. \tag{23d}$$

In the case of a stack of plane layers one indexes the media and the interfaces. At the interface, whose abscissa is z_{j-1} and which separates the layer $j-1$ from the layer j, the continuity relations give the set of recurrent equations

$$\begin{aligned} M_{j-1}[z_{j-1}]\begin{pmatrix} T \\ R \end{pmatrix}_{j-1} &+ \begin{pmatrix} S_\perp \\ S_\parallel \end{pmatrix}_{j-1} e^{+ik_z z_{j-1}} \\ = M_j[z_{j-1}]\begin{pmatrix} T \\ R \end{pmatrix}_j &+ \begin{pmatrix} S_\perp \\ S_\parallel \end{pmatrix}_j e^{+ik_z z_{j-1}}. \end{aligned} \tag{24}$$

By introducing the following quantities:

$$\Delta_{j-1} = \begin{pmatrix} \Delta_\perp \\ \Delta_\parallel \end{pmatrix}_{j-1} = \begin{pmatrix} S_\perp \\ S_\parallel \end{pmatrix}_{j-1} - \begin{pmatrix} S_\perp \\ S_\parallel \end{pmatrix}_j, \tag{25}$$

one has

$$M_j[z_{j-1}]\begin{pmatrix} T \\ R \end{pmatrix}_j = M_{j-1}[z_{j-1}]\begin{pmatrix} T \\ R \end{pmatrix}_{j-1} + \begin{pmatrix} \Delta_\perp \\ \Delta_\parallel \end{pmatrix}_{j-1} e^{+ik_z z_{j-1}}, \tag{26}$$

which can be rewritten

$$\begin{aligned} M_j[z_{j-1}]e^{-ik_z z_j}\begin{pmatrix} T \\ R \end{pmatrix}_j = \begin{pmatrix} \Delta_\perp \\ \Delta_\parallel \end{pmatrix}_{j-1} & e^{-ik_z(z_j - z_{j-1})} \\ & + e^{-ik_z(z_j - z_{j-1})} M_{j-1}[z_{j-1}] \\ & \times M_{j-1}^{-1}[z_{j-2}] M_{j-1}[z_{j-2}] \\ & \times e^{-ik_z z_{j-1}} \begin{pmatrix} T \\ R \end{pmatrix}_{j-1}. \end{aligned} \tag{27}$$

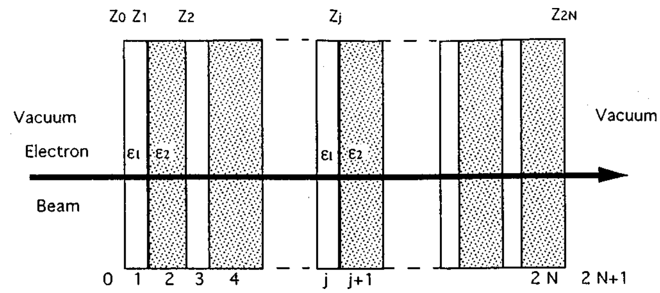

FIG. 2. Scheme of a periodic multilayer stack with the relevant notations. The stack consists of alternated layers of a material of dielectric constant ε_1 and thickness d_1 and of a material of dielectric constant ε_2 and thickness d_2.

By introducing the following notations:

$$\mathbf{F}_j = \begin{pmatrix} F_\perp \\ F_\parallel \end{pmatrix}_j = M_j[z_{j-1}]e^{-ik_z z_j}\begin{pmatrix} T \\ R \end{pmatrix}_j \tag{28}$$

and

$$d_j = z_j - z_{j-1}, \tag{29}$$

one gets the recursive formula

$$\mathbf{F}_j = \Delta_{j-1}e^{-ik_z d_j} + e^{-ik_z d_j} M_{j-1}[z_{j-1}] M_{j-1}^{-1}[z_{j-2}] \mathbf{F}_{j-1}. \tag{30}$$

IV. PROPAGATION IN A PERIODIC MULTILAYER STACK

Now, one handles the case of a periodic multilayer stack immersed in the vacuum (see Fig. 2). We suppose that the stack consists of an alternate arrangement of two materials characterized by the dielectric constants ε_1 and ε_2 and the thicknesses d_1 and d_2, respectively. We denote by d the period $d_1 + d_2$ and by N the number of periods. The number of the first medium is 0 while the one of the last medium is equal to $2N+1$.

Using Eq. (30) twice one obtains

$$\mathbf{F}_{2j+1} = -\Delta e^{-ik_z d_1} + e^{-ik_z d} A\Delta + e^{-ik_z d} B \mathbf{F}_{2j-1}, \tag{31}$$

where A and B are two matrices defined by

$$A = M_{2j}[z_{2j}] M_{2j}^{-1}[z_{2j-1}], \tag{32}$$

$$B = M_{2j}[z_{2j}] M_{2j}^{-1}[z_{2j-1}] M_{2j-1}[z_{2j-1}] M_{2j-1}^{-1}[z_{2j-2}], \tag{33}$$

and

$$\Delta = \Delta_1.$$

These matrices depend only upon the thicknesses of the layers. In this derivation, we have used the fact that $\Delta_{n+1} = -\Delta_n$. By setting

$$\delta = -\Delta e^{-ik_z d_1} + e^{-ik_z d} A\Delta \tag{34}$$

and

$$\mu = e^{-ik_z d} B, \tag{35}$$

Eq. (31) becomes

$$\mathbf{F}_{2j+1} = \delta + \mu \mathbf{F}_{2j-1}. \tag{36}$$

A little algebra leads to a linear relation in terms of $\mathbf{F}_1$:

$$\mathbf{F}_{2j+1} = S_j[\mu]\delta + P_j[\mu]\mathbf{F}_1, \tag{37}$$

where

$$S_j[\mu] = \sum_{i=0}^{j-1} \mu^i \quad \text{and} \quad P_j[\mu] = \mu^j. \tag{38}$$

V. DETERMINATION OF THE RADIATION INTENSITY FAR FROM THE STACK

The field seen by a detector can always be calculated by Fourier transforms,

$$\mathbf{E}_D[\rho,z,t] = \int\int\int \tilde{\mathbf{E}}_D[k_\parallel,k_\perp] \times e^{i(\mathbf{k}_\parallel\cdot\boldsymbol{\rho}+k_\perp z-\omega[\mathbf{k}_\parallel,k_\perp]t)} d^2\mathbf{k}_\parallel \, dk_\perp. \tag{39}$$

In Eq. (39), z must be taken from the boundaries of the stack and we have taken into account that the Fourier component of the electric field depends on the tangential component of the wave vector only through its modulus $k_\parallel$ and that the electric field depends on the radial component ρ of the position vector only through its modulus ρ.

