Stochastic resonance in free-electron lasers

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We present evidence of stochastic resonance in free-electron lasers. In order to do that, we have analyzed theoretically the dynamics of a free-electron laser oscillator. A weak modulation and a noise source have been applied to the initial energy of the electron beam. We have found stochastic resonance for different frequencies and amplitudes of the modulation. A threshold crossing mechanism leads to the stochastic resonance in this system.

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I. INTRODUCTION

Stochastic resonance (SR) has been the focus of intensive research over the past decade. The term is given to a phenomenon that is manifest in nonlinear systems whereby generally feeble input information (such as a weak signal) can be amplified and optimized by the assistance of noise, i.e., adding noise to a system can increase the output signal-to-noise ratio (SNR). The effect requires three basic ingredients: (i) an energetic activation barrier or, more generally, a form of threshold; (ii) a weak coherent input (such as a periodic signal); (iii) a source of noise that is inherent in the system, or that adds to the coherent input. Given these features, the response of the system undergoes resonancelike behavior as a function of the noise level [1,2]. The first experimental observation of this effect was performed by Fauve *et al.* by observing the switching of a saturated operational amplifier in a Schmitt trigger circuit driven by both modulation and noise [3]. The first observation of SR in an optical device was in a bidirectional ring laser [4,5]. It was also the first instance of this phenomenon in a system with a double-well potential where the depth of the wells could be controlled by a technique that employs an acousto-optic modulator with a tunable-frequency acoustic signal [6]. More recently, SR has been observed in lasers with saturable absorbers [7] and semiconductor diode laser [8], where both systems present bistability.

In this paper, we describe theoretically the stochastic resonance in a free-electron laser (FEL) oscillator. FEL physics has become an active area of research since the first FEL was operated at Stanford in 1976 [9]. As a result of the interaction between the relativistic electron beam and the electromagnetic fields, FEL becomes a highly nonlinear system in which instabilities and chaos can easily be found [10]. In an FEL, relativistic electrons travel through a static periodic magnetic field (undulator or wiggler) and oscillate to amplify coherent optical radiation with the same polarization as the magnet. The electron trajectories are primarily determined by the magnet, but the laser radiation causes ''bunching'' on the optical wavelength scale and leads to gain [11,12]. Several theoretical approaches have been used to describe the free-electron laser. The picture of single-particle currents driving Maxwell's nonlinear wave equation provides a clear, intuitive description of both electron and wave dynamics [13,14].

II. THEORETICAL MODEL

The starting point for our analysis is the well-known FEL equations for an electron bunch larger than the slippage length λN_w, which is the length overtaken by the radiation of wavelength λ over the electrons after the N_w wiggler periods [14].

$$\frac{d\xi}{d\tau}=\nu, \tag{2.1}$$

$$\frac{d\nu}{d\tau}=|a|\cos(\xi+\phi), \tag{2.2}$$

$$\frac{da}{d\tau}=-r\langle\exp(-i\xi)\rangle. \tag{2.3}$$

We use the following definitions:

$$\xi\equiv(k+k_w)z-\omega t,\quad \nu\equiv 2\pi N_w\left(1-\frac{\gamma_R^2}{\gamma^2}\right), \tag{2.4}$$

$$|a|\equiv\left(\frac{4\pi N_w eKL_wE}{\gamma_R^2 mc^2}\right),\quad \tau\equiv\frac{c}{L_w}t, \tag{2.5}$$

$$r\equiv\left(\frac{16\pi^2 N_w e^2K^2L_w^2\rho_0}{\gamma_R^3 mc^2}\right),\quad \gamma_R^2\equiv\frac{k}{2k_0}\left(1+\frac{1}{2}K^2\right), \tag{2.6}$$

$$K\equiv\frac{eB_w\lambda_w}{2\pi mc^2}, \tag{2.7}$$

The meaning of the dimensionless variables and parameters in Eqs. (2.1)–(2.3) is the following: ξ and ν are the phase variable and the dimensionless velocity of the electrons, respectively; $a=|a|\exp(i\phi)$ is the scaled complex field amplitude of the radiation field, τ is the scaled time, and r is a rough measure of the gain.

