Anomalous scaling of a passive scalar in the presence of strong anisotropy

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Field theoretic renormalization group and the operator product expansion are applied to a model of a passive scalar quantity $\theta(t,\mathbf{x})$, advected by the Gaussian strongly anisotropic velocity field with the covariance $\propto \delta(t-t')|\mathbf{x}-\mathbf{x}'|^\varepsilon$. Inertial-range anomalous scaling behavior is established, and explicit asymptotic expressions for the structure functions $S_n(\mathbf{r}) \equiv \langle [\theta(t,\mathbf{x}+\mathbf{r}) - \theta(t,\mathbf{x})]^n \rangle$ are obtained. They are represented by superpositions of power laws; the corresponding anomalous exponents, which depend explicitly on the anisotropy parameters, are calculated to the first order in ε in any space dimension d. In the limit of vanishing anisotropy, the exponents are associated with tensor composite operators built of the scalar gradients, and exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank, the less is the dimension and, consequently, the more important is the contribution to the inertial-range behavior. The leading terms of the even (odd) structure functions are given by the scalar (vector) operators. For the finite anisotropy, the exponents cannot be associated with individual operators (which are essentially ''mixed'' in renormalization), but the aforementioned hierarchy survives for all the cases studied. The second-order structure function S_2 is studied in more detail using the renormalization group and zero-mode techniques; the corresponding exponents and amplitudes are calculated within the perturbation theories in ε, $1/d$, and in the anisotropy parameters. If the anisotropy of the velocity is strong enough, the skewness factor $S_3/S_2^{3/2}$ *increases* going down towards the depth of the inertial range; the higher-order odd ratios increase even if the anisotropy is weak.

I. INTRODUCTION

Recently, much effort has been invested to understand inertial-range anomalous scaling of a passive scalar. Both the natural and numerical experiments suggest that the violation of the classical Kolmogorov–Obukhov theory [1,2] is even more strongly pronounced for a passively advected scalar field than for the velocity field itself; see, e.g., [3–5] and literature cited therein. At the same time, the problem of passive advection appears to be easier tractable theoretically. The most remarkable progress has been achieved for the so-called Kraichnan's rapid-change model [6]: for the first time, the anomalous exponents have been calculated on the basis of a microscopic model and within regular expansions in formal small parameters; see, e.g., [7–13] and references therein.

Within the ''zero-mode approach,'' developed in [7–11], nontrivial anomalous exponents are related to the zero modes (unforced solutions) of the closed exact equations satisfied by the equal-time correlations. Within the approach based on the field theoretic renormalization group (RG) and operator product expansion (OPE), the anomalous scaling emerges as a consequence of the existence in the model of composite operators with *negative* critical dimensions, which determine the anomalous exponents [12–16].

Another important question recently addressed is the effects of large-scale anisotropy on inertial-range statistics of passively advected fields [10,11,15–20] and the velocity field itself [21,22]. According to the classical Kolmogorov–Obukhov theory, the anisotropy introduced at large scales by the forcing (boundary conditions, geometry of an obstacle, etc.) dies out when the energy is transferred down to the smaller scales owing to the cascade mechanism [1,2]. A

number of recent works confirms this picture for the *even* correlation functions, thus giving some quantitative support to the aforementioned hypothesis on the restored local isotropy of the inertial-range turbulence for the velocity and passive fields [15,17–22]. More precisely, the exponents describing the inertial-range scaling exhibit universality and hierarchy related to the degree of anisotropy, and the leading contribution to an even function is given by the exponent from the isotropic shell [15,16,18–20,22]. Nevertheless, the anisotropy survives in the inertial range and reveals itself in *odd* correlation functions, in disagreement with what was expected on the basis of the cascade ideas. The so-called skewness factor decreases down the scales much slower than expected [3,4,10,11], while the higher-order odd dimensionless ratios (hyperskewness, etc.) increase, thus signalling of persistent small-scale anisotropy [15,17,19]. The effect seems rather universal, being observed for the scalar [15] and vector [19] fields, advected by the Gaussian rapid-change velocity, and for the scalar advected by the two-dimensional Navier-Stokes velocity field [17].

In the present paper, we study the anomalous scaling behavior of a passive scalar advected by the time-decorrelated strongly anisotropic Gaussian velocity field. In contradistinction with the studies of [4,10,11,15], where the velocity was isotropic and the large-scale anisotropy was introduced by the imposed linear mean gradient, the uniaxial anisotropy in our model persists for all scales, leading to nonuniversality of the anomalous exponents through their dependence on the anisotropy parameters.

The aim of our paper is twofold.

First, we obtain explicit inertial-range expressions for the structure functions and correlation functions of the scalar gradients and calculate the corresponding anomalous expo-

nents to the first order of the ε expansion.

It was argued on phenomenological grounds in [23] that the anomalous exponents in the Gaussian model can depend on more details of the velocity statistics than the exponent ε. In the case at hand, the exponents become nonuniversal through the dependence on the parameters describing the anisotropy of the velocity field. Owing to the anisotropy of the velocity statistics, the composite operators of different ranks mix strongly in renormalization, and the corresponding anomalous exponents are given by the eigenvalues of the matrices which are neither diagonal nor triangular (in contrast with the case of large-scale anisotropy). In the language of the zero-mode technique this means that the $SO(d)$ decompositions of the correlation functions (employed, e.g., in Refs. [20]) do not lead to the diagonalization of the differential operators in the corresponding exact equations.

In spite of the mixing and nonuniversality, the hierarchy obeyed by the exponents for the vanishing anisotropy survives when the anisotropy becomes strong. Furthermore, the leading exponents of the even structure functions appear less sensitive to the anisotropy than the subleading ones. These facts are the main qualitative results of the paper.

Another scope of the paper is to give the detailed account of the RG approach and to compare the results obtained within both the RG and zero-mode techniques on the example of the second-order structure function.

The paper is organized as follows. In Sec. II, we give the precise formulation of the model and outline briefly the general strategy of the RG approach. In Sec. III, we give the field theoretic formulation of the model and derive the exact equations for the response function and pair correlator. In Sec. IV, we perform the ultraviolet (UV) renormalization of the model and derive the corresponding RG equations with exactly known RG functions (β functions and anomalous dimensions). These equations possess an infrared (IR) stable fixed point, which establishes the existence of IR scaling with exactly known critical dimensions of the basic fields and parameters of the model. In Sec. V, we discuss the solution of the RG equations for various correlation functions and determine the dependence of the latter on the UV scale. In Sec. VI, we discuss the renormalization of various composite operators. In particular, we derive the one-loop result for the critical dimensions of the tensor operators, constructed of the scalar gradients. In Sec. VII, we show that these dimensions play the part of the anomalous exponents, and present explicit inertial-range expressions for the structure functions and correlations of the scalar gradients. In Sec. VIII, we give the detailed comparison of the RG and zero-mode techniques on the example of the second-order structure function, compare the representation obtained using RG and OPE with the ordinary perturbation theory, and calculate the amplitude factors in the corresponding power laws. The results obtained are reviewed in Sec. IX.

II. DEFINITION OF THE MODEL. ANOMALOUS SCALING AND ''DANGEROUS'' COMPOSITE OPERATORS

The advection of a passive scalar field $\theta(x) \equiv \theta(t, \mathbf{x})$ in the rapid-change model is described by the stochastic equation

$$\nabla_t \theta = \nu_0 \Delta \theta + f, \quad \nabla_t \equiv \partial_t + v_i \partial_i, \quad (1)$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, ν_0 is the molecular diffusivity coefficient, Δ is the Laplace operator, $\mathbf{v}(x)$ is the transverse (owing to the incompressibility) velocity field, and $f \equiv f(x)$ is a Gaussian scalar noise with zero mean and correlator

$$\langle f(x) f(x') \rangle = \delta(t - t') C(r/\ell), \quad r \equiv |\mathbf{x} - \mathbf{x}'|. \quad (2)$$

Here ℓ is an integral scale related to the scalar noise and $C(r/\ell)$ is a function finite as $\ell \to \infty$. With no loss of generality, we take $C(0) = 1$ in what follows. The velocity $\mathbf{v}(x)$ obeys a Gaussian distribution with zero mean and correlator

$$\langle v_i(x) v_j(x') \rangle = D_0 \frac{\delta(t - t')}{(2\pi)^d} \int d\mathbf{k} \, T_{ij}(\mathbf{k})$$
$$\times (k^2 + m^2)^{-d/2 - \varepsilon/2} \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}')]. \quad (3)$$

In the isotropic case, the tensor quantity $T_{ij}(\mathbf{k})$ in Eq. (3) is taken to be the ordinary transverse projector, $T_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) \equiv \delta_{ij} - k_i k_j / k^2$, $k \equiv |\mathbf{k}|$, $D_0 > 0$ is an amplitude factor, $1/m$ is another integral scale, and d is the dimensionality of the $\mathbf{x}$ space; $0 < \varepsilon < 2$ is a parameter with the real (Kolmogorov) value $\varepsilon = 4/3$. The relations

$$D_0 / \nu_0 \equiv g_0 \equiv \Lambda^\varepsilon \quad (4)$$

define the coupling constant g_0 (i.e., the formal expansion parameter in the ordinary perturbation theory) and the characteristic UV momentum scale Λ. In what follows, we shall not distinguish the two IR scales, setting $m \simeq 1/\ell$.

The issue of interest is, in particular, the behavior of the equal-time structure functions

$$S_N(r) \equiv \langle [\theta(t, \mathbf{x}) - \theta(t, \mathbf{x}')]^N \rangle \quad (5)$$

in the inertial range, specified by the inequalities $1/\Lambda \ll r \ll 1/m \simeq \ell$. In the isotropic case, the odd functions S_{2n+1} vanish, while for S_{2n} dimensionality considerations give

$$S_{2n}(r) = \nu_0^{-n} r^{2n} R_{2n}(\Lambda r, mr), \quad (6)$$

where R_{2n} are some functions of dimensionless parameters. In principle, they can be calculated within the ordinary perturbation theory (i.e., as series in g_0), but this is not useful for studying inertial-range behavior: the coefficients are singular in the limits $\Lambda r \to \infty$ and/or $mr \to 0$, which compensate the smallness of g_0 (assumed in ordinary perturbation theory), and in order to find correct IR behavior we have to sum the entire series. The desired summation can be accomplished using the field theoretic renormalization group (RG) and operator product expansion (OPE); see Refs. [13–15].

The RG analysis consists of two main stages. On the first stage, the multiplicative renormalizability of the model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behavior of the

functions like Eq. (5) for $\Lambda r \gg 1$ and any fixed mr is given by IR stable fixed points of the RG equations and has the form

$$S_{2n}(r) = \nu_0^{-n} r^{2n} (\Lambda r)^{-\gamma_n} R_{2n}(mr), \quad \Lambda r \gg 1, \qquad (7)$$

with certain, as yet unknown, "scaling functions" $R_{2n}(mr)$. In the theory of critical phenomena [24,25], the quantity $\Delta[S_{2n}] \equiv -2n + \gamma_n$ is termed the "critical dimension," and the exponent γ_n, the difference between the critical dimension $\Delta[S_{2n}]$ and the "canonical dimension" $-2n$, is called the "anomalous dimension." In the case at hand, the latter has an extremely simple form: $\gamma_n = n\varepsilon$. Whatever be the functions $R_n(mr)$, the representation (7) implies the existence of a scaling (scale invariance) in the IR region ($\Lambda r \gg 1$, mr fixed) with definite critical dimensions of all "IR relevant" parameters, $\Delta[S_{2n}] = -2n + n\varepsilon$, $\Delta_r = -1$, $\Delta_m = 1$ and fixed "irrelevant" parameters ν_0 and Λ. This means that the structure functions (5) scale as $S_{2n} \rightarrow \lambda^{\Delta[S_{2n}]} S_{2n}$ upon the substitution $m \rightarrow \lambda^{\Delta_m} m$, $r \rightarrow \lambda^{\Delta_r} r$. In general, the exponent $\Delta[S_{2n}]$ is replaced by the critical dimension of the corresponding correlation function. This dimension is calculated as a series in ε, so that the exponent ε plays the part analogous to that played by the parameter $\varepsilon = 4 - d$ in the RG theory of critical phenomena, while ℓ is an analog of the correlation length $\ell \equiv r_c$; see [24,25].