Two cases must be considered: (i) the forward one—the point of observation, assumed to be in vacuum, is beyond the stack and $z = z_B + D$ (z_B, abscissa of the exit surface of the stack, equals the thickness of the stack while D is the distance of the detector taken this exit surface), and (ii) the backward one—the point of observation is in front of the stack and $z = z_B - D$ (z_B, the abscissa of the entrance surface of the stack, equals 0).

From the above considerations and by changing the integration variable using Eq. (15), it can be shown that $\mathbf{E}_D[\rho,D,t]$, the field at the observation point, is given by

$$\mathbf{E}_D[\rho,D,t] = \int\int d^2\mathbf{k}_\parallel \int d\omega \, \tilde{\mathbf{E}}_{B\eta}[k_\parallel,\omega] \times e^{i\eta k_\perp[\mathbf{k}_\parallel,\omega]z_B} e^{i(\mathbf{k}_\parallel\cdot\boldsymbol{\rho}+k_\perp[\mathbf{k}_\parallel,\omega]D-\omega t)} \frac{\partial k_\perp[k_\parallel,\omega]}{\partial\omega}, \tag{40}$$

where $\tilde{\mathbf{E}}_{B\eta}[k_\parallel,\omega]$ is the partial Fourier transform calculated on the appropriate boundary. The presence of the symbol η is due to the fact that the dispersion relation, Eq. (15), admits the two solutions $+k_\perp$ and $-k_\perp$. In fact, we shall be interested in the time Fourier transform,

$$\hat{\mathbf{E}}_D[\rho,z,\omega'] = \frac{1}{2\pi}\int dt \, e^{i\omega' t} \int\int d^2\mathbf{k}_\parallel \int d\omega \, \tilde{\mathbf{E}}_{B\eta}[k_\parallel,\omega] \times e^{i\eta k_\perp[\mathbf{k}_\parallel,\omega]z_B} e^{i(\mathbf{k}_\parallel\cdot\boldsymbol{\rho}+k_\perp[\mathbf{k}_\parallel,\omega]D-\omega t)} \times \frac{\partial k_\perp[k_\parallel,\omega]}{\partial\omega}. \tag{41}$$

The integrations over t and ω give

$$\hat{\mathbf{E}}_D[\rho,z,\omega'] = \int\int d^2\mathbf{k}_\parallel \, \tilde{\mathbf{E}}_{B\eta}[k_\parallel,\omega'] \times e^{i\eta k_\perp[\mathbf{k}_\parallel,\omega']z_B} e^{i(\mathbf{k}_\parallel\cdot\boldsymbol{\rho}+k_\perp[\mathbf{k}_\parallel,\omega']D)} \frac{\partial k_\perp[k_\parallel,\omega']}{\partial\omega'}. \tag{42}$$

Now we calculate the field at a far point, the spherical coordinates of which are $D,\alpha,0$ without loss of the generality of the problem.

By introducing the projections $\tilde{\mathbf{E}}_{B\parallel\eta}[k_\parallel,\omega']$ and $\tilde{\mathbf{E}}_{B\perp\eta}[k_\parallel,\omega']$ of $\tilde{\mathbf{E}}_{B\eta}[k_\parallel,\omega']$, one has

$$\hat{\mathbf{E}}_D[\rho,z,\omega'] = \int\int \begin{pmatrix} \tilde{E}_{B\parallel\eta}[k_\parallel,\omega']\cos[\Phi] \\ \tilde{E}_{B\parallel\eta}[k_\parallel,\omega']\sin[\Phi] \\ \tilde{E}_{B\perp\eta}[k_\parallel,\omega'] \end{pmatrix} e^{i\eta k_\perp[\mathbf{k}_\parallel,\omega']z_B} \times e^{iD(k_\parallel\sin[\alpha]\cos[\Phi]+k_\perp[\mathbf{k}_\parallel,\omega']\cos[\alpha])} \times \frac{\partial k_\perp[k_\parallel,\omega']}{\partial\omega'} k_\parallel \, dk_\parallel \, d\Phi. \tag{43}$$

Integration over Φ ends in the Bessel functions J_m and gives

$$\hat{\mathbf{E}}_D[\rho,D,\omega'] = 2\pi\int_0^\infty \begin{pmatrix} i\tilde{E}_{B\parallel\eta}[k_\parallel,\omega']J_{+1}(k_\parallel D\sin[\alpha]) \\ 0 \\ \tilde{E}_{B\perp\eta}[k_\parallel,\omega']J_0(k_\parallel D\sin[\alpha]) \end{pmatrix} \times e^{i\eta k_\perp[\mathbf{k}_\parallel,\omega']z_B} e^{iDk_\perp[k_\parallel,\omega']\cos(\alpha)} \times \frac{\partial k_\perp[k_\parallel,\omega']}{\partial\omega'} k_\parallel \, dk_\parallel. \tag{44}$$

If we consider large values of $D\sin[\alpha]$ we can replace the Bessel functions by their asymptotic expressions

$$\hat{\mathbf{E}}_D[D,\alpha,\omega'] = 2\pi \int_0^\infty \begin{pmatrix} i\tilde{E}_{B\parallel\eta}[k_\parallel,\omega'] \sqrt{\dfrac{2}{\pi k_\parallel D \sin(\alpha)}} \cos\left[k_\parallel D \sin(\alpha) - 3\dfrac{\pi}{4}\right] \\ 0 \\ \tilde{E}_{B\perp\eta}[k_\parallel,\omega'] \sqrt{\dfrac{2}{\pi k_\parallel D \sin(\alpha)}} \cos\left[k_\parallel D \sin(\alpha) - \dfrac{\pi}{4}\right] \end{pmatrix}$$
$$\times e^{i\eta k_\perp[\mathbf{k}_\parallel,\omega'] z_B} e^{iDk_\perp[\mathbf{k}_\parallel,\omega']\cos(\alpha)} \frac{\partial k_\perp[k_\parallel,\omega']}{\partial\omega'} k_\parallel \, dk_\parallel . \tag{45}$$