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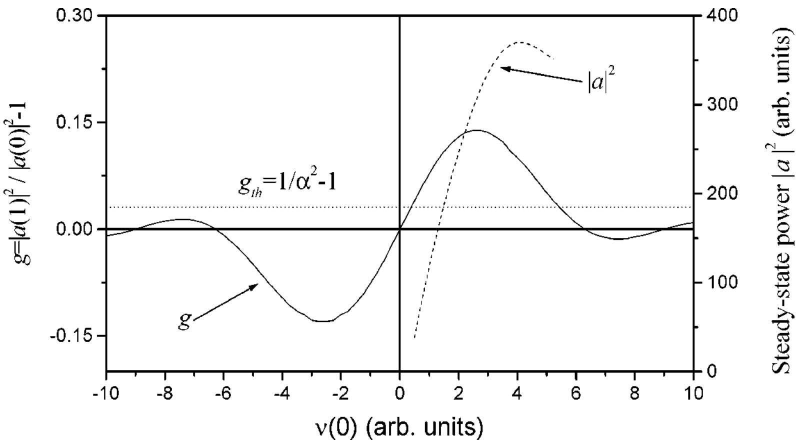

FIG. 1. Dimensionless gain curve g (solid line) versus initial electron dimensionless velocity ν_0 for single-pass operation. Steady-state power (dashed line) versus ν_0 for an FEL oscillator case. The threshold gain g_{th} (dotted line) is also shown. The parameters are $r=1$ and $\alpha=0.985$.

The rest of the parameters that appear in Eqs. (2.4)–(2.7) are the amplitude E and wave number k of the radiation field ($\omega=ck$), the magnetostatic amplitude B_w, and the wave number k_w associated with the undulator periodicity ($k_w=2\pi/\lambda_w$). K is the undulator parameter, ρ_0 is the electron density, γ is the electron energy, and γ_R is the resonant energy. It means that when $\gamma=\gamma_R$, i.e., $\nu=0$ (resonant condition), one wavelength of light passes over the electrons as they pass through one period of the undulator magnet. Here z represents the direction of propagation of the electron beam and the electromagnetic wave. Note that if the undulator length is $L_w=\lambda_w N_w$, the scaled time is confined between the interval $0\leq\tau\leq 1$, corresponding to the time interval $0\leq t\leq L_w/c$ in which the FEL process takes place.

It has been shown that the complete FEL dynamics can be approximately described in terms of a few relevant collective variables [15]. Introducing the bunching parameter $B=\langle\exp(-i\xi)\rangle$, the phase-momentum average $P=\langle\nu\exp(-i\xi)\rangle$, the average momentum $V=\langle\nu\rangle$, and the average kinetic energy $S=\langle\nu^2\rangle$, Eqs. (2.1)–(2.3) reduce to the following set of closed equations:

$$\frac{da}{d\tau}=-rB, \tag{2.8}$$

$$\frac{dB}{d\tau}=-iP, \tag{2.9}$$

$$\frac{dP}{d\tau}=\frac{1}{2}a-iSB-2iVP+2iV^2B, \tag{2.10}$$

$$\frac{dV}{d\tau}=\frac{1}{2}[aB^*+\text{c.c.}], \tag{2.11}$$

$$\frac{dS}{d\tau}=[aP^*+\text{c.c.}]. \tag{2.12}$$

In an FEL oscillator, an optical pulse circulates in a cavity of length L_c between two reflecting mirrors and during each pass interacts with a new electron pulse along the wiggler following Eqs, (2.8)–(2.12). The radiation is reflected backward after amplification and then forward for the next round trip, so that the input field for the $(n+1)$th pass is