On the second stage, the small mr behavior of the functions $R_{2n}(mr)$ is studied within the general representation (7) using the OPE. It shows that, in the limit $mr \rightarrow 0$, the functions $R_{2n}(mr)$ have the asymptotic forms

$$R_{2n}(mr) = \sum_F C_F(mr) (mr)^{\Delta[F]}, \qquad (8)$$

where C_F are coefficients regular in mr. In general, the summation is implied over all possible renormalized composite operators F allowed by the symmetry (more precisely, see Sec. VII); $\Delta[F]$ being their critical dimensions. Without loss of generality, it can be assumed that the expansion is made in irreducible tensors; in the isotropic case only scalar operators have nonvanishing mean values and contribute to the right hand side of Eq. (8).

The peculiarity of the models describing turbulence is the existence of the so-called "dangerous" composite operators with *negative* critical dimensions [13–15,26,27]. Their contributions into the OPE give rise to a singular behavior of the scaling functions for $mr \rightarrow 0$, and the leading term is given by the operator with minimal $\Delta[F]$. The leading contributions to S_{2n} are determined by scalar gradients $F_n = (\partial_i \theta \partial_i \theta)^n$ and have the form

$$S_{2n}(r) \simeq D_0^{-n} r^{n(2-\varepsilon)} (mr)^{\Delta_n}, \qquad (9)$$

where the critical dimensions Δ_n of the operators F_n are given by

$$\tilde{\Delta}_n = -2n(n-1)\varepsilon/(d+2) + O(\varepsilon^2)$$
$$= -2n(n-1)\varepsilon/d + O(1/d^2). \qquad (10)$$

The expression (9) agrees with the results obtained earlier in [8,9] using the zero-mode techniques; the $O(\varepsilon^2)$ contribution to Δ_n is obtained in [13].

In the theory of turbulence, the singular m dependence of correlation functions with the exponents nonlinear in n is referred to as anomalous scaling, and Δ_n themselves are termed the anomalous exponents; see, e.g., [1,2]. The above discussion shows that the anomalous exponents (in the sense of the turbulence theory) are not simply related to the anomalous or critical dimensions (in the sense of the theory of phase transitions) of the structure functions themselves; they are determined by the critical dimensions of certain composite operators entering into the corresponding OPE.

The OPE and the concept of dangerous operators in the stochastic hydrodynamics were introduced and investigated in detail in [28]; see also [26,27]. For the Kraichnan model, the relationship between the anomalous exponents and dimensions of composite operators was anticipated in [7–9,12] within certain phenomenological formulation of the OPE, the so-called "additive fusion rules," typical to the models with multifractal behavior [29]. A similar picture has been discussed in [30,31] in connection with the Burgers turbulence and growth phenomena.

In a number of papers, e.g., [4,10,11,15], the artificial stirring force in Eq. (1) was replaced by the term $(\mathbf{hv})$, where $\mathbf{h}$ is a constant vector that determines the distinguished direction and therefore introduces large-scale anisotropy. The anisotropy gives rise to nonvanishing odd functions S_{2n+1}. The critical dimensions of all composite operators remain unchanged, but the irreducible tensor operators acquire nonzero mean values and their contributions appear on the right hand side of Eq. (8); see [15]. This is easily understood in the language of the zero-mode approach: the noise f and the term $(\mathbf{hv})$ do not affect the differential operators in the equations satisfied by the equal-time correlation functions; the zero modes (homogeneous solutions) coincide in the two cases, but in the latter case the modes with nontrivial angular dependence should be taken into account.

The direct calculation to the order $O(\varepsilon)$ has shown that the leading exponent associated with a given rank contribution to Eq. (1) decreases monotonically with the rank [15]. Hence, the leading term of the inertial-range behavior of an even structure function is determined by the same exponent (10), while the exponents related to the tensor operators determine only subleading corrections. A similar hierarchy has been established recently in Ref. [18] (see also [19]) for the magnetic field advected passively by the rapid-change velocity in the presence of a constant background field, and in [22] within the context of the Navier–Stokes turbulence.

Below we shall take the velocity statistics to be anisotropic also at small scales. We replace the ordinary transverse projector in Eq. (3) with the general transverse structure that possesses the uniaxial anisotropy:

$$T_{ij}(\mathbf{k}) = a(\psi) P_{ij}(\mathbf{k}) + b(\psi) \tilde{n}_i(\mathbf{k}) \tilde{n}_j(\mathbf{k}). \qquad (11)$$

Here the unit vector $\mathbf{n}$ determines the distinguished direction ($\mathbf{n}^2 = 1$), $\tilde{n}_i(\mathbf{k}) \equiv P_{ij}(\mathbf{k}) n_j$, and ψ is the angle between the vectors $\mathbf{k}$ and $\mathbf{n}$, so that $(\mathbf{nk}) = k \cos \psi$ [note that $(\tilde{\mathbf{n}}\mathbf{k}) = 0$].

The scalar functions can be decomposed into the Gegenbauer polynomials (the d-dimensional generalization of the Legendre polynomials, see Ref. [39]):

$$a(\psi) = \sum_{l=0}^{\infty} a_l P_{2l}(\cos\psi), \quad b(\psi) = \sum_{l=0}^{\infty} b_l P_{2l}(\cos\psi) \tag{12}$$

(we shall see later that odd polynomials do not affect the scaling behavior). The positivity of the correlator (3) leads to the conditions

$$a(\psi) > 0, \quad a(\psi) + b(\psi)\sin^2\psi > 0. \tag{13}$$

In practical calculations, we shall mostly confine ourselves with the special case

$$T_{ij}(\mathbf{k}) = (1 + \rho_1 \cos^2\psi)P_{ij}(\mathbf{k}) + \rho_2 \tilde{n}_i(\mathbf{k})\tilde{n}_j(\mathbf{k}). \tag{14}$$

Then the inequalities (13) reduce to $\rho_{1,2} > -1$. We shall see that this case represents nicely all the main features of the general model (11).

We note that the quantities (11), (14) possess the symmetry $\mathbf{n} \to -\mathbf{n}$. The anisotropy makes it possible to introduce mixed correlator $\langle \mathbf{v}f \rangle \propto \mathbf{n}\delta(t-t')C'(r/\ell)$ with some function $C'(r/\ell)$ analogous to $C(r/\ell)$ from Eq. (2). This violates the evenness in $\mathbf{n}$ and gives rise to nonvanishing odd functions S_{2n+1}. However, this leads to no serious alterations in the RG analysis; we shall discuss this case in Sec. VII, and for now we assume $\langle \mathbf{v}f \rangle = 0$.

In a number of papers, e.g., [32–36], the RG techniques were applied to the anisotropically driven Navier–Stokes equation, including passive advection and magnetic turbulence, with the expression (14) entering into the stirring force correlator. The detailed account can be found in Ref. [27], where some errors of the previous treatments are also corrected. However, these studies have up to now been limited to the first stage, i.e., investigation of the existence and stability of the fixed points and calculation of the critical dimensions of basic quantities. Calculation of the anomalous exponents in those models remains an open problem.

III. FIELD THEORETIC FORMULATION AND THE DYSON-WYLD EQUATIONS

The stochastic problem (1)–(3) is equivalent to the field theoretic model of the set of three fields $\Phi \equiv \{\theta', \theta, \mathbf{v}\}$ with action functional

$$S(\Phi) = \theta' D_\theta \theta'/2 + \theta'[-\partial_t - (\mathbf{v}\partial) + \nu_0\Delta]\theta - \mathbf{v}D_v^{-1}\mathbf{v}/2. \tag{15}$$

The first four terms in Eq. (15) represent the Martin–Siggia–Rose-type action for the stochastic problem (1), (2) at fixed $\mathbf{v}$ (see, e.g., [24,25]), and the last term represents the Gaussian averaging over $\mathbf{v}$. Here D_θ and D_v are the correlators (2) and (3), respectively, the required integrations over $x = (t, \mathbf{x})$ and summations over the vector indices are implied.

The formulation (15) means that statistical averages of random quantities in stochastic problem (1)–(3) can be rep-

resented as functional averages with the weight $\exp S(\Phi)$, so that the generating functionals of total $[G(A)]$ and connected $[W(A)]$ Green functions of the problem are given by the functional integral

$$G(A) = \exp W(A) = \int \mathcal{D}\Phi \exp[S(\Phi) + A\Phi] \tag{16}$$

with arbitrary sources $A \equiv A^{\theta'}, A^{\theta}, A^{\mathbf{v}}$ in the linear form

$$A\Phi \equiv \int dx[A^{\theta'}(x)\theta'(x) + A^{\theta}(x)\theta(x) + A_i^{\mathbf{v}}(x)v_i(x)].$$

The model (15) corresponds to a standard Feynman diagrammatic technique with the triple vertex $-\theta'(\mathbf{v}\partial)\theta \equiv \theta' V_j v_j \theta$ with vertex factor (in the momentum-frequency representation)

$$V_j = -ik_j, \tag{17}$$

where $\mathbf{k}$ is the momentum flowing into the vertex via the field θ. The bare propagators in the momentum-frequency representation have the form

$$\langle \theta\theta' \rangle_0 = \langle \theta'\theta \rangle_0^* = (-i\omega + \nu_0 k^2)^{-1},$$
$$\langle \theta\theta \rangle_0 = C(k)(\omega^2 + \nu_0^2 k^4)^{-1}, \quad \langle \theta'\theta' \rangle_0 = 0, \tag{18}$$

where $C(k)$ is the Fourier transform of the function $C(r/\ell)$ from Eq. (2) and the bare propagator $\langle \mathbf{vv} \rangle_0 \equiv \langle \mathbf{vv} \rangle$ is given by Eq. (3) with the transverse projector from Eqs. (11) or (14).

The pair correlation functions $\langle \Phi\Phi \rangle$ of the multicomponent field $\Phi \equiv \{\theta', \theta, \mathbf{v}\}$ satisfy standard Dyson equation, which in the component notation reduces to the system of two equations, cf. [1]

$$G^{-1}(\omega, \mathbf{k}) = -i\omega + \nu_0 k^2 - \Sigma_{\theta'\theta}(\omega, \mathbf{k}), \tag{19a}$$

$$D(\omega, \mathbf{k}) = |G(\omega, \mathbf{k})|^2 [C(k) + \Sigma_{\theta'\theta'}(\omega, \mathbf{k})], \tag{19b}$$

where $G(\omega, \mathbf{k}) \equiv \langle \theta\theta' \rangle$ and $D(\omega, \mathbf{k}) \equiv \langle \theta\theta \rangle$ are the exact response function and pair correlator, respectively, and $\Sigma_{\theta'\theta}$, $\Sigma_{\theta'\theta'}$ are self-energy operators represented by the corresponding 1-irreducible diagrams; all the other functions $\Sigma_{\Phi\Phi}$ in the model (15) vanish identically.