Expanding the trigonometric functions leads to

$$\hat{\mathbf{E}}_D[D,\alpha,\omega']$$
$$= \pi \int_0^\infty \begin{pmatrix} i\tilde{E}_{B\parallel\eta}[k_\parallel,\omega'] \sqrt{\dfrac{2}{\pi k_\parallel D \sin[\alpha]}} \left(e^{i\{k_\perp[k_\parallel,\omega']\cos[\alpha]D + k_\parallel D \sin[\alpha] - 3(\pi/4)\}} + e^{i\{k_\perp[k_\parallel,\omega']\cos[\alpha]D - k_\parallel D \sin[\alpha] + (\pi/4)\}}\right) \\ 0 \\ \tilde{E}_{B\perp\eta}[k_\parallel,\omega'] \sqrt{\dfrac{2}{\pi k_\parallel D \sin[\alpha]}} \left(e^{i\{k_\perp[k_\parallel,\omega']\cos[\alpha]D + k_\parallel D \sin[\alpha] - (\pi/4)\}} + e^{i\{k_\perp[k_\parallel,\omega']\cos[\alpha]D - k_\parallel D \sin[\alpha] + (\pi/4)\}}\right) \end{pmatrix}$$
$$\times e^{i\eta k_\perp[k_\parallel,\omega'] z_B} \frac{\partial k_\perp[k_\parallel,\omega']}{\partial\omega'} k_\parallel \, dk_\parallel . \tag{46}$$

Now we have to deal with integrals of the form $\int e^{iDf(\mu)} g(\mu) d\mu$, where f is a real-valued function. When D becomes very large, it can be estimated by the method of stationary phase. The integration is presented in Appendix C and gives

$$\hat{\mathbf{E}}_D[D,\alpha,\omega'] = -i\sigma[\alpha] \frac{2\pi}{D} \frac{\omega'}{c^2} \begin{pmatrix} \tilde{E}_{D\parallel\alpha}[k_{\parallel S},\omega'] \\ 0 \\ \tilde{E}_{D\perp\alpha}[k_{\parallel S},\omega'] \end{pmatrix}$$
$$\times e^{iD(\omega'/c)} e^{i\eta k_\perp[\mathbf{k}_\parallel,\omega'] z_B}. \tag{47}$$

$k_{\parallel S}$ is the tangential component of the wave vector which goes from the stack to the point of observation. In Eq. (47) the derivative of $k_\perp^2$ has been calculated using the point of observation in the vacuum.

Expressing $\tilde{E}'_D$ in terms of $\overline{E}'_D$ by means of Eqs. (18), one obtains

$$\hat{\mathbf{E}}_D[D,\alpha,\omega'] = -i\sigma[\alpha] \frac{2\pi}{D} \frac{\omega'}{c} \cos[\alpha] \begin{pmatrix} \overline{E}'_{D\parallel\alpha}[k_{\parallel S},\omega'] \\ 0 \\ \overline{E}'_{D\perp\alpha}[k_{\parallel S},\omega'] \end{pmatrix}$$
$$\times e^{iD(\omega'/c)} e^{i\eta k_\perp[\mathbf{k}_\parallel,\omega'] z_B}. \tag{48}$$

Using Eq. (22) to express the amplitudes of the fields on the two external surfaces of the stack, one obtains the following: (i) in the forward case where the α angle lies between 0 and $\pi/2$

$$\hat{\mathbf{E}}_{Df}[D,\alpha,\omega'] = i\frac{2\pi}{D}\frac{\omega'}{c}\cos[\alpha] t_B[z_B,k_{\parallel S},\omega']$$
$$\times \begin{pmatrix} \cos[\alpha] \\ 0 \\ -\sin[\alpha] \end{pmatrix} e^{iD(\omega'/c)}, \tag{49}$$

where

$$t_B[z_B,k_{\parallel S},\omega'] = T_B[k_{\parallel S},\omega'] e^{i\eta k_\perp[k_{\parallel S},\omega'] z_B} \tag{50}$$

denotes the transmitted field near the exit surface of the stack; and (ii) in the backward case where the α angle lies between $\pi/2$ and π

$$\hat{\mathbf{E}}_{Db}[D,\alpha,\omega'] = i\frac{2\pi}{D}\frac{\omega'}{c}\cos[\alpha] r_B[z_B,k_{\parallel S},\omega']$$
$$\times \begin{pmatrix} -\cos[\alpha] \\ 0 \\ \sin[\alpha] \end{pmatrix} e^{iD(\omega'/c)}, \tag{51}$$

where

$$r_B[z_B,k_{\parallel S},\omega'] = R_B[k_{\parallel S},\omega'] e^{i\eta k_\perp[\mathbf{k}_{\parallel S},\omega'] z_B} \tag{52}$$

denotes the reflected field near the entrance surface of the stack.

Now the problem is the determination of t_B and r_B. The algebraic procedure that we have used is given in Appendix D. It gives their expression in terms of the eigenvalues λ_1 and λ_2 and the components of the eigenvectors of matrix μ

$$\mathbf{u}_1=\begin{pmatrix} u_{11}\\ u_{12}\end{pmatrix},$$

$$\mathbf{u}_2=\begin{pmatrix} u_{21}\\ u_{22}\end{pmatrix}.$$

Note that the module of the product of the eigenvalues equals 1. It can be admitted that λ_1 has a module less than 1 while λ_2 has a module greater than 1. In these conditions the expressions of the modules of t_B and r_B become simple when the number of bilayers is infinite

$$|t_B|=\left|\frac{(u_{11}u_{22}-u_{12}u_{21})\delta_1}{(u_{22}\sin[\alpha]+u_{21}\cos[\alpha])(1-\lambda_1)}\right|, \tag{53}$$

$$|r_B|=\left|\frac{(u_{11}u_{22}-u_{12}u_{21})\delta_2}{(u_{12}\sin[\alpha]-u_{11}\cos[\alpha])(1-\lambda_2)}\right|, \tag{54}$$

where δ_1 and δ_2 are defined by

$$\delta=\delta_1\mathbf{u}_1+\delta_2\mathbf{u}_2. \tag{55}$$

Let us emphasize that these expressions of t_B and r_B remain finite as long as there is a tiny bit of absorption in the media. In this case the photon yield tends towards a finite limit when the number N of bilayers becomes infinite. When the absorption vanishes the yield diverges as N becomes large as shown in Sec. VI.