$$a_0^{(n+1)}=\alpha a_f^{(n)}, \tag{2.13}$$

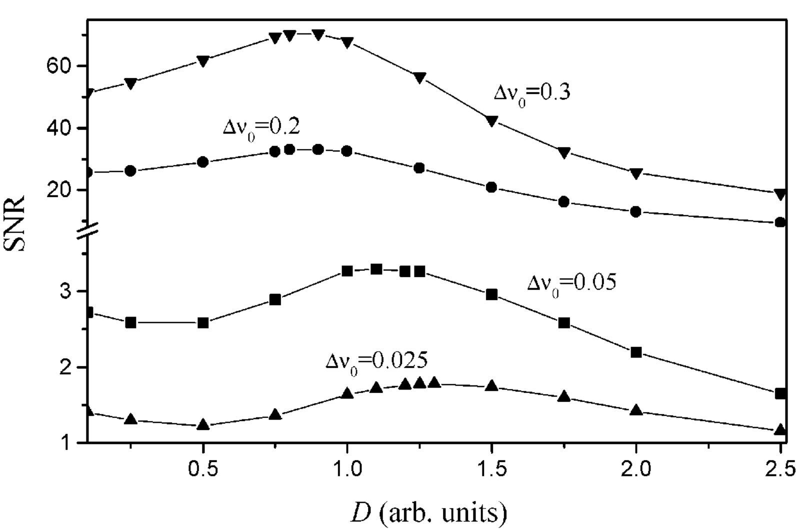

FIG. 2. Signal-to-noise ratio as a function of the input noise power for a modulation amplitude $\Delta\nu_0=0.3$ (down triangles), $\Delta\nu_0=0.2$ (circles), $\Delta\nu_0=0.05$ (squares), and $\Delta\nu_0=0.025$ (up triangles). The parameters are $\nu_0=2.5$ and $f_m=50$ kHz.

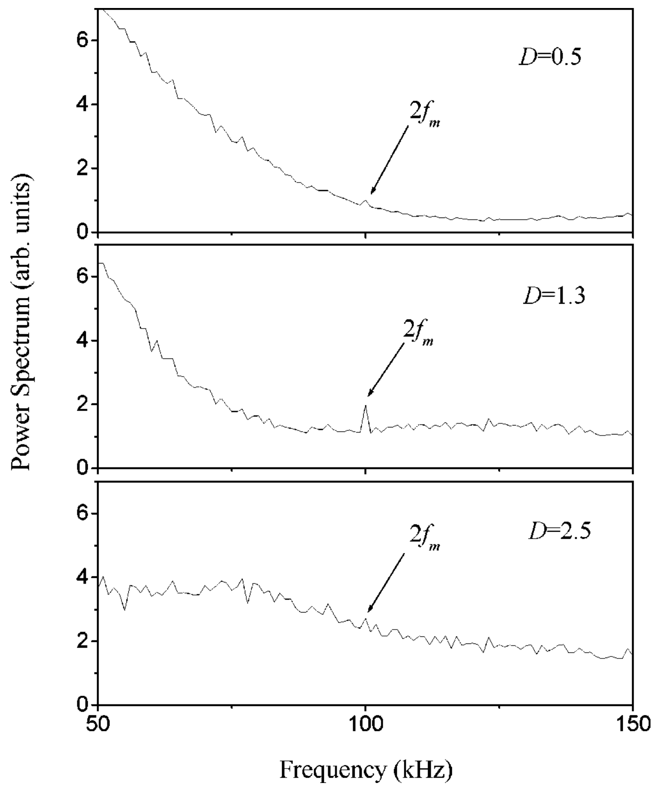

FIG. 3. Power spectrum (averaged over 200 realizations) of the optical power versus frequency at different noise strength values. The parameters are $\overline{\nu_0}=2.5$, $f_m=50$ kHz, and $\Delta\nu_0=0.025$.