The feature characteristic of the models like Eq. (15) is that all the skeleton multiloop diagrams entering into the self-energy operators $\Sigma_{\theta'\theta}$, $\Sigma_{\theta'\theta'}$ contain effectively closed circuits of retarded propagators $\langle \theta\theta' \rangle$ (it is crucial here that the propagator $\langle \mathbf{vv} \rangle_0$ in Eq. (3) is proportional to the δ function in time) and therefore vanish.

Therefore the self-energy operators in Eq. (19) are given by the one-loop approximation exactly and have the form

$$\Sigma_{\theta'\theta}(\omega,\mathbf{k}) = \quad \overset{\frown}{\longleftarrow} \quad = -D_0\, k_i k_j \int \frac{d\omega'}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{T_{ij}(\mathbf{q})}{(q^2+m^2)^{d/2+\varepsilon/2}} \, G(\omega',\mathbf{q}'), \tag{20a}$$

$$\Sigma_{\theta'\theta'}(\omega,\mathbf{k}) = \quad \overset{\frown}{\longrightarrow} \quad = D_0\, k_i k_j \int \frac{d\omega'}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{T_{ij}(\mathbf{q})}{(q^2+m^2)^{d/2+\varepsilon/2}} \, D(\omega',\mathbf{q}'), \tag{20b}$$

where $\mathbf{q}' \equiv \mathbf{k} - \mathbf{q}$. The solid lines in the diagrams denote the exact propagators $\langle\theta\theta'\rangle$ and $\langle\theta\theta\rangle$, the ends with a slash correspond to the field θ', and the ends without a slash correspond to θ; the dashed lines denote the bare propagator (3); the vertices correspond to the factor (17).

The integrations over ω' in the right hand sides of Eqs. (20) give the equal-time response function $G(\mathbf{q}) = (1/2\pi)\int d\omega'\, G(\omega',\mathbf{q})$ and the equal-time pair correlator $D(\mathbf{q}) = (1/2\pi)\int d\omega' D(\omega',\mathbf{q})$; note that both the self-energy operators are in fact independent of ω. The only contribution to $G(\mathbf{q})$ comes from the bare propagator $\langle\theta\theta'\rangle_0$ from Eq. (18), which in the t representation is discontinuous at coincident times. Since the correlator (3), which enters into the one-loop diagram for $\Sigma_{\theta'\theta}$, is symmetric in t and t', the response function must be defined at $t = t'$ by half the sum of the limits. This is equivalent to the convention $G(\mathbf{q}) = (1/2\pi)\int d\omega'(-i\omega' + \nu_0 q^2)^{-1} = 1/2$ and gives

$$\Sigma_{\theta'\theta}(\omega,\mathbf{k}) = -\frac{D_0 k_i k_j}{2} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{T_{ij}(\mathbf{q})}{(q^2+m^2)^{d/2+\varepsilon/2}}. \tag{21}$$

The integration of Eq. (19b) over the frequency ω gives a closed equation for the equal-time correlator. Using Eq. (21) it can be written in the form

$$2\nu_0 k^2 D(\mathbf{k}) = C(k) + D_0 k_i k_j \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{T_{ij}(\mathbf{q})}{(q^2+m^2)^{d/2+\varepsilon/2}}$$
$$\times [D(\mathbf{q}') - D(\mathbf{k})]. \tag{22}$$

Equation (22) can also be rewritten as a partial differential equation for the pair correlator in the coordinate representation, $D(\mathbf{r}) \equiv \langle\theta(t,\mathbf{x})\theta(t,\mathbf{x}+\mathbf{r})\rangle$ (we use the same notation D for the coordinate function and its Fourier transform). Noting that the integral in Eq. (22) involves convolutions of the functions $D(\mathbf{k})$ and $D_0 T_{ij}(\mathbf{q})/(q^2+m^2)^{d/2+\varepsilon/2}$, and replacing the momenta by the corresponding derivatives, $ik_i \to \partial_i$ and so on, we obtain:

$$2\nu_0 \partial^2 D(\mathbf{r}) + C(r/\ell) + D_0 S_{ij}(\mathbf{r})\partial_i\partial_j D(\mathbf{r}) = 0, \tag{23}$$

where the "effective eddy diffusivity" is given by

$$S_{ij}(\mathbf{r}) \equiv \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{T_{ij}(\mathbf{q})}{(q^2+m^2)^{d/2+\varepsilon/2}} [1 - \exp(i\mathbf{q}\cdot\mathbf{r})] \tag{24}$$

(note that $\partial_i S_{ij} = 0$). For $0 < \varepsilon < 2$, Eqs. (22)–(24) allow for the limit $m \to 0$: the possible IR divergence of the integrals at $\mathbf{q} = 0$ is suppressed by the vanishing of the expressions in the square brackets. For the isotropic case (i.e., after the substitution $T_{ij} \to P_{ij}$) Eq. (23) coincides (up to the notation) with the well-known equation for the equal-time pair correlator in the model [6].

Equation (21) will be used in the next section for the exact calculation of the RG functions; solution of Eq. (23) will be discussed in Sec. VIII in detail.

IV. RENORMALIZATION, RG FUNCTIONS, AND RG EQUATIONS

The analysis of the UV divergences is based on the analysis of canonical dimensions. Dynamical models of the type Eq. (15), in contrast to static models, have two scales, so that the canonical dimension of some quantity F (a field or a parameter in the action functional) is described by two numbers, the momentum dimension d_F^k and the frequency dimension d_F^ω. They are determined so that $[F] \sim [L]^{-d_F^k}[T]^{-d_F^\omega}$, where L is the length scale and T is the time scale. The dimensions are found from the obvious normalization conditions $d_k^k = -d_\mathbf{x}^k = 1$, $d_k^\omega = d_\mathbf{x}^\omega = 0$, $d_\omega^k = d_t^k = 0$, $d_\omega^\omega = -d_t^\omega = 1$, and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension $d_F = d_F^k + 2d_F^\omega$ (in the free theory, $\partial_t \propto \Delta$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems.

The dimensions for the model (15) are given in Table I, including the parameters which will be introduced later on. From Table I it follows that the model is logarithmic (the

TABLE I. Canonical dimensions of the fields and parameters in the model (26).

F	θ	θ'	$\mathbf{v}$	ν, ν_0	m, μ, Λ	g_0	g, α, α_0
d_F^k	0	d	-1	-2	1	ε	0
d_F^ω	$-1/2$	$1/2$	1	1	0	0	0
d_F	-1	$d+1$	1	0	1	ε	0

coupling constant g_0 is dimensionless) at $\varepsilon=0$, so that the UV divergences have the form of the poles in ε in the Green functions.

The total canonical dimension of an arbitrary 1-irreducible Green function $\Gamma=\langle\Phi\cdots\Phi\rangle_{1-\mathrm{ir}}$ is given by the relation

$$d_\Gamma=d_\Gamma^k+2d_\Gamma^\omega=d+2-N_\Phi d_\Phi,\tag{25}$$

where $N_\Phi=\{N_{\theta'},N_\theta,N_\mathbf{v}\}$ are the numbers of corresponding fields entering into the function Γ, and the summation over all types of the fields is implied. The total dimension d_Γ is the formal index of the UV divergence. Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which d_Γ is a non-negative integer.

Analysis of the divergences should be based on the following auxiliary considerations.

(i) From the explicit form of the vertex and bare propagators in the model (15) it follows that $N_{\theta'}-N_\theta=2N_0$ for any 1-irreducible Green function, where $N_0\geqslant0$ is the total number of bare propagators $\langle\theta\theta\rangle_0$ entering into the function (obviously, no diagrams with $N_0<0$ can be constructed). Therefore, the difference $N_{\theta'}-N_\theta$ is an even non-negative integer for any nonvanishing function.

(ii) If for some reason a number of external momenta occurs as an overall factor in all the diagrams of a given Green function, the real index of divergence d_Γ' is smaller than d_Γ by the corresponding number (the Green function requires counterterms only if d_Γ' is a non-negative integer). In the model (15), the derivative $\boldsymbol{\partial}$ at the vertex $\theta'(\mathbf{v}\boldsymbol{\partial})\theta$ can be moved onto the field θ' by virtue of the transversality of the field $\mathbf{v}$. Therefore, in any 1-irreducible diagram it is always possible to move the derivative onto any of the external "tails" θ or θ', which decreases the real index of divergence: $d_\Gamma'=d_\Gamma-N_\theta-N_{\theta'}$. The fields θ, θ' enter into the counterterms only in the form of derivatives $\partial\theta$, $\partial\theta'$.

From the dimensions in Table I we find $d_\Gamma=d+2-N_\mathbf{v}+N_\theta-(d+1)N_{\theta'}$ and $d_\Gamma'=(d+2)(1-N_{\theta'})-N_\mathbf{v}$. It then follows that for any d, superficial divergences can exist only in the 1-irreducible functions $\langle\theta'\theta\cdots\theta\rangle_{1-\mathrm{ir}}$ with $N_{\theta'}=1$ and arbitrary value of N_θ, for which $d_\Gamma=2$, $d_\Gamma'=0$. However, all the functions with $N_\theta>N_{\theta'}$ vanish (see above) and obviously do not require counterterms. We are left with the only superficially divergent function $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$; the corresponding counterterms must contain two symbols ∂ and in the isotropic case reduce to the only structure $\theta'\Delta\theta$. In the presence of anisotropy, it is necessary to also introduce new counterterm of the form $\theta'(\mathbf{n}\boldsymbol{\partial})^2\theta$, which is absent in the unrenormalized action functional (15). Therefore, the model (15) in its original formulation is not multiplicatively renormalizable, and in order to use the standard RG techniques it is necessary to extend the model by adding the new contribution to the unrenormalized action:

$$S(\Phi)=\theta'D_\theta\theta'/2+\theta'[-\partial_t-(\mathbf{v}\boldsymbol{\partial})+\nu_0\Delta+\alpha_0\nu_0(\mathbf{n}\boldsymbol{\partial})^2]\theta$$
$$-\mathbf{v}D_v^{-1}\mathbf{v}/2.\tag{26}$$

Here α_0 is a new dimensionless unrenormalized parameter. The stability of the system implies the positivity of the total viscous contribution $\nu_0k^2+\alpha_0\nu_0(\mathbf{nk})^2$, which leads to the inequality $\alpha_0>-1$. Its real ("physical") value is zero, but this fact does not hinder the use of the RG techniques, in which it is first assumed to be arbitrary, and the equality $\alpha_0=0$ is imposed as the initial condition in solving the equations for invariant variables (see Sec. V). Below we shall see that the zero value of α_0 corresponds to certain nonzero value of its renormalized analog, which can be found explicitly.