The intensity of the resonant transition radiation per unit positive angular frequency interval and per unit solid angle can be deduced straightforwardly from the modulus of the field by the relation

$$\frac{\partial^2 I[\alpha,\omega']}{\partial\omega'\,\partial\Omega}=cD^2|\hat{E}_D[D,\alpha,\omega']|^2. \tag{56}$$

For the forward case one has

$$\frac{\partial^2 I_f}{\partial\omega'\,\partial\Omega}=4\pi^2\frac{\omega'^2}{c}\cos^2[\alpha]|t_B[z_B,k_{\parallel S},\omega']|^2. \tag{57}$$

while in the backward case one has

$$\frac{\partial^2 I_b}{\partial\omega'\,\partial\Omega}=4\pi^2\frac{\omega'^2}{c}\cos^2[\alpha]|r_B[z_B,k_{\parallel S},\omega']|^2. \tag{58}$$

VI. APPROXIMATION OF HARD X RAYS

In the case of hard x rays the optical indices are close to unity and if the absorption by materials can be neglected, important simplifications arise in the formulation of the radiated intensity. In these conditions it can be shown that the two eigenvalues λ_1 and λ_2 and their associated normalized eigenvectors $\mathbf{u}_1,\mathbf{u}_2$ are, respectively,

$$\lambda_1=e^{-id(k_z+k_\perp)}, \tag{59a}$$

$$\lambda_2=e^{-id(k_z-k_\perp)}, \tag{59b}$$

$$\mathbf{u}_1=\frac{1}{k}\begin{pmatrix} -k_\parallel\\ k_\perp\end{pmatrix}, \tag{59c}$$

$$\mathbf{u}_2=\frac{1}{k}\begin{pmatrix} k_\parallel\\ k_\perp\end{pmatrix}. \tag{59d}$$

One assumes that the emission takes place practically along the direction of the electrons, then a tedious calculation using series expansion at the first order in unit decrement of the optical indices leads to the following expression of the modulus of the projections δ_1 and δ_2 of δ following the basis $\mathbf{u}_1,\mathbf{u}_2$:

$$|\delta_1|=\left|\frac{\beta q\,\delta\varepsilon\tan[\alpha](1-\beta^2-\beta\cos[\alpha])\sin\left[\frac{d_2k}{2}\left(\cos[\alpha]-\frac{1}{\beta}\right)\right]}{2\pi^2ck(1-\beta^2\cos^2[\alpha])(1-\beta\cos[\alpha])}\right|, \tag{60a}$$

$$|\delta_2|=\left|\frac{\beta q\,\delta\varepsilon\tan[\alpha](1-\beta^2+\beta\cos[\alpha])\sin\left[\frac{d_2k}{2}\left(\cos[\alpha]+\frac{1}{\beta}\right)\right]}{2\pi^2ck(1-\beta^2\cos^2[\alpha])(1+\beta\cos[\alpha])}\right|. \tag{60b}$$

From Eqs. (D15), (D16), and (59a)–(59d), it follows that

$$|t_B|=|\delta_1||S_1|, \tag{61a}$$

$$|r_B|=|\delta_2||S_2|. \tag{61b}$$

In the hard-x-ray approximation the sums of the power of the eigenvalues are

$$|S_1|=\left|\frac{\sin\left[N\frac{dk}{2}\left(\cos[\alpha]-\frac{1}{\beta}\right)\right]}{\sin\left[\frac{dk}{2}\left(\cos[\alpha]-\frac{1}{\beta}\right)\right]}\right|, \tag{62a}$$

$$|S_2| = \left| \frac{\sin\left[N \frac{dk}{2} \left(\cos[\alpha] + \frac{1}{\beta} \right) \right]}{\sin\left[\frac{dk}{2} \left(\cos[\alpha] + \frac{1}{\beta} \right) \right]} \right|. \tag{62b}$$

The use of Eqs. (57), (58), (60a) and (60b), (61a) and (61b), and (62a) and (62b) allows one to write the intensity as the product of three factors,

$$\frac{\partial^2 I}{\partial E \partial \Omega} = I_1 I_2 I_3. \tag{63a}$$

Here the intensity is given by the unit photon energy interval instead of the unit angular frequency interval.

For the forward case, the first factor reads

$$I_1 = \frac{\bar{\alpha}}{4\pi^2} (\delta\varepsilon)^2 \beta^2 \left| \sin[\alpha] \frac{(1-\beta^2-\beta\cos[\alpha])}{(1-\beta^2\cos^2[\alpha])(1-\beta\cos[\alpha])} \right|^2, \tag{63b}$$

where $\bar{\alpha}$ is the fine-structure constant. It gives the intensity of the transition radiation by a single interface. This formula is in agreement with the result of Garibian [16].

The second factor reads

$$I_2 = 4\sin^2\left[\frac{d_2 k}{2} \left(\cos[\alpha] - \frac{1}{\beta} \right) \right]. \tag{63c}$$

It accounts for the interferences between two successive interfaces. It is nothing else but the structure factor of a bilayer.

Finally, the third factor reads

$$I_3 = \left| \frac{\sin\left[N \frac{dk}{2} \left(\cos[\alpha] - \frac{1}{\beta} \right) \right]}{\sin\left[\frac{dk}{2} \left(\cos[\alpha] - \frac{1}{\beta} \right) \right]} \right|^2. \tag{63d}$$

It accounts for the interferences between the bilayers.

Similar formulas could be deduced for the backward case. It is worth discussing the behavior of these different factors. When the particle is very relativistic, the first factor I_1 is proving to peak for an angle α, expressed in radian, close to $1/\gamma = \sqrt{1-\beta^2}$, which means that the radiation is mainly emitted along a cone centered on the trajectory of the particle with an opening angle equal to $2/\gamma$. To compare with previous works we consider the case when the number N of bilayers becomes very large. In this case, the factor I_3 behaves as a sum of δ functions, as pointed out in [11] and [12],

$$I_3 \cong 2\pi N \sum_p \delta\left[dk \left(\cos[\alpha] - \frac{1}{\beta} \right) - 2\pi p \right]. \tag{64}$$

As mentioned before, one sees from the above equation that, in case of no absorption, the photon yield diverges when N tends towards infinity. The integration over the solid angle leads, for the intensity, to the classical formula given by Eq. (11) and is in agreement with Eq. (23) of Ref. [13] applied far from the Bragg domains.