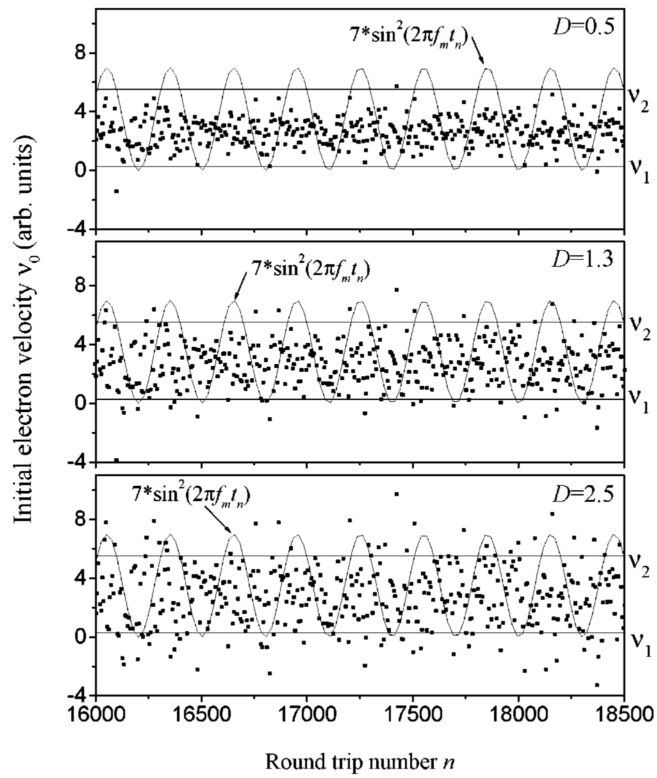

FIG. 4. Initial electron dimensionless velocity ν_0 (points) versus round trip number n at different noise strength values. The threshold values ν_1, ν_2, and an oscillatory function proportional to the deterministic component of ν_0 are also shown. The parameters are $\overline{\nu_0}=2.5$, $f_m=50$ kHz, and $\Delta\nu_0=0.025$.

where $a_0\equiv a(\tau=0)$ and $a_f\equiv a(\tau=1)$ are the radiation fields at the entrance and at the end of the wiggler, respectively, and α accounts for the reduction in amplitude due to energy losses at the mirrors.

III. DETERMINISTIC CASE

First, we study the problem in the absence of noise. We use the physical parameters $N_w=72$, $\lambda_w=3$ cm, $Bw=3\times10^3$ G, $\gamma_R=80$, and $\rho_0=0.5\times10^{10}$ cm^{-3}, and we obtain an undulator parameter $K=0.8$, a radiation wavelength $\lambda=3$ μm, and $r=1$. We consider a cavity length $L_c=500$ cm and $\alpha=0.985$, which corresponds to a cavity $Q=30$ ($\alpha=0.5[1+\exp(-1/Q)]$). The initial electron beam is assumed to be monochromatic and unbunched, and we consider an initial energy γ_0, i.e., $\nu_0\equiv\nu(0)$. This initial dimensionless velocity of electrons is crucial in determining the phase-space evolution. Then the initial conditions for the electron beam at each pass are $B^{(n)}=P^{(n)}=0$, $V^{(n)}=\nu_0$, and $S^{(n)}=\nu_0^2$ at $\tau=0$. The initial field is assumed to match the level of spontaneous emission as we take $a^{(0)}(\tau=0)=0.001$.

We consider first the problem of only one round trip. This is the case of an FEL single-pass amplifier, the simplest FEL mode of operation. We define the gain curve as $g\equiv|a(1)|^2/|a(0)|^2-1$. Figure 1 shows the gain versus the initial electron velocity ν_0 (solid line). This is the well-known antisymmetric gain curve [14]. The maximum gain is placed at $\nu_0\simeq2.6$. On one hand, with higher gains, $r\gg1$, gain becomes somewhat more symmetric about the resonance ($\nu_0=0$), and on the other hand, as the fields become stronger, all modes in the free-electron laser eventually saturate. An important feature of this last process is that the point of maximum gain moves away from resonance; it starts at 2.6 in weak fields and moves to 5 at high-field values.