For the action (26), the nontrivial bare propagators in Eq. (18) are replaced with

$$\langle\theta\theta'\rangle_0=\langle\theta'\theta\rangle_0^*=(-i\omega+\nu_0k^2+\alpha_0\nu_0(\mathbf{nk})^2)^{-1},$$

$$\langle\theta\theta\rangle_0=\frac{C(k)}{|-i\omega+\nu_0k^2+\alpha_0\nu_0(\mathbf{nk})^2|^2}.\tag{27}$$

After the extension, the model has become multiplicatively renormalizable: inclusion of the counterterms is reproduced by the inclusion of two independent renormalization constants $Z_{1,2}$ as coefficients in front of the counterterms. This leads to the renormalized action of the form

$$S_R(\Phi)=\theta'D_\theta\theta'/2+\theta'[-\partial_t-(\mathbf{v}\boldsymbol{\partial})+\nu Z_1\Delta+\alpha\nu Z_2(\mathbf{n}\boldsymbol{\partial})^2]\theta$$
$$-\mathbf{v}D_v^{-1}\mathbf{v}/2,\tag{28}$$

or, equivalently, to the multiplicative renormalization of the parameters ν_0, g_0, and α_0 in the action functional (26):

$$\nu_0=\nu Z_\nu,\quad g_0=g\mu^\varepsilon Z_g,\quad \alpha_0=\alpha Z_\alpha.\tag{29}$$

Here μ is the reference mass in the minimal subtraction (MS) scheme, which we always use in what follows, ν, g, and α are renormalized counterparts of the corresponding bare parameters, and the renormalization constants $Z=Z(g,\alpha;\rho_{1,2},\varepsilon,d)$ depend only on the dimensionless parameters. The correlator (3) in Eq. (28) is expressed in renormalized variables using Eqs. (29). The comparison of Eqs. (26), (28), and (29) leads to the relations

$$Z_1=Z_\nu,\quad Z_2=Z_\alpha Z_\nu,\quad Z_g=Z_\nu^{-1}.\tag{30}$$

The last relation in Eq. (30) results from the absence of renormalization of the contribution with D_0, so that $D_0\equiv g_0\nu_0=g\mu^\varepsilon\nu$; see Eq. (4). No renormalization of the fields, anisotropy parameters and the "mass" m is required, i.e., $Z_\Phi=1$ for all Φ, and so on.

The relation $S(\Phi,e_0)=S_R(\Phi,e,\mu)$ (where $e_0\equiv\{\nu_0,g_0,\alpha_0\}$ is the complete set of bare parameters, and $e\equiv\{\nu,g,\alpha\}$ is the set of their renormalized counterparts) for the generating functional $W(A)$ in Eq. (16) yields $W(A,e_0)=W_R(A,e,\mu)$. We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ at fixed e_0 and operate on both sides of this equation with it. This gives the basic RG differential equation

$$\mathcal{D}_{RG}W_R(A,e,\mu)=0.\tag{31}$$

Here $\mathcal{D}_{RG}$ is the operation $\tilde{\mathcal{D}}_\mu$ expressed in the renormalized variables

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta_g \partial_g + \beta_\alpha \partial_\alpha - \gamma_\nu \mathcal{D}_\nu, \tag{32}$$

where we have written $\mathcal{D}_x \equiv x \partial_x$ for any variable x, and the RG functions (the β functions and the anomalous dimension γ) are defined as

$$\gamma_F(g, \alpha) \equiv \tilde{\mathcal{D}}_\mu \ln Z_F \tag{33a}$$

for any renormalization constant Z_F and

$$\beta_g(g, \alpha) \equiv \tilde{\mathcal{D}}_\mu g = g(-\varepsilon - \gamma_g) = g(-\varepsilon + \gamma_\nu) = g(-\varepsilon + \gamma_1), \tag{33b}$$

$$\beta_\alpha(g, \alpha) \equiv \tilde{\mathcal{D}}_\mu \alpha = -\alpha \gamma_\alpha = \alpha(\gamma_1 - \gamma_2). \tag{33c}$$

The relation between β_g and γ_ν in Eq. (33b) results from the definitions and the last relation in Eq. (30).

Now let us turn to the calculation of the renormalization constants $Z_{1,2}$ in the MS scheme. They are determined by the requirement that the Green function $G = \langle \theta' \theta \rangle$, when expressed in renormalized variables, be UV finite, i.e., have no poles in ε. It satisfies the Dyson equation of the form

$$G^{-1}(\omega, \mathbf{k}) = -i\omega + \nu_0 k^2 + \alpha_0 \nu_0 (\mathbf{nk})^2 - \Sigma_{\theta' \theta}(\omega, \mathbf{k}), \tag{34}$$

which is obtained from Eq. (19a) with the obvious substitution $\nu_0 k^2 \to \nu_0 k^2 + \alpha_0 \nu_0 (\mathbf{nk})^2$; the self-energy operator $\Sigma_{\theta' \theta}$ is given by the same expression (21). With T_{ij} from Eq. (14) the integration in Eq. (21) yields

$$\Sigma_{\theta' \theta}(k) = -\frac{D_0 J(m)}{2d(d+2)}[(d-1)(d+2)k^2 + \rho_1(d+1)k^2$$
$$+ \rho_2 k^2 - 2\rho_1(\mathbf{nk})^2 + (d^2-2)\rho_2(\mathbf{nk})^2], \tag{35}$$

where we have written

$$J(m) \equiv \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{(q^2+m^2)^{d/2+\varepsilon/2}} = \frac{\Gamma(\varepsilon/2)m^{-\varepsilon}}{(4\pi)^{d/2}\Gamma(d/2+\varepsilon/2)}. \tag{36}$$

In deriving Eqs. (35), (36), we used the relations

$$\int d\mathbf{q} \, f(q) \frac{q_i q_j}{q^2} = \frac{\delta_{ij}}{d} \int d\mathbf{q} \, f(q),$$

$$\int d\mathbf{q} \, f(q) \frac{q_i q_j q_l q_p}{q^4} = \frac{\delta_{ij}\delta_{lp} + \delta_{ip}\delta_{lj} + \delta_{il}\delta_{pj}}{d(d+2)} \int d\mathbf{q} \, f(q). \tag{37}$$

The renormalization constants $Z_{1,2}$ are found from the requirement that the function $\langle \theta' \theta \rangle$ expressed in renormalized variables be finite for $\varepsilon \to 0$. This requirement determines $Z_{1,2}$ up to an UV finite contribution; the latter is fixed by the choice of a renormalization scheme. In the MS scheme all renormalization constants have the form ''1 + only poles in

ε.'' We substitute Eqs. (29), (30) into Eqs. (34), (35) and choose $Z_{1,2}$ to cancel the pole in ε in the integral $J(m)$. This gives

$$Z_1 = 1 - \frac{gC_d}{2d(d+2)\varepsilon}[(d-1)(d+2) + \rho_1(d+1) + \rho_2], \tag{38}$$

$$Z_2 = 1 - \frac{gC_d}{2d(d+2)\alpha\varepsilon}[-2\rho_1 + \rho_2(d^2-2)],$$

where we have written $C_d \equiv S_d/(2\pi^d)$ and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in d-dimensional space.

For the anomalous dimension $\gamma_1(g) \equiv \tilde{\mathcal{D}}_\mu \ln Z_1 = \beta_g \partial_g \ln Z_1$ from the relations (33b) and (38) one obtains (note that Z_1 is independent of α)

$$\gamma_1(g) = \frac{-\varepsilon \mathcal{D}_g \ln Z_\nu}{1 - \mathcal{D}_g \ln Z_\nu}$$

$$= \frac{gC_d}{2d(d+2)}[(d-1)(d+2) + \rho_1(d+1) + \rho_2], \tag{39}$$

and for $\gamma_2(g, \alpha) \equiv \tilde{\mathcal{D}}_\mu \ln Z_2 = (\beta_g \partial_g + \beta_\alpha \partial_\alpha)\ln Z_2$ one has

$$\gamma_2(g, \alpha) = \frac{[(-\varepsilon + \gamma_1)\mathcal{D}_g + \gamma_1)\mathcal{D}_\alpha]\ln Z_2}{1 + \mathcal{D}_\alpha \ln Z_2} = \frac{-\varepsilon \mathcal{D}_g \ln Z_2}{1 + \mathcal{D}_\alpha \ln Z_2}$$

$$= \frac{gC_d}{2d(d+2)\alpha}[-2\rho_1 + \rho_2(d^2-2)] \tag{40}$$

[note that $(\mathcal{D}_g + \mathcal{D}_\alpha)\ln Z_2 = 0$]. The cancellation of the poles in ε in Eqs. (39), (40) is a consequence of the UV finiteness of the anomalous dimensions γ_F; their independence of ε is a property of the MS scheme. Note also that the expressions (38)–(40) are exact, i.e., have no corrections of order g^2 and higher; this is a consequence of the fact that the one-loop approximation (21) for the self-energy operator is exact.

The fixed points of the RG equations are determined from the requirement that all the beta functions of the model vanish. In our model the coordinates g_*, α_* of the fixed points are found from the equations

$$\beta_g(g_*, \alpha_*) = \beta_\alpha(g_*, \alpha_*) = 0, \tag{41}$$

with the beta functions from Eqs. (33). The type of the fixed point is determined by the eigenvalues of the matrix $\Omega = \{\Omega_{ik} = \partial \beta_i / \partial g_k\}$, where β_i denotes the full set of the beta functions and g_k is the full set of charges. The IR asymptotic behavior is governed by the IR stable fixed points, i.e., those for which all the eigenvalues are positive. From the explicit expressions (39), (40) it then follows that the RG equations of the model have the only IR stable fixed point with the coordinates

$$g_* C_d = \frac{2d(d+2)\varepsilon}{(d-1)(d+2) + \rho_1(d+1) + \rho_2},$$

$$\alpha_* = \frac{-2\rho_1 + \rho_2(d^2-2)}{(d-1)(d+2)+\rho_1(d+1)+\rho_2}. \qquad (42)$$

For this point, both the eigenvalues of the matrix Ω are equal to ε; the values $\gamma_1^* = \gamma_2^* = \gamma_\nu^* = \varepsilon$ are also found exactly from Eqs. (33) [here and below, $\gamma_F^* \equiv \gamma_F(g_*, \alpha_*)$]. The fixed point (42) is degenerate in the sense that its coordinates depend continuously on the anisotropy parameters $\rho_{1,2}$. Formally, $\rho_{1,2}$ can be treated as the additional coupling constants. The corresponding beta functions $\beta_{1,2} \equiv \widetilde{\mathcal{D}}_\mu \rho_{1,2}$ vanish identically owing to the fact that $\rho_{1,2}$ are not renormalized. Therefore the equations $\beta_{1,2}=0$ give no additional constraints on the values of the parameters g, α at the fixed point.

V. SOLUTION OF THE RG EQUATIONS. INVARIANT VARIABLES

The solution of the RG equations is discussed in Refs. [24–28] in detail; below we confine ourselves to only the information we need.

Consider the solution of the RG equation on the example of the even different-time structure functions

$$S_{2n}(\mathbf{r},\tau) \equiv \langle [\theta(t,\mathbf{x}) - \theta(t',\mathbf{x}')]^{2n}\rangle, \quad \mathbf{r} \equiv \mathbf{x}-\mathbf{x}', \quad \tau \equiv t-t'. \qquad (43)$$

It satisfies the RG equation $\mathcal{D}_{RG}S_{2n}=0$ with the operator $\mathcal{D}_{RG}$ from Eq. (32) [this fact does not follow automatically from Eq. (31) and will be justified below in Sec. VI].