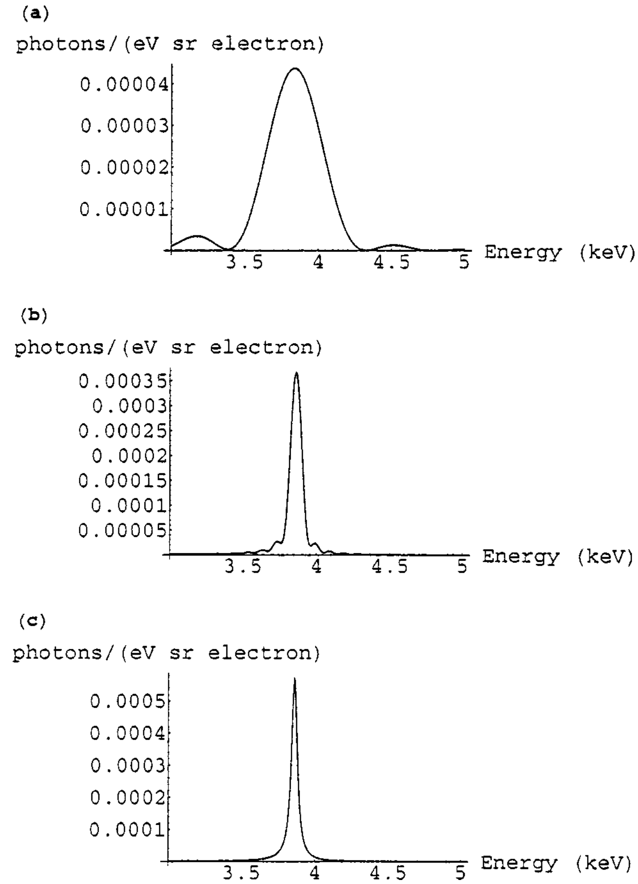

FIG. 3. Calculated photon yield versus the photon energy of the RTR emitted by Ni/C stacks crossed by 15-MeV electrons. The thickness of the Ni layer is 176 nm while the thickness of the carbon layer is 221 nm. These parameters correspond to the experiment of Ref. [10]. Number of bilayers: (a) $N=10$, (b) $N=50$, (c) $N=\infty$.

Always in these conditions, the emission of the RTR takes place when the following well-known resonance condition is fulfilled:

$$\cos[\alpha] = \frac{1}{\beta} - p\frac{\lambda}{d}. \tag{65}$$

Then, the second term I_2 becomes

$$I_2 = 4\sin^2[\Gamma \pi p], \tag{66}$$

where Γ is the ration $d_2/(d_1+d_2)$. The relation (60) gives conditions of extinction which are similar to the ones of a Bragg reflector [17]. One can consider the matrix method to be tested by its behavior in the hard-x-ray case.

VII. NUMERICAL APPLICATION AND DISCUSSION

In this section we consider the experiment recently performed by Yamada *et al.* [10]. They observed the soft x rays produced by the RTR from a Ni/C multilayer with a submicrometer period. This target consisted of only ten bilayers of 176-nm-thick Ni and 221-nm-thick C stacked on a SiN membrane. The energy of the electrons was 15 MeV. For different

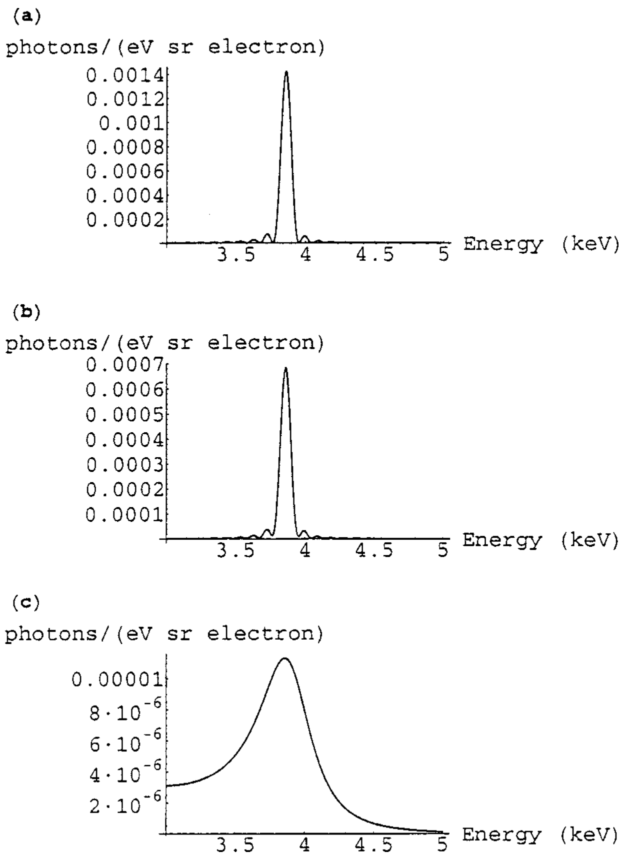

FIG. 4. Calculated photon yield versus the photon energy of the RTR emitted by targets, the parameters of which are the same as those of Fig. 3(b) for different values of the imaginary part of the optical index: (a) 0, (b) 0.5, (c) 10 times the very values.

values of the number of bilayers N, Fig. 3 shows the photon yield versus the RTR photon energy for an observation angle α equal to 17 mrad, calculated by our approach. The calculations are performed with an energy step of 4×10^{-4} keV. The optical constants used in the calculations are borrowed from Ref. [18] which uses the theory developed in Ref. [19]. In practice, the variations of the complex index $n=1-\delta+i\beta$ versus the photon energy E, expressed in eV, can be calculated with a reasonable approximation by means of the following empirical formula:

$$\delta=\exp(a_1-b_1\ln[E]),$$

$$\beta=\exp(a_2-b_2\ln[E]).$$

The parameters for nickel are

$$a_1=4.2511,\quad b_1=1.5796,\quad a_2=16.8626,\quad b_2=3.4487$$

and the parameters for carbon are

$$a_1=6.4609,\quad b_1=2.0372,\quad a_2=16.8646,\quad b_2=3.8762.$$

We note that the values of the photon yield calculated by our method are slightly different from the measured ones; this discrepancy can be attributed to different points.