Next we study the case of an FEL oscillator. In order to achieve laser emission, it is necessary that gain per pass is larger than the loss per round trip; that is the laser condition

$$[g(\nu_0)+1]\alpha^2>1. \tag{3.1}$$

Then, only initial velocities with gain values $g(\nu_0)$ satisfying Eq. (3.1) will reach laser emission. In our case, the threshold gain is $g_{\rm th}=0.03$, which leads to $\nu_1\equiv0.375<\nu_0<\nu_2\equiv5.375$. During each pass, the field increases until saturation. The maximum gain point moves away from resonance. The steady-state power, $|a|^2$, versus the initial electron velocity is shown in Fig. 1 (dashed line). The laser emission occurs only between $0.375\leqslant\nu_0\leqslant5.375$, in agreement with the prediction of the laser condition [Eq. (3.1)], and the maximum steady-state power is placed at $\nu_0\simeq4$.

Energy modulation of the electron beam in FEL oscillators has been utilized in some recent works [16–18]. Energy modulation has been used to generate very short optical pulses in the Stanford FEL oscillator [16,17]. In this FEL, the slippage distance is comparable with the electron pulse

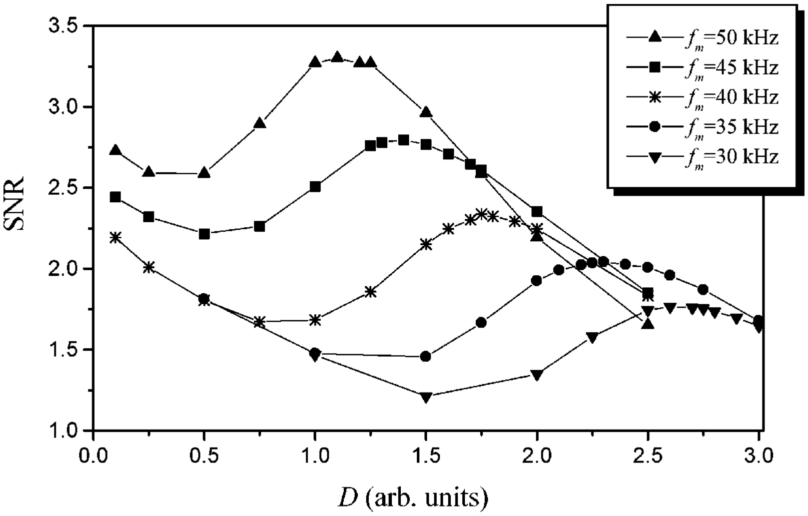

FIG. 5. Signal-to-noise ratio as a function of the input noise power for a modulation frequency $f_m = 50$ kHz (up triangles), $f_m = 45$ kHz (squares), $f_m = 40$ kHz (starts), $f_m = 35$ kHz (circles), and $f_m = 30$ kHz (down triangles). The parameters are $\overline{\nu_0} = 2.5$ and $\Delta\nu_0 = 0.05$.

length, thus longitudinal overlap effects (desynchronism) dominate its dynamics. The magnetic chicanes present in the Stanford FEL beam line are nonisochronous, i.e., higher-energy electrons pass through them quicker than lower-energy ones. Therefore, modulation of the beam energy is translated into modulation of the electron bunch repetition frequency and finally into a modulation of the desynchronism. It was found that the optical power oscillates at the same frequency as that of the modulation. Another work analyzed the effect of energy modulation for harmonic generation [18]. It was shown that this modulation produces an efficient cavity depletion and provides an output optical beam, chopped at the same frequency as the beam energy modulation. This phenomenon provides a technique to enhance the power radiated at higher harmonics.