In renormalized variables, dimensionality considerations give

$$S_{2n}(\mathbf{r},\tau) = \nu^{-n}r^{2n}R_{2n}(\mu r, \tau\nu/r^2, mr, g, \alpha), \qquad (44)$$

where R_{2n} is a function of completely dimensionless arguments (the dependence on d, ε, $\rho_{1,2}$ and the angle between the vectors $\mathbf{r}$ and $\mathbf{n}$ is also implied). From the RG equation the identical representation follows,

$$S_{2n}(\mathbf{r},\tau) = (\bar{\nu})^{-n}r^{2n}R_{2n}(1,\tau\bar{\nu}/r^2, mr, \bar{g}, \bar{\alpha}), \qquad (45)$$

where the invariant variables $\bar{e} = \bar{e}(\mu r, e)$ satisfy the equation $\mathcal{D}_{RG}\bar{e}=0$ and the normalization conditions $\bar{e}=e$ at $\mu r=1$ (we recall that $e \equiv \{\nu, g, \alpha, m\}$ denotes the full set of renormalized parameters). The identity $\bar{m} \equiv m$ is a consequence of the absence of $\mathcal{D}_m$ in the operator $\mathcal{D}_{RG}$ owing to the fact that m is not renormalized. Equation (45) is valid because both sides of it satisfy the RG equation and coincide for $\mu r=1$ owing to the normalization of the invariant variables. The relation between the bare and invariant variables has the form

$$\nu_0 = \bar{\nu}Z_\nu(\bar{g}), \quad g_0 = \bar{g}r^{-\varepsilon}Z_g(\bar{g}), \quad \alpha_0 = \bar{\alpha}Z_\alpha(\bar{g},\bar{\alpha}). \qquad (46)$$

Equation (46) determines implicitly the invariant variables as functions of the bare parameters; it is valid because both

sides of it satisfy the RG equation, and because Eq. (46) at $\mu r=1$ coincides with Eq. (29) owing to the normalization conditions.

In general, the large μr behavior of the invariant variables is governed by the IR stable fixed point: $\bar{g} \to g_*$, $\bar{\alpha} \to \alpha_*$ for $\mu r \to \infty$. However, in multicharge problems one has to take into account that even when the IR point exists, not every phase trajectory [i.e., solution of Eq. (46)] reaches it in the limit $\mu r \to \infty$. It may first pass outside the natural region of stability (in our case, $g>0$, $\alpha>-1$) or go to infinity within this region. Fortunately, in our case the constants Z_F entering into Eq. (46) are known exactly from Eqs. (38), and it is readily checked that the RG flow indeed reaches the fixed point (42) for any initial conditions $g_0 >0$, $\alpha_0 >-1$, including the physical case $\alpha_0=0$. Furthermore, the large μr behavior of the invariant viscosity $\bar{\nu}$ is also found explicitly from Eq. (46) and the last relation in Eq. (30): $\bar{\nu} = D_0 r^\varepsilon/\bar{g} \to D_0 r^\varepsilon/g_*$ (we recall that $D_0 = g_0\nu_0$). Then for $\mu r \to \infty$ and any fixed mr we obtain

$$S_{2n}(\mathbf{r},\tau) = D_0^{-n}r^{n(2-\varepsilon)}g_*^n\xi_{2n}(\tau D_0 r^{\Delta_t}, mr), \qquad (47)$$

where

$$\xi_{2n}(D_0\tau r^{\Delta_t}, mr) \equiv R_{2n}(1, D_0\tau r^{\Delta_t}, mr, g_*, \alpha_*), \qquad (48)$$

and $\Delta_t \equiv -2 + \gamma_\nu^* = -2 + \varepsilon$ is the critical dimension of time. The dependence of the scaling function ξ_{2n} on its arguments is *not* determined by the RG equation (31) itself. For the equal-time structure function (5), the first argument of ξ_{2n} in the representation (48) is absent:

$$S_{2n}(\mathbf{r}) = D_0^{-n}r^{n(2-\varepsilon)}g_*^n\xi_{2n}(mr), \qquad (49)$$

where the definition of ξ_{2n} is obvious from Eq. (48). It is noteworthy that Eqs. (47)–(49) prove the independence of the structure functions in the IR range (large μr and any mr) of the viscosity coefficient or, equivalently, of the UV scale: the parameters g_0 and ν_0 enter into Eq. (47) only in the form of the combination $D_0 = g_0\nu_0$. A similar property was established in Ref. [37] for the stirred Navier–Stokes equation and is related to the second Kolmogorov hypothesis; see also the discussion in [27,28].

Now let us turn to the general case. Let $F(r,\tau)$ be some multiplicatively renormalized quantity (say, a correlation function involving composite operators), i.e., $F=Z_F F_R$ with certain renormalization constant Z_F. It satisfies the RG equation of the form $[\mathcal{D}_{RG}+\gamma_F]F_R=0$ with γ_F from Eq. (33a). Dimensionality considerations give

$$F_R(r,\tau) = \nu^{d_F^\omega}r^{-d_F}R_F(\mu r, \tau\nu/r^2, mr, g, \alpha), \qquad (50)$$

where d_F^ω and d_F are the frequency and total canonical dimensions of F (see Sec. IV) and R_F is a function of dimensionless arguments. The analog of Eq. (45) has the form

$$F(r,\tau)=Z_F(g,\alpha)F_R$$
$$=Z_F(\bar{g},\bar{\alpha})(\bar{\nu})^{d_F^{\omega}}r^{-d_F}R_F(1,\tau\bar{\nu}/r^2,mr,\bar{g},\bar{\alpha}).$$
$$(51)$$

In the large μr limit, one has $Z_F(\bar{g},\bar{\alpha})\simeq\text{const}(\Lambda r)^{-\gamma_F^*}$; see, e.g., [38]. The UV scale appears in this relation from Eq. (4). Then in the IR range ($\Lambda r \sim \mu r$ large, mr arbitrary) Eq. (51) takes on the form

$$F(r,\tau)\simeq\text{const}\,\Lambda^{-\gamma_F^*}D_0^{d_F^{\omega}}r^{-\Delta[F]}\xi_F(D_0\tau r^{\Delta_t},mr). \quad (52)$$

Here

$$\Delta[F]\equiv\Delta_F=d_F^k-\Delta_t d_F^{\omega}+\gamma_F^*, \quad \Delta_t=-2+\varepsilon \quad (53)$$

is the critical dimension of the function F and the scaling function ξ_F is related to R_F as in Eq. (47). For nontrivial γ_F^*, the function F in the IR range retains the dependence on Λ or, equivalently, on ν_0.

VI. RENORMALIZATION AND CRITICAL DIMENSIONS OF COMPOSITE OPERATORS

Any local (unless stated to be otherwise) monomial or polynomial constructed of primary fields and their derivatives at a single spacetime point $x\equiv(t,\mathbf{x})$ is termed a composite operator. Examples are $\theta^N(x)$, $[\partial_i\theta(x)\partial_i\theta(x)]^N$, $\partial_i\theta(x)\partial_j\theta(x)$, $\theta'(x)\nabla_t\theta(x)$ and so on.

Since the arguments of the fields coincide, correlation functions with such operators contain additional UV divergences, which are removed by additional renormalization procedure; see, e.g., [24,25]. For the renormalized correlation functions standard RG equations are obtained, which describe IR scaling of certain ''basis'' operators F with definite critical dimensions $\Delta_F\equiv\Delta[F]$. Due to the renormalization, $\Delta[F]$ does not coincide in general with the naive sum of critical dimensions of the fields and derivatives entering into F. As a rule, composite operators mix in renormalization, i.e., an UV finite renormalized operator F^R has the form $F^R=F+$ counterterms, where the contribution of the counterterms is a linear combination of F itself and, possibly, other unrenormalized operators which ''admix'' to F in renormalization.

Let $F\equiv\{F_\alpha\}$ be a closed set, all of whose monomials mix only with each other in renormalization. The renormalization matrix $Z_F\equiv\{Z_{\alpha\beta}\}$ and the matrix of anomalous dimensions $\gamma_F\equiv\{\gamma_{\alpha\beta}\}$ for this set are given by

$$F_\alpha=\sum_\beta Z_{\alpha\beta}F_\beta^R, \quad \gamma_F=Z_F^{-1}\tilde{\mathcal{D}}_\mu Z_F, \quad (54)$$

and the corresponding matrix of critical dimensions $\Delta_F\equiv\{\Delta_{\alpha\beta}\}$ is given by Eq. (53), in which d_F^k and d_F^ω are understood as the diagonal matrices of canonical dimensions of the operators in question (with the diagonal elements equal to sums of corresponding dimensions of all fields and derivatives constituting F) and $\gamma_F^*\equiv\gamma_F(g_*,\alpha_*)$ is the matrix (54) at the fixed point (42).

Critical dimensions of the set $F\equiv\{F_\alpha\}$ are given by the eigenvalues of the matrix Δ_F. The ''basis'' operators that possess definite critical dimensions have the form

$$F_\alpha^{bas}=\sum_\beta U_{\alpha\beta}F_\beta^R, \quad (55)$$

where the matrix $U_F=\{U_{\alpha\beta}\}$ is such that $\Delta_F'=U_F\Delta_F U_F^{-1}$ is diagonal.

In general, counterterms to a given operator F are determined by all possible 1-irreducible Green functions with one operator F and arbitrary number of primary fields, $\Gamma=\langle F(x)\Phi(x_1)\dots\Phi(x_2)\rangle_{1-\text{ir}}$. The total canonical dimension (formal index of UV divergence) for such function is given by

$$d_\Gamma=d_F-N_\Phi d_\Phi, \quad (56)$$

with the summation over all types of fields entering into the function. For superficially divergent diagrams, d_Γ is a nonnegative integer, cf. Sec. IV.

Let us consider operators of the form $\theta^N(x)$ with the canonical dimension $d_F=-N$, entering into the structure functions (5). From Table I in Sec. IV and Eq. (56) we obtain $d_\Gamma=-N+N_\theta-N_{\mathbf{v}}-(d+1)N_{\theta'}$, and from the analysis of the diagrams it follows that the total number of the fields θ entering into the function Γ can never exceed the number of the fields θ in the operator θ^N itself, i.e., $N_\theta\leq N$ [cf. item (i) in Sec. IV]. Therefore, the divergence can only exist in the functions with $N_{\mathbf{v}}=N_{\theta'}=0$, and arbitrary value of $N=N_\theta$, for which the formal index vanishes, $d_\Gamma=0$. However, at least one of N_θ external ''tails'' of the field θ is attached to a vertex $\theta'(\mathbf{v}\partial)\theta$ (it is impossible to construct a nontrivial, superficially divergent diagram of the desired type with all the external tails attached to the vertex F), at least one derivative ∂ appears as an extra factor in the diagram, and, consequently, the real index of divergence d_Γ' is necessarily negative.

This means that the operator θ^N requires no counterterms at all, i.e., it is in fact UV finite, $\theta^N=Z[\theta^N]^R$ with $Z=1$. It then follows that the critical dimension of $\theta^N(x)$ is simply given by the expression (53) with no correction from γ_F^* and is therefore reduced to the sum of the critical dimensions of the factors:

$$\Delta[\theta^N]=N\Delta[\theta]=N(-1+\varepsilon/2). \quad (57)$$

Since the structure functions (5) or (43) are linear combinations of pair correlators involving the operators θ^N, Eq. (57) shows that they indeed satisfy the RG equation of the form (31), discussed in Sec. V. We stress that the relation (57) was not clear *a priori*; in particular, it is violated if the velocity field becomes nonsolenoidal [14].

In the following, an important role will be also played by the tensor composite operators $\partial_{i_1}\theta\cdots\partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^n$ constructed solely of the scalar gradients. It is convenient to deal with the scalar operators obtained by contracting the tensors with the appropriate number of the vectors $\mathbf{n}$,

$$F[N,p] \equiv [(\mathbf{n}\boldsymbol{\partial})\theta]^p (\partial_i \theta \partial_i \theta)^n, \quad N \equiv 2n+p. \quad (58)$$

Their canonical dimensions depend only on the total number of the fields θ and have the form $d_F = 0$, $d_F^\omega = -N$.