(i) The multilayer target used in the experiment is not ideal; there is interface roughness which degrades the sharpness of the interface and consequently tends to diminish the photon yield.

(ii) Pile-up effects in the detection of the radiation are difficult to account for and probably the number of recorded counts are less than the number of emitted ones.

(iii) The optical constants retained in the calculation correspond to bulk materials and are likely larger than the true ones.

To get an insight of the influence of the absorption upon the photon yield, we have computed this quantity for imaginary parts of the dielectric constants equal to 0, 0.5, 10 times the very values. The calculations have been carried out for the same kind of target as in Ref. [10], but for 50 bilayers instead of 10. Their results are shown in Fig. 4. As expected, the bandwidth increases when the absorption becomes large.

VIII. CONCLUSION

A theory of the transition radiation by a stack of layers crossed normally by an electron has been developed. It rigorously takes into account absorption and the dynamical effects such as the multiple scattering. Extension to the oblique case corresponding to the so-called parametric radiation is planned; this problem will be more difficult to handle because of the lack of the cylindrical symmetry.

APPENDIX A

The transverse spatial and temporal Fourier transform of the field is defined by

$$\overline{\mathbf{E}}(\mathbf{k}'_{\parallel},\omega',z)=\frac{1}{(2\pi)^3}\int\int\int \mathbf{E}(\rho,zt)e^{i(-\mathbf{k}'_{\parallel}\cdot\boldsymbol{\rho}+\omega' t)}dt\,d^2\rho. \tag{A1}$$

Combining Eqs. (9), (10), and (12) yields

$$\overline{\mathbf{E}}(\mathbf{k}'_{\parallel},\omega',z)=\frac{1}{(2\pi)^3}\int\int\int\int\int\int\int \frac{qi}{2\pi^2\varepsilon[\omega]} \times\frac{\frac{\omega\varepsilon[\omega]}{c^2}\mathbf{v}-\mathbf{k}}{k^2-\frac{\omega^2\varepsilon[\omega]}{c^2}}\delta[\omega-\mathbf{k}\cdot\mathbf{v}] \times e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d\omega\,d\mathbf{k}\,e^{i(-\mathbf{k}'_{\parallel}\cdot\boldsymbol{\rho}+\omega' t)}dt\,d^2\rho. \tag{A2}$$

Performing the integration over the time t and the variable ρ gives

$$\overline{\mathbf{E}}(\mathbf{k}'_{\parallel},\omega',z)=\int\int\int\int \frac{qi}{2\pi^2\varepsilon[\omega]}\frac{\dfrac{\omega\varepsilon[\omega]}{c^2}\mathbf{v}-\mathbf{k}}{k^2-\dfrac{\omega^2\varepsilon[\omega]}{c^2}}\times\delta[\omega-k_\perp v]e^{ik_\perp z}\delta[\mathbf{k}_\parallel-\mathbf{k}'_\parallel]\times\delta[\omega'-\omega]d\omega\, d^2\mathbf{k}_\parallel\, dk_\perp . \tag{A3}$$

The integration over ω leads to

$$\overline{\mathbf{E}}(\mathbf{k}'_{\parallel},\omega',z)=\int\int\int \frac{qi}{2\pi^2\varepsilon[\omega']}\frac{\dfrac{\omega\varepsilon[\omega']}{c^2}\mathbf{v}-\mathbf{k}}{k^2-\dfrac{\omega^2\varepsilon[\omega']}{c^2}}\times\delta[\omega'-k_\perp v]e^{ik_\perp z}\delta[\mathbf{k}_\parallel-\mathbf{k}'_\parallel]d^2\mathbf{k}_\parallel\, dk_\perp . \tag{A4}$$

The integration over $\mathbf{k}_\parallel$ gives

$$\overline{\mathbf{E}}(\mathbf{k}'_{\parallel},\omega',z)=\int \frac{qi}{2\pi^2\varepsilon[\omega']}\frac{\dfrac{\omega'\varepsilon[\omega']}{c^2}\mathbf{v}-\mathbf{k}'_\parallel-k_\perp\dfrac{\mathbf{v}}{v}}{k'^2_\parallel+k_\perp^2-\dfrac{\omega'^2\varepsilon[\omega']}{c^2}}\times\delta[\omega'-k_\perp v]e^{ik_\perp z}\, dk_\perp . \tag{A5}$$

After a last integration over $k_\perp$ one obtains the expression of $\overline{\mathbf{E}}$,

$$\overline{\mathbf{E}}(\mathbf{k}'_{\parallel},\omega',z)=\frac{1}{v}\frac{qi}{2\pi^2\varepsilon[\omega']}\times\frac{\omega'\left(\dfrac{\varepsilon[\omega']}{c^2}-\dfrac{\omega'}{v^2}\right)\mathbf{v}-\mathbf{k}'_\parallel}{k'^2_\parallel+\left(\dfrac{\omega'}{v}\right)^2-\dfrac{\omega'^2\varepsilon[\omega']}{c^2}}e^{i(\omega'/v)z}. \tag{A6}$$

APPENDIX B

One starts from Eq. (17)

$$\overline{\mathbf{E}}_0[\mathbf{k}'_{\parallel},\omega',z]=\int\int\int \tilde{\mathbf{E}}_0[\mathbf{k}_\parallel,k_\perp]e^{ik_\perp z}\delta[\mathbf{k}_\parallel-\mathbf{k}'_\parallel]\times\delta[\omega-\omega']d^2\mathbf{k}_\parallel\, dk_\perp . \tag{B1}$$

From the dispersion equation

$$k_\perp^2=\frac{\omega^2}{c^2}\varepsilon[\omega]-\mathbf{k}_\parallel^2, \tag{B2}$$

one obtains

$$dk_\perp=\pm\frac{2\omega\varepsilon[\omega]+\omega^2\dfrac{d\varepsilon[\omega]}{d\omega}}{2c^2\sqrt{\dfrac{\omega^2}{c^2}\varepsilon[\omega]-\mathbf{k}_\parallel^2}}d\omega. \tag{B3}$$