IV. STOCHASTIC CASE

In this case we modulate the initial energy of the electron beam, and add a noise source. Then the initial electron velocity for the nth round trip can be written as

$$\nu_0^{(n)} = \overline{\nu_0} + \Delta\nu_0 \sin^2(2\pi f_m t_n) + \psi^{(n)}, \tag{4.1}$$

where f_m is the modulation frequency, $\Delta\nu_0$ is the amplitude of the periodic modulation, and $t_n = (2L_c/c)n$ is the electron-beam injection time, where we have considered a synchronism between this time and the round-trip time of the radiation in the cavity $(2L_c/c)$. Here $\psi^{(n)}$ denotes a zero-mean, Gaussian white noise with autocorrelation function

$$\langle \psi^{(n)} \psi^{(n')} \rangle = 2D\,\delta(n-n') \tag{4.2}$$

and intensity D. We take an average initial velocity $\overline{\nu_0} = 2.5$ and choose a small value of the modulation amplitude $(\Delta\nu_0 \leqslant 0.5)$ to avoid ν_0 reaching the threshold values ν_1, ν_2.

In order to observe stochastic resonance, we numerically integrate the model [Eqs. (2.8)–(2.13)]. We calculate the power spectrum from the time evolution of the optical power $|a|^2$. In particular, for a given set of parameters $(f_m, \Delta\nu_0, D)$, we calculate an averaged power spectrum over 200 realizations. These spectra show a principal peak above the noise background located at double the modulation frequency $(2f_m)$, which indicates the existence of a coherent movement. To analyze the dependence of the noise-induced coherent motion on the noise strength, we calculate the ratio of the peak of the signal and the broadband noise level at the signal frequency, which is the signal-to-noise ratio (SNR). Figure 2 shows the SNR versus D for a modulation frequency $f_m = 50$ kHz. Each curve corresponds to a different modulation amplitude. We observe a maximum in the SNR for all of the amplitudes, which indicates the presence of the stochastic resonance.

To understand how the stochastic resonance takes place in our system, we analyze the dynamics at $\Delta\nu_0 = 0.025$ in more detail. In this case, the value of the noise strength that gives a maximum SNR is $D \equiv D_r \simeq 1.3$. Figure 3 shows the power spectrum for different noise strengths. At $D \leqslant D_r$ and $D \geqslant D_r$, the peak of the signal at $2f_m$ is not appreciable. However, at $D \simeq D_r$ the signal peak is much larger than the noise level. Figure 4 shows the temporal evolution of the initial velocity, ν_0, for an individual realization using the same values of D as in Fig. 3. For $D < D_r$, the initial velocity does not reach the threshold values. However, at $D \simeq D_r$, ν_0 crosses the laser threshold, due to the noise source, roughly once every half-period. We can define an average time between threshold crossing induced by the noise as $T_N(D)$, which is a function of the noise strength. A statistical synchronization takes place when this time T_N is comparable with half the period $1/(2f_m)$ of the modulation signal. This yields the time-scale matching condition for stochastic resonance, i.e.,

$$T_N(D_r) = \frac{1}{4f_m}. \tag{4.3}$$

For higher noise levels $(D > D_r)$, ν_0 crosses the threshold many times but no correlation between the average time between threshold crossing and the signal period is observed.

Equation (4.3) implies that D_r shifts as we move the modulation frequency. To check this behavior, we plot in Fig. 5 the SNR at $\Delta\nu_0 = 0.05$ for different modulation frequencies. We can see in this figure that the maximum of the SNR shifts to larger *D* values as the frequency decreases roughly following a linear behavior.

V. CONCLUSIONS

We have studied theoretically the temporal dynamics of a free-electron laser oscillator for an electron bunch larger than the slippage length. In this type of FEL, a new electron pulse interacts with an optical pulse along the wiggler during each pass. In the regular operation, that means that with an initial energy or velocity of the electron-beam constant, a well-known steady-state optical power is found. This final value depends on the initial velocity.

The modulation of the initial energy of the electron beam presents different effects, as has been studied recently in some works [16–18]. It was shown that this phenomenon provides an output optical beam chopped at the same frequency of the beam energy modulation. In this paper we have consider a weak modulation and a noise source in the initial electron velocity. We have analyzed the signal-to-noise ratio versus noise power for different frequencies and amplitudes of the modulation, and in all the cases a maximum in the SNR has been found. This indicates the presence of a stochastic resonance. A crossing of the initial electron velocity threshold can explain the behavior of the system. This is evidence of stochastic resonance in free-electron lasers.

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