In this case, from Table I and Eq. (56) we obtain $d_\Gamma = N_\theta - N_\mathbf{v} - (d+1)N_{\theta'}$, with the necessary condition $N_\theta \leq N$, which follows from the structure of the diagrams. It is also clear from the analysis of the diagrams that the counterterms to these operators can involve the fields θ, θ' only in the form of derivatives, $\partial\theta$, $\partial\theta'$, so that the real index of divergence has the form $d_\Gamma' = d_\Gamma - N_\theta - N_{\theta'} = -N_\mathbf{v} - (d+2)N_{\theta'}$. It then follows that superficial divergences can exist only in the Green functions with $N_\mathbf{v} = N_{\theta'} = 0$ and any $N_\theta \leq N$, and that the corresponding operator counterterms reduce to the form $F[N',p']$ with $N' \leq N$. Therefore, the operators (58) can mix only with each other in renormalization, and the corresponding infinite renormalization matrix

$$F[N,p] = \sum_{N',p'} Z_{[N,p][N',p']} F^R[N',p'] \quad (59)$$

is in fact block-triangular, i.e., $Z_{[N,p][N',p']} = 0$ for $N' > N$. It is then obvious that the critical dimensions associated with the operators $F[N,p]$ are completely determined by the eigenvalues of the finite subblocks with $N' = N$. In the following, we shall not be interested in the precise form of the basis operators (55), we rather shall be interested in the anomalous dimensions themselves. Therefore, we can neglect all the elements of the matrix (59) other than $Z_{[N,p][N,p']}$.

In the isotropic case, as well as in the presence of large-scale anisotropy, the elements $Z_{[N,p][N,p']}$ vanish for $p < p'$, and the block $Z_{[N,p][N,p']}$ is triangular along with the corresponding blocks of the matrices U_F and Δ_F from Eqs. (55), (53). In the isotropic case it can be diagonalized by changing to irreducible operators (scalars, vectors, and traceless tensors), but even for nonzero imposed gradient its eigenvalues are the same as in the isotropic case. Therefore, the inclusion of large-scale anisotropy does not affect critical dimensions of the operators (58); see [15]. In the case of small-scale anisotropy, the operators with different values of p mix heavily in renormalization, and the matrix $Z_{[N,p][N,p']}$ is neither diagonal nor triangular here.

Now let us turn to the calculation of the renormalization constants $Z_{[N,p][N,p']}$ in the one-loop approximation. Let $\Gamma(x;\theta)$ be the generating functional of the 1-irreducible Green functions with one composite operator $F[N,p]$ from Eq. (58) and any number of fields θ. Here $x \equiv (t,\mathbf{x})$ is the argument of the operator and θ is the functional argument, the "classical counterpart" of the random field θ. We are interested in the N-th term of the expansion of $\Gamma(x;\theta)$ in θ, which we denote $\Gamma_N(x;\theta)$; it has the form

$$\Gamma_N(x;\theta) = \frac{1}{N!} \int dx_1 \cdots \int dx_N \, \theta(x_1) \cdots \theta(x_N) \langle F[N,p]$$

$$\times (x) \theta(x_1) \cdots \theta(x_N) \rangle_{1-\mathrm{ir}}. \quad (60)$$

In the one-loop approximation the function (60) is represented diagrammatically in the following manner:

$$\Gamma_N = F[N,p] + \frac{1}{2} \; \triangle \;\;\; . \quad (61)$$

Here the solid lines denote the *bare* propagator $\langle \theta\theta' \rangle_0$ from Eq. (27), the ends with a slash correspond to the field θ', and the ends without a slash correspond to θ; the dashed line denotes the bare propagator (3); the vertices correspond to the factor (17).

The first term is the "tree" approximation, and the black circle with two attached lines in the diagram denotes the variational derivative $V(x;x_1,x_2) \equiv \delta^2 F[N,p]/\delta\theta(x_1)\delta\theta(x_2)$. It is convenient to represent it in the form

$$V(x;x_1,x_2) = \partial_i \delta(x-x_1)\partial_j \delta(x-x_2) \frac{\partial^2}{\partial a_i \partial a_j}[(\mathbf{n}a)^p(a^2)^n], \quad (62)$$

where a_i is a constant vector, which *after the differentiation* is substituted with $\partial_i\theta(x)$. The diagram in Eq. (61) is written analytically in the form

$$\int dx_1 \cdots \int dx_4 V(x;x_1,x_2)\langle \theta(x_1)\theta'(x_3)\rangle_0 \langle \theta(x_2)\theta'(x_4)\rangle_0$$

$$\times \langle v_k(x_3)v_l(x_4)\rangle_0 \partial_k\theta(x_3)\partial_l\theta(x_4), \quad (63)$$

with the bare propagators from Eqs. (3), (27); the derivatives appear from the ordinary vertex factors (17).

In order to find the renormalization constants, we need not the entire exact expression (63), rather we need its UV divergent part. The latter is proportional to a polynomial built of N factors $\partial\theta$ at a single spacetime point x. The needed N gradients have already been factored out from the expression (63): $(N-2)$ factors from the vertex (62) and 2 factors from the ordinary vertices (17). Therefore, we can neglect the spacetime inhomogeneity of the gradients and replace them with the constant vectors a_i. Expression (63) then can be written, up to an UV finite part, in the form

$$a_k a_l \frac{\partial^2}{\partial a_i \partial a_j}[(\mathbf{n}a)^p(a^2)^n]X_{ij,kl}, \quad (64)$$

where we have denoted

$$X_{ij,kl} \equiv \int dx_3 \int dx_4 \, \partial_i\langle \theta(x)\theta'(x_3)\rangle_0 \partial_j\langle \theta(x)\theta'(x_4)\rangle_0$$

$$\times \langle v_k(x_3)v_l(x_4)\rangle_0, \quad (65)$$

or, in the momentum-frequency representation, after the integration over the frequency,

$$X_{ij,kl} = \frac{D_0}{2\nu_0}\int \frac{d\mathbf{q}}{(2\pi)^d}\frac{q_i q_j}{(q^2 + \alpha(\mathbf{qn})^2)}\frac{T_{kl}(\mathbf{q})}{(k^2+m^2)^{-d/2-\varepsilon/2}}, \quad (66)$$

with D_0 from Eq. (3) and T_{kl} from Eq. (14). The tensor $X_{ij,kl}$ can be decomposed in certain basic structures:

$$X_{ij,kl} = \sum_{s=1}^{6} B_s S_{ij,kl}^{(s)}, \quad (67)$$

where B_s are scalar coefficients and $S^{(s)}$ are all possible fourth rank tensors constructed of the vector $\mathbf{n}$ and Kronecker delta symbols, and symmetric with respect to the permutations within the pairs $\{ij\}$ and $\{kl\}$:

$$S^{(1)}_{ij,kl}=\delta_{ij}\delta_{kl}, \quad S^{(2)}_{ij,kl}=(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/2,$$

$$S^{(3)}_{ij,kl}=\delta_{ij}n_k n_l, \quad S^{(4)}_{ij,kl}=\delta_{kl}n_i n_j,$$

$$S^{(5)}_{ij,kl}=(n_i n_k\delta_{jl}+n_i n_l\delta_{jk}+n_j n_l\delta_{ik}+n_j n_k\delta_{il})/4,$$

$$S^{(6)}_{ij,kl}=n_i n_j n_k n_l.$$

(68)

It is now convenient to change from the tensors (66), (67) to the scalar integrals A_s defined by the relations

$$A_s=\sum_{ijkl} X_{ij,kl}S^{(s)}_{ij,kl}, \quad s=1,\dots,6 \tag{69}$$

[it is obvious from Eqs. (67), (68) that the coefficients A_2 and A_5 vanish identically by virtue of the relation $q_l T_{lk}(\mathbf{q})=0$]. The UV divergence and the angular integration in Eqs. (66), (67) can be disentangled using the relations

$$A_s=\frac{D_0}{2\nu_0}J(m)H_s, \tag{70}$$

where the integral $J(m)$ from Eq. (36) contains a pole in ε, and H_s are UV finite and ε independent quantities given by only the angular integrations:

$$H_1=\left\langle\frac{T_{ii}}{1+\alpha\cos^2\psi}\right\rangle_S, \quad H_3=\left\langle\frac{n_i n_j T_{ij}}{1+\alpha\cos^2\psi}\right\rangle_S,$$

$$H_4=\left\langle\frac{T_{ii}\cos^2\psi}{1+\alpha\cos^2\psi}\right\rangle_S, \quad H_6=\left\langle\frac{n_i n_j T_{ij}\cos^2\psi}{1+\alpha\cos^2\psi}\right\rangle_S,$$

(71)

where ψ is the angle between the vectors $\mathbf{q}$ and $\mathbf{n}$, so that $(\mathbf{nq})=q\cos\psi$, and the quantities T_{ii} and $n_i n_j T_{ij}$ with T_{ij} from Eq. (14) depend only on ψ (the dependence on d and the anisotropy parameters $\rho_{1,2}$ is of course also implied). The brackets $\langle\cdots\rangle_S$ denote averaging over the unit d-dimensional sphere, which for a function dependent only on ψ takes on the form

$$\langle f(\psi)\rangle_S=\frac{S_{d-1}}{S_d}\int_0^\pi d\psi\sin^{d-2}\psi f(\psi), \tag{72}$$

with $S_d=2\pi^{d/2}/\Gamma(d/2)$. Finally, the change of variables $y=\cos^2\psi$ in the integrals (71) gives:

$$\int_0^1 dy\frac{y^{a-1}(1-y)^{b-1}}{(1+\alpha y)^c}=B(a,b)F(c,a;a+b;-\alpha),$$

(73)

where B is the Euler beta function and F is the hypergeometric function, see, e.g., [39]. This gives

$$H_1=(d-1+\rho_2)F_0+[(d-1)\rho_1-\rho_2]\frac{F_1}{d},$$

$$H_3=(1+\rho_2)F_0+(-1+\rho_1-2\rho_2)\frac{F_1}{d}+(\rho_2-\rho_1)\frac{3F_2}{d(d+2)},$$

$$H_4=(d-1+\rho_2)\frac{F_1}{d}+[(d-1)\rho_1-\rho_2]\frac{3F_2}{d(d+2)}, \tag{74}$$

$$H_6=(1+\rho_2)\frac{F_1}{d}+(-1+\rho_1-2\rho_2)\frac{3F_2}{d(d+2)}$$

$$+(\rho_2-\rho_1)\frac{15F_3}{d(d+2)(d+4)},$$

where we have denoted $F_n\equiv F(1,1/2+n;d/2+n;-\alpha)$.

In principle, all the hypergeometric functions entering into Eqs. (74) can be reduced to the only function, for example F_3, but the coefficients then become too complex.