Inserting Eq. (B3) into Eq. (B1) gives

$$\overline{\mathbf{E}}_0[\mathbf{k}'_{\parallel},\omega',z]=\pm\int\int\int \tilde{\mathbf{E}}_0[\mathbf{k}_\parallel,\pm k_\perp[\mathbf{k}_\parallel,\omega]]\times e^{\pm ik_\perp[\mathbf{k}_\parallel,\omega]z}\delta[\mathbf{k}_\parallel-\mathbf{k}'_\parallel]\times\delta[\omega-\omega']d^2\mathbf{k}_\parallel\frac{2\omega\varepsilon[\omega]+\omega^2\dfrac{d\varepsilon[\omega]}{d\omega}}{2c^2\sqrt{\dfrac{\omega^2}{c^2}\varepsilon[\omega]-\mathbf{k}_\parallel^2}}. \tag{B4}$$

Performing the integration one obtains

$$\overline{\mathbf{E}}_0[\mathbf{k}'_{\parallel},\omega',z]=\pm\left(\tilde{\mathbf{E}}_0[\mathbf{k}'_\parallel,\pm k_\perp[\omega',\mathbf{k}'_\parallel]]e^{\pm ik_\perp[\omega',\mathbf{k}'_\parallel]z}\times\frac{2\omega'\varepsilon[\omega']+\omega'^2\dfrac{d\varepsilon[\omega']}{d\omega'}}{2c^2\sqrt{\dfrac{\omega'^2}{c^2}\varepsilon[\omega']-\mathbf{k}'^2_\parallel}}\right). \tag{B5}$$

APPENDIX C

According to the method of stationary phase, integrals of the following form:

$$\int e^{iDf[\mu]}g[\mu]d\mu, \tag{C1}$$

where f is a real-valued function, integrals are given when D becomes very large by

$$\int e^{iDf[\mu]}g[\mu]d\mu=\sum_j\sqrt{\frac{2\pi}{D|f''[\mu_j]|}}\times e^{iDf[\mu_j]}e^{i\,\mathrm{sgn}(f''[\mu_j])(\pi/4)}g[\mu_j], \tag{C2}$$

where μ_j are the roots of the equation $f'[\mu]=0$ which expresses the stationarity of the phase. Introducing the following quantities:

$$g_{\parallel\eta}[k_\parallel,\alpha,\omega']=\tilde{E}_{0\parallel\eta}[k_\parallel,\omega']i \times\sqrt{\frac{2}{\pi k_\parallel D\sin[\alpha]}}\frac{\partial k_\perp[k_\parallel,\omega']}{\partial\omega'}k_\parallel,$$
$$g_{\perp\eta}[k_\parallel,\alpha,\omega']=\tilde{E}_{0\perp\eta}[k_\parallel,\omega'] \times\sqrt{\frac{2}{\pi k_\parallel D\sin[\alpha]}}\frac{\partial k_\perp[k_\parallel,\omega']}{\partial\omega'}k_\parallel, \tag{C3}$$

and

$$p_\eta[k_\parallel,\alpha,\omega']=\eta k_\perp[k_\parallel,\omega']\cos[\alpha]+k_\parallel\sin[\alpha],$$
$$q_\eta[k_\parallel,\alpha,\omega']=\eta k_\perp[k_\parallel,\omega']\cos[\alpha]-k_\parallel\sin[\alpha], \tag{C4}$$

the integral (53) can be rewritten as

$$\hat{\mathbf{E}}_0[D,\alpha,\omega']=\sum_{\eta\in\{-1,1\}}2\pi^2\eta\int\begin{pmatrix} g_{\parallel\eta}[k_\parallel,\alpha,\omega'](e^{-i(3/4)\pi}e^{iDp_\eta[k_\parallel,\alpha,\omega']}+e^{i(3/4)\pi}e^{iDq_\eta[k_\parallel,\alpha,\omega']})\\ 0\\ g_{\perp\eta}[k_\parallel,\alpha,\omega'](e^{-i(\pi/4)}e^{iDp_\eta[k_\parallel,\alpha,\omega']}+e^{i(\pi/4)}e^{iDq_\eta[k_\parallel,\alpha,\omega']})\end{pmatrix}dk_\parallel. \tag{C5}$$

The first derivative of the phases are

$$p'_\eta[k_\parallel,\alpha,\omega']=\eta\frac{-k_\parallel}{k_\perp[k_\parallel,\omega']}\cos[\alpha]+\sin[\alpha],$$
$$q'_\eta[k_\parallel,\alpha,\omega']=\eta\frac{-k_\parallel}{k_\perp[k_\parallel,\omega']}\cos[\alpha]-\sin[\alpha], \tag{C6}$$

that is, in terms of the angle θ between the electron axis and the wave vector $\mathbf{k}$,

$$p'_\theta[k_\parallel,\alpha,\omega']=\frac{-\sin[\theta]\cos[\alpha]+\cos[\theta]\sin[\alpha]}{\cos[\theta]},$$
$$q'_\theta[k_\parallel,\alpha,\omega']=\frac{-\sin[\theta]\cos[\alpha]-\cos[\theta]\sin[\alpha]}{\cos[\theta]}. \tag{C7}$$

The zeros of the first derivative of the phases which are the stationary points $k_{\parallel S}$ are obtained, respectively, for

$$\theta=\alpha,$$
$$\theta=\pi-\alpha. \tag{C8}$$

The second derivatives are

$$p''_\theta[k_{\parallel S},\alpha,\omega']=q''_\theta[k_{\parallel S},\alpha,\omega']=-\frac{\cos[\alpha]}{k\cos^3[\theta]}=-\frac{c}{\omega'}\frac{\cos[\alpha]}{\cos^3[\theta]}. \tag{C9}$$

At the stationary points they take the values

$$p''[k_{\parallel S},\alpha,\omega']=q''[k_{\parallel S},\alpha,\omega']=-\frac{1}{k\cos^2[\alpha]}=-\frac{c}{\omega'}\frac{1}{\cos^2[\alpha]}. \tag{C10}$$

It should be noted that these second derivatives are negative. The phases p and q are not simultaneously stationary according to the value of θ and α. The stationary phases are

$$p[k_{\parallel S},\alpha,\omega']=k[k_{\parallel S},\omega']=\frac{\omega'}{c},$$
$$q[k_{\parallel S},\alpha,\omega']=-k[k_{\parallel S},\omega']=-\frac{\omega'}{c}. \tag{C11}$$