After quite lengthy but straightforward calculations, the quantity in Eq. (64) can be expressed through the coefficients B_s, then through A_s, and finally through H_s:

$$\frac{D_0}{2\nu_0}J(m)\{Q_1 F[N,p-2]+Q_2 F[N,p]+Q_3 F[N,p+2]$$

$$+Q_4 F[N,p+4]\}, \tag{75}$$

where we have substituted a_i back with the gradients $\partial_i\theta(x)$, used the notation (58), and denoted

$$Q_1\equiv\frac{p(p-1)}{(d-1)}(H_4-H_6),$$

$$Q_2\equiv\frac{p(p-1)}{(d-1)}(dH_6-H_4)+\frac{(N-p)}{(d-1)(d+1)}\{(N-p+d-1)$$

$$\times(H_1-H_3)+H_4[-(N-2)+p(2d+3)]$$

$$+3H_6[(N-2)-p(d+2)]\},$$

(76)

$$Q_3\equiv\frac{(N-p)}{(d-1)(d+1)}\{-(2N-2p+d-3)H_1+H_3[(N-p)$$

$$\times(d+3)+(d+2)(d-3)]+H_4[(N-2)(d+3)$$

$$-p(3d+8)]+2H_6(d+2)[-3(N-2)+p(d+4)]\},$$

$$Q_4\equiv\frac{(N-p)(N-p-2)}{(d-1)(d+1)}[H_1-(d+2)(H_3+H_4)$$

$$+(d+2)(d+4)H_6].$$

The renormalization constants $Z_{[N,p][N,p']}$ can be found from the requirement that the function (61) after the replacement $F[N,p]\to F^R[N,p]$ with $F^R[N,p]$ from Eq. (59) be UV finite, i.e., have no poles in ε, when expressed in renormalized variables using the formulas (29). In the MS scheme, the diagonal element has the form $Z_{[N,p][N,p']}=1$

+ only poles in ε, while the nondiagonal ones contain only poles. Then from Eqs. (61), (70), and (75) one obtains

$$Z_{[N,p][N,p]} = 1 + \frac{g\,C_d}{4\varepsilon}Q_2, \quad Z_{[N,p][N,p-2]} = \frac{g\,C_d}{4\varepsilon}Q_1,$$

(77)

$$Z_{[N,p][N,p+2]} = \frac{g\,C_d}{4\varepsilon}Q_3, \quad Z_{[N,p][N,p+4]} = \frac{g\,C_d}{4\varepsilon}Q_4,$$

with coefficients Q_i from Eq. (76) and C_d from Eq. (38). The corresponding anomalous dimensions $\gamma_{[N,p][N',p']}$ $= Z^{-1}_{[N,p][N'',p'']}\tilde{\mathcal{D}}_\mu Z_{[N'',p''][N',p']}$ have the form

$$\gamma_{[N,p][N,p]} = -g\,C_dQ_2/4, \quad \gamma_{[N,p][N,p-2]} = -g\,C_dQ_1/4,$$

(78)

$$\gamma_{[N,p][N,p+2]} = -g\,C_dQ_3/4, \quad \gamma_{[N,p][N,p+4]} = -g\,C_dQ_4/4.$$

In contrast to exact expressions (38), the quantities in Eqs. (77), (78) have nontrivial corrections of order g^2 and higher. The matrix of critical dimensions (53) is given by

$$\Delta_{[N,p][N,p']} = N\varepsilon/2 + \gamma^*_{[N,p][N,p']},$$

(79)

where the asterisk implies the substitution (42). The set of equations (42), (74), (76), (78), (79) gives the desired expression for the matrix of critical dimensions of the composite operators (58) in the one-loop approximation, i.e., in the first order in ε [we recall that $g_* = O(\varepsilon)$ and $\alpha_* = O(1)$]. In contrast to simpler expressions like (57), the dimensions in Eq. (79) depend on the anisotropy parameters $\rho_{1,2}$ and have nontrivial corrections of order ε^2 and higher.

In the one-loop approximation, the expression (75) contains only four terms, and therefore all the matrix elements $\gamma_{[N,p][N,p']}$ other than Eq. (78) vanish. However, they become nontrivial in higher orders, so that the basis operators (55), associated with the eigenvalues of the matrix (79), are given by mixtures of the monomials (58) with all possible values of the index p, or, in other words, mixtures of the tensors belonging to different representations of the rotation group.

As already said above, the critical dimensions themselves are given by the eigenvalues of the matrix (79). One can check that for the isotropic case ($\rho_{1,2}=0$), its elements with $p'>p$ vanish, the matrix becomes triangular, and its eigenvalues are simply given by the diagonal elements $\Delta[N,p]$ $\equiv \Delta_{[N,p][N,p]}$. They are found explicitly and have the form

$$\Delta[N,p] = N\varepsilon/2 + \frac{2p(p-1)-(d-1)(N-p)(d+N+p)}{2(d-1)(d+2)}\varepsilon$$

$$+ O(\varepsilon^2).$$

(80)

It is easily seen from Eq. (80) that for fixed N and any d ≥ 2, the dimension $\Delta[N,p]$ decreases monotonically with p and reaches its minimum for the minimal possible value of $p = p_N$, i.e., $p_N = 0$ if N is even and $p_N = 1$ if N is odd:

$$\Delta[N,p] > \Delta[N,p'] \quad \text{if} \quad p > p'.$$

(81a)

Furthermore, this minimal value $\Delta[N,p_N]$ decreases monotonically as N increases for odd and even values of N separately, i.e.,

$$0 \geq \Delta[2n,0] > \Delta[2n+2,0], \quad \Delta[2n+1,1] > \Delta[2n+3,1].$$

(81b)

A similar hierarchy is demonstrated by the critical dimensions of certain tensor operators in the stirred Navier–Stokes turbulence; see Ref. [40] and Sec. 2.3 of [27]. However, no clear hierarchy is demonstrated by neighboring even and odd dimensions: from the relations

$$\Delta[2n+1,1] - \Delta[2n,0] = \frac{\varepsilon(d+2-4n)}{2(d+2)},$$

(81c)

$$\Delta[2n+2,0] - \Delta[2n+1,1] = \frac{\varepsilon(2-d)}{2(d+2)}$$

it follows that the inequality $\Delta[2n+1,1] > \Delta[2n+2,0]$ holds for any $d > 2$, while the relation $\Delta[2n,0] > \Delta[2n+1,1]$ holds only if n is sufficiently large, $n > (d+2)/4$. [The situation is slightly different in the presence of the linear mean gradient: the first term $N\varepsilon/2$ in Eq. (80) is then absent owing to the difference in canonical dimensions, and the complete hierarchy relations hold, $\Delta[2n,0] > \Delta[2n+1,1] > \Delta[2n+2,0]$; see Ref. [15].]

In what follows, we shall use the notation $\Delta[N,p]$ for the eigenvalue of the matrix (79) which coincides with Eq. (80) for $\rho_{1,2} = 0$. Since the eigenvalues depend continuously on $\rho_{1,2}$, this notation is unambiguous at least for small values of $\rho_{1,2}$.

The dimension $\Delta[2,0]$ vanishes identically for any $\rho_{1,2}$ and to all orders in ε. Like in the isotropic model, this can be demonstrated using the Schwinger equation of the form

$$\int \mathcal{D}\Phi\,\delta[\theta(x)\exp S_R(\Phi) + A\Phi]/\delta\theta'(x) = 0$$

(82)

(in the general sense of the word, Schwinger equations are any relations stating that any functional integral of a total variational derivative is equal to zero; see, e.g., [24,25]). In Eq. (82), S_R is the renormalized action (28), and the notation introduced in Eq. (16) is used. Equation (82) can be rewritten in the form

$$\langle \theta'D_\theta\theta - \nabla_t[\theta^2/2] + \nu Z_1\Delta[\theta^2/2] + \alpha\nu Z_2(\mathbf{n}\boldsymbol{\partial})^2[\theta^2/2]$$

$$- \nu Z_1 F[2,0] - \alpha\nu Z_2 F[2,2]\rangle_A = -A_{\theta'}\delta W_R(A)/\delta A_\theta.$$

(83)

Here D_θ is the correlator (2), $\langle\cdots\rangle_A$ denotes the averaging with the weight $\exp[S_R(\Phi) + A\Phi]$, W_R is determined by Eq. (16) with the replacement $S \to S_R$, and the argument x common to all the quantities in Eq. (83) is omitted.

The quantity $\langle F\rangle_A$ is the generating functional of the correlation functions with one operator F and any number of the

primary fields Φ, therefore the UV finiteness of the operator F is equivalent to the finiteness of the functional $\langle F \rangle_A$. The quantity in the right hand side of Eq. (83) is UV finite (a derivative of the renormalized functional with respect to finite argument), and so is the operator in the left hand side. Our operators $F[2,0]$, $F[2,2]$ do not admix in renormalization to $\theta' D_\theta \theta$ (no needed diagrams can be constructed), and to the operators $\nabla_t[\theta^2/2]$ and $\triangle[\theta^2/2]$ (they have the form of total derivatives, and $F[N,p]$ do not reduce to this form). On the other hand, all the operators in Eq. (83) other than $F[N,p]$ do not admix to $F[N,p]$, because the counterterms of the operators (58) can involve only operators of the same type; see above. Therefore, the operators $F[N,p]$ entering into Eq. (83) are independent of the others, and so they must be UV finite separately: $\nu Z_1 F[2,0] + \alpha \nu Z_2 F[2,2] =$ UV finite. Since the operator in Eq. (83) is UV finite, it coincides with its finite part,

$$\nu Z_1 F[2,0] + \alpha \nu Z_2 F[2,2] = \nu F^R[2,0] + \alpha \nu F^R[2,2],$$

which along with the relation (59) gives

$$Z_1 Z_{[2,0][2,0]} + \alpha Z_2 Z_{[2,2][2,0]} = 1,$$

$$Z_1 Z_{[2,0][2,2]} + \alpha Z_2 Z_{[2,2][2,2]} = \alpha,$$

and therefore for the anomalous dimensions in the MS scheme one obtains

$$\gamma_1 + \gamma_{[2,0][2,0]} + \alpha \gamma_{[2,2][2,0]} = 0,$$

$$\gamma_{[2,0][2,2]} + \alpha \gamma_2 + \alpha \gamma_{[2,2][2,2]} = 0.$$

These relations can indeed be checked from the explicit first-order expressions (77), (78). Bearing in mind that $\gamma_1^* = \gamma_2^* = \varepsilon$ (see Sec. IV), we conclude that among the four elements of the matrix γ_F^* only two, which we take to be $\gamma_{[2,2][2,0]}^*$ and $\gamma_{[2,2][2,2]}^*$, are independent. Then the matrix of critical dimensions (80) takes on the form

$$
\Delta_{[2,p][2,p']}
$$
$$
= \varepsilon + \begin{pmatrix} -\varepsilon - \alpha_* \gamma_{[2,2][2,0]}^* & -\alpha_* \varepsilon - \alpha_* \gamma_{[2,2][2,2]}^* \\ \gamma_{[2,2][2,0]}^* & \gamma_{[2,2][2,2]}^* \end{pmatrix}.
$$
$$(84)$$

It is then easily checked that the eigenvalue of the matrix (84), which is identified with $\Delta[2,0]$, does not involve unknown anomalous dimensions and vanishes identically, $\Delta[2,0] \equiv 0$, while the second one is represented as

$$\Delta[2,2] = \varepsilon - \alpha_* \gamma_{[2,2][2,0]}^* + \gamma_{[2,2][2,2]}^*.$$

Using the explicit $O(\varepsilon)$ expressions (76), (78), one obtains to the order $O(\varepsilon)$:

$$\Delta[2,2]/\varepsilon = 2 + \{-(d-2)d(d+2)(d+4)F_0^*$$
$$-(d+2)(d+4)(2+(d-2)\rho_1 + d\rho_2)F_1^*$$
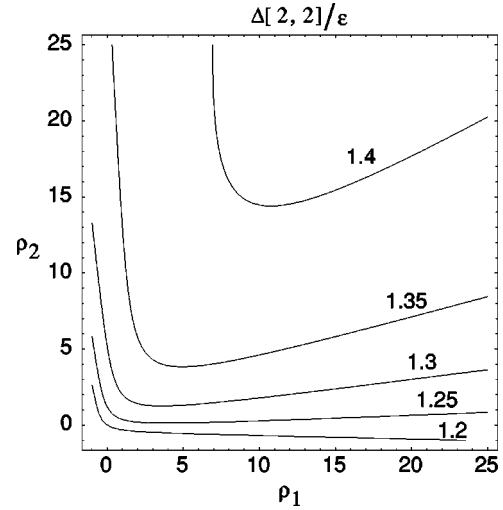$$+3(d+4)(d-2\rho_1 + 2d\rho_2)F_2^*$$

FIG. 1. Levels of the dimension $\Delta[2,2]$ for $d=3$ on the plane $\rho_1 - \rho_2$. Value changes from 1.2 (left-bottom) to 1.4 (right-top) with step 0.05.

$$+15d(\rho_1 - \rho_2)F_3^*\}/\{(d-1)(d+4)$$
$$\times [(d-1)(d+2) + (d+1)\rho_1 + \rho_2]\}, \qquad (85)$$

where $F_n^* \equiv F(1,1/2+n;d/2+n;-\alpha_*)$ with α_* from Eq. (42) and the hypergeometric function from Eq. (73).