The value of g's at the stationary points are, in terms of the angle θ,

$$g_{\parallel\theta}[k_{\parallel S},\alpha,\omega']=\tilde{E}_{0\parallel\theta}[k_{\parallel S},\omega']i \times\sqrt{\frac{2}{\pi kD}}\frac{\partial k_\perp^2[k_{\parallel S},\omega']}{\partial\omega'}\frac{1}{2|\cos[\alpha]|},$$
$$g_{\perp\theta}[k_{\parallel S},\alpha,\omega']=\tilde{E}_{0\perp\theta}[k_{\parallel S},\omega'] \times\sqrt{\frac{2}{\pi kD}}\frac{\partial k_\perp^2[k_{\parallel S},\omega']}{\partial\omega'}\frac{1}{2|\cos[\alpha]|}. \tag{C12}$$

Collecting the preceding results gives the expression of the field

$$\hat{\mathbf{E}}_0[D,\alpha,\omega']=\sigma[\alpha]\frac{2\pi^2}{D}\frac{\partial k_\perp^2[k_{\parallel S},\omega']}{\partial\omega'}\begin{pmatrix}-i\tilde{E}_{0\parallel\alpha}[k_{\parallel S},\omega']e^{iD(\omega'/c)}+\tilde{E}_{0\parallel(\pi-\alpha)}[k_{\parallel S},\omega']e^{-iD(\omega'/c)}\\ 0\\ -i\tilde{E}_{0\perp\alpha}[k_{\parallel S},\omega']e^{iD(\omega'/c)}-\tilde{E}_{0\perp(\pi-\alpha)}[k_{\parallel S},\omega']e^{-iD(\omega'/c)}\end{pmatrix}, \quad \text{(C13)}$$

where $\sigma[\alpha]$ stands for the sign of $\cos[\alpha]$. However, there are only waves outgoing from the stack, so that the solution $\theta=\pi-\alpha$ must be rejected,

$$\hat{\mathbf{E}}_0[D,\alpha,\omega']=-i\sigma[\alpha]\frac{2\pi^2}{D}\frac{\partial k_\perp^2[k_{\parallel S},\omega']}{\partial\omega'}\times\begin{pmatrix}\tilde{E}_{0\parallel\alpha}[k_{\parallel S},\omega']\\ 0\\ \tilde{E}_{0\perp\alpha}[k_{\parallel S},\omega']\end{pmatrix}e^{iD(\omega'/c)}. \quad \text{(C14)}$$

APPENDIX D

One starts from the formula (37) with $j=N$,

$$\mathbf{F}_{2N+1}=S_N[\mu]\delta+P_N[\mu]\mathbf{F}_1. \quad \text{(D1)}$$

One decomposes the F's and δ in terms of the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of the matrix μ,

$$\mathbf{F}_{2N+1}=\varphi_{o1N}\mathbf{u}_1+\varphi_{o2N}\mathbf{u}_2, \quad \text{(D2)}$$

$$\mathbf{F}_1=\varphi_{i1N}\mathbf{u}_1+\varphi_{i2N}\mathbf{u}_2, \quad \text{(D3)}$$

$$\delta=\delta_1\mathbf{u}_1+\delta_2\mathbf{u}_2. \quad \text{(D4)}$$

This decomposition applied to Eq. (D1) leads to the two following equations:

$$\varphi_{o1N}=\delta_1 S_1+P_1\varphi_{i1N}, \quad \text{(D5)}$$

$$\varphi_{o2N}=\delta_2 S_2+P_2\varphi_{i2N}, \quad \text{(D6)}$$

where S and P are given in terms of the eigenvalues λ of the matrix μ by

$$S=\frac{1-\lambda^N}{1-\lambda}, \quad \text{(D7)}$$

$$P=\lambda^N. \quad \text{(D8)}$$

The limit conditions (no incoming wave from vacuum into the multilayer structure) lead to the four following equations involving the boundary field amplitudes t_B and r_B:

$$\varphi_{i1N}u_{11}+\varphi_{i2N}u_{21}=\phi_{i1}r_B, \quad \text{(D9)}$$

$$\varphi_{i1N}u_{12}+\varphi_{i2N}u_{22}=\phi_{i2}r_B, \quad \text{(D10)}$$

$$\varphi_{o1N}u_{11}+\varphi_{o2N}u_{21}=\phi_{o1}t_B, \quad \text{(D11)}$$

$$\varphi_{o1N}u_{12}+\varphi_{o2N}u_{22}=\phi_{o2}t_B. \quad \text{(D12)}$$

The ϕ's can be determined from Eq. (28). The variables u_{k1}'s are defined as the components of the eigenvectors by

$$\mathbf{u}_1=\begin{pmatrix}u_{11}\\ u_{12}\end{pmatrix}, \quad \text{(D13)}$$

$$\mathbf{u}_2=\begin{pmatrix}u_{21}\\ u_{22}\end{pmatrix}. \quad \text{(D14)}$$

Solving the system of six equations [(D5), (12), (15)–(18)] gives

$$t_B=\frac{(u_{12}u_{21}-u_{11}u_{22})[P_2S_1\delta_1(u_{12}\phi_{i1}-u_{11}\phi_{i2})+P_1S_2\delta_2(u_{22}\phi_{i1}-u_{21}\phi_{i2})]}{P_1(u_{22}\phi_{i1}-u_{21}\phi_{i2})(u_{12}\phi_{o1}-u_{11}\phi_{o2})-P_2(u_{12}\phi_{i1}-u_{11}\phi_{i2})(u_{22}\phi_{o1}-u_{21}\phi_{o2})}, \quad \text{(D15)}$$

$$r_B=\frac{(u_{12}u_{21}-u_{11}u_{22})[S_1\delta_1(u_{12}\phi_{o1}-u_{11}\phi_{o2})+S_2\delta_2(u_{22}\phi_{o1}-u_{21}\phi_{o2})]}{P_1(u_{22}\phi_{i1}-u_{21}\phi_{i2})(u_{12}\phi_{o1}-u_{11}\phi_{o2})-P_2(u_{12}\phi_{i1}-u_{11}\phi_{i2})(u_{22}\phi_{o1}-u_{21}\phi_{o2})}. \quad \text{(D16)}$$

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