In Fig. 1, we present the levels of the dimension (85) on the $\rho_1 - \rho_2$ plane for $d=3$. We note that the dependence on $\rho_{1,2}$ is quite smooth, and that $\Delta[2,2]$ remains positive on the whole of the $\rho_{1,2}$ plane, i.e., the first of the hierarchy relations (81) remains valid also in the presence of anisotropy. A similar behavior takes place also for $d=2$.

For $N>2$, the eigenvalues can be found analytically only within the expansion in $\rho_{1,2}$. Below we give such expansions up to the terms $O(\rho_{1,2}^2)$ for all the eigenvalues with $N \leq 4$ (with the notation $\rho_3 \equiv 2\rho_1 + d\rho_2$):

$$\gamma^*[2,2]/\varepsilon = \frac{2}{(d-1)(d+2)} + \frac{4(d-2)(d+1)\rho_3}{(d-1)^2(d+2)^2(d+4)}$$

$$+ \frac{2(d+1)}{(d-1)^3(d+2)^3(d+4)(d+6)}$$

$$\times [-6d(d^2+6d-15)\rho_1^2 + (d^5+6d^4-41d^3$$

$$+42d^2+16d+24)\rho_1\rho_2$$

$$+(-11d^4+23d^3+2d^2-26d+24)\rho_2^2],$$

$$(86a)$$

$$\gamma^*[3,1]/\varepsilon = -\frac{(d+4)}{(d+2)} - \frac{2(d-2)(d+1)\rho_3}{(d-1)(d+2)^3}$$

$$+ \frac{2(d-2)(d+1)}{(d-1)^2(d+2)^6(d+4)^2}$$

$$\times [2(d^5+15d^4+88d^3+204d^2+16d-96)\rho_1^2$$

$$+ d(-d^6 + d^5 + 80d^4 + 296d^3 + 64d^2 - 48d$$
$$+ 64)\rho_1\rho_2 + (3d^6 + 29d^5 + 74d^4 - 32d^3$$
$$- 72d^2 + 48d + 64)\rho_2^2], \tag{86b}$$

$$\gamma^*[3,3]/\varepsilon = \frac{6}{(d-1)(d+2)} + \frac{12(d-2)(d+1)^2\rho_3}{(d-1)^2(d+2)^3(d+4)}$$
$$- \frac{6(d-2)(d+1)}{(d-1)^3(d+2)^6(d+4)^2(d+6)}$$
$$\times [2(3d^6 + 54d^5 + 321d^4 + 722d^3 + 420d^2$$

$$+ 184d + 96)\rho_1^2 + d(-d^7 - 16d^6 - 49d^5$$
$$+ 158d^4 + 844d^3 + 904d^2 + 1120d + 640)\rho_1\rho_2$$
$$+ (13d^7 + 111d^6 + 258d^5 + 26d^4 - 124d^3$$
$$+ 88d^2 + 336d + 192)\rho_2^2], \tag{86c}$$

$$\gamma^*[4,0]/\varepsilon = \frac{-2(d+4)}{(d+2)} - \frac{4(d-2)^2(d+1)(d+6)\rho_3^2}{(d-1)^2d(d+2)^4(d+4)}, \tag{86d}$$

$$\gamma^*[4,2]/\varepsilon = \frac{-d^2 - 5d + 8}{(d-1)(d+2)} - \frac{4(d-2)(d+1)(d^2 + 3d - 16)\rho_3}{(d-1)^2(d+2)^2(d+4)^2} - \frac{2(d+1)}{(d-1)^3d(d+2)^3(d+4)^5(d+6)} [-2(2d^9 + 47d^8$$
$$+ 422d^7 + 1253d^6 - 3684d^5 - 23344d^4 - 4032d^3 + 57856d^2 + 86016d - 73728)\rho_1^2 + d(d^9 - 28d^8 - 759d^7$$
$$- 4654d^6 - 2712d^5 + 40232d^4 + 42592d^3 - 76928d^2 - 202240d + 122880)\rho_1\rho_2 - d(8d^9 + 149d^8 + 699d^7$$
$$- 1074d^6 - 10994d^5 - 2240d^4 + 29248d^3 + 29696d^2 - 49664d + 24576)\rho_2^2], \tag{86e}$$

$$\gamma^*[4,4]/\varepsilon = \frac{12}{(d-1)(d+2)} + \frac{24(d-2)(d+1)\rho_3}{(d-1)^2(d+2)(d+4)^2} - \frac{12(d-2)}{(d-1)^3(d+2)^4(d+4)^5(d+6)}$$
$$\times [2d(3d^7 + 81d^6 + 823d^5 + 4023d^4 + 10230d^3 + 14536d^2 + 11040d + 3456)\rho_1^2$$
$$+ (-d^{10} - 23d^9 - 163d^8 - 225d^7 + 2040d^6 + 9948d^5 + 21240d^4 + 25424d^3 + 18112d^2 + 8960d + 3072)\rho_1\rho_2$$
$$+ (15d^9 + 232d^8 + 1309d^7 + 3306d^6 + 3782d^5 + 780d^4 - 768d^3 + 3712d^2 + 6656d + 3072)\rho_2^2]. \tag{86f}$$

These expressions illustrate two facts which seem to hold for all N:

(i) The leading anisotropy correction is of order $O(\rho_{1,2})$ for $p \neq 0$ and $O(\rho_{1,2}^2)$ for $p = 0$, so that the dimensions $\gamma^*[N,0]$ are anisotropy independent in the *linear* approximation, and

(ii) this leading contribution depends on $\rho_{1,2}$ only through the combination $\rho_3 \equiv 2\rho_1 + d\rho_2$.

This conjecture is confirmed by the following expressions for $N = 6,8$ and $p = 0$:

$$\gamma^*[6,0]/\varepsilon = \frac{-2(d+6)}{(d+2)}$$
$$- \frac{12(d-2)^2(d+1)(d^2 + 14d + 48)\rho_3^2}{(d-1)^2d(d+2)^4(d+4)^2}, \tag{87a}$$

$$\gamma^*[8,0]/\varepsilon = \frac{-4(d+8)}{(d+2)}$$
$$- \frac{24(d-2)^2(d+1)(d^2 + 18d + 80)\rho_3^2}{(d-1)^2d(d+2)^4(d+4)^2}. \tag{87b}$$

The eigenvalues beyond the small $\rho_{1,2}$ expansion have been obtained numerically. Some of them are presented in Figs. 2–5, namely, the dimensions $\Delta[n,p]$ for $n = 3,4,5,6$ vs ρ_1 for $\rho_2 = 0$, vs $\rho_1 = \rho_2$, and vs ρ_2 for $\rho_1 = 0$. The main conclusion that can be drawn from these diagrams is that the hierarchy (81) demonstrated by the dimensions for the isotropic case ($\rho_{1,2} = 0$) holds valid for all the values of the anisotropy parameters.

VII. OPERATOR PRODUCT EXPANSION AND ANOMALOUS SCALING

Representations (49) for any scaling functions $\xi(mr)$ describe the behavior of the correlation functions for $\Lambda r \gg 1$ and any fixed value of mr; see Sec. V. The inertial range corresponds to the additional condition $mr \ll 1$. The form of the functions $\xi(mr)$ is not determined by the RG equations themselves; in analogy with the theory of critical phenomena, its behavior for $mr \to 0$ is studied using the well-known Wilson operator product expansion (OPE); see, e.g., [24–28].

According to the OPE, the equal-time product $F_1(x)F_2(x')$ of two renormalized operators for $\mathbf{x} \equiv (\mathbf{x}$
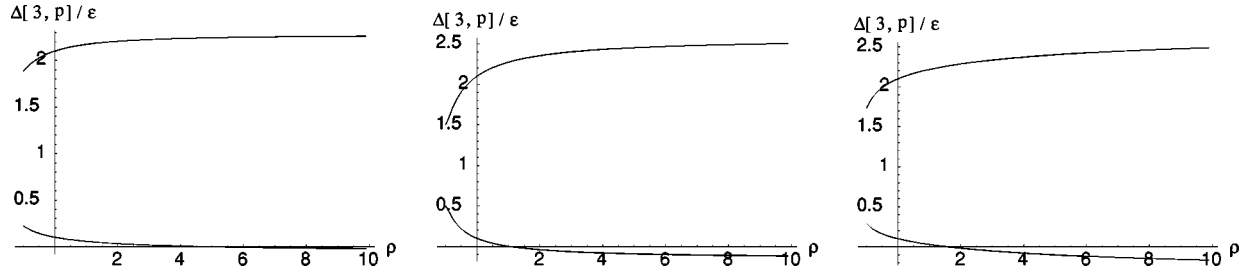
FIG. 2. Behavior of the critical dimension $\Delta[3,p]/\varepsilon$ for $d=3$ with $p=1,3$ (from below to above) vs $\rho\equiv\rho_1$ for $\rho_2=0$—left, vs $\rho\equiv\rho_1=\rho_2$—center, vs $\rho\equiv\rho_2$ for $\rho_1=0$—right.

$+\mathbf{x}')/2=$ const and $\mathbf{r}\equiv\mathbf{x}-\mathbf{x}'\to0$ has the representation

$$F_1(x)F_2(x')=\sum_F C_F(\mathbf{r})F(t,\mathbf{x}), \qquad (88)$$

where the functions C_F are coefficients regular in m^2 and F are all possible renormalized local composite operators allowed by symmetry (more precisely, see below). Without loss of generality, it can be assumed that the expansion is made in basis operators of the type Eq. (55), i.e., those having definite critical dimensions Δ_F. The renormalized correlator $\langle F_1(x)F_2(x')\rangle$ is obtained by averaging Eq. (88) with the weight $\exp S_R$ with the renormalized action (28); the quantities $\langle F\rangle\propto m^{\Delta_F}$ appear on the right hand side.

From the operator product expansion (88) we therefore find the following expression for the scaling function $\xi(mr)$ in the representation (49) for the correlator $\langle F_1(x)F_2(x')\rangle$:

$$\xi(mr)=\sum_F A_F(mr)^{\Delta_F}, \quad mr\ll1, \qquad (89)$$

with the coefficients A_F regular in $(mr)^2$.

Now let us turn to the equal-time structure functions S_N from Eq. (5). From now on, we assume that the mixed correlator $\langle\mathbf{v}f\rangle$ differs from zero (see Sec. II); this does not affect the critical dimensions, but gives rise to nonvanishing odd structure functions. In general, the operators entering into the OPE are those which appear in the corresponding Taylor expansions, and also all possible operators that admix to them in renormalization [24,25]. The leading term of the Taylor expansion for the function S_N is obviously given by the operator $F[N,N]$ from Eq. (58); the renormalization gives rise to all the operators $F[N',p]$ with $N'\leq N$ and all possible values of p. The operators with $N'>N$ (whose con-

tributions would be more important) do not appear in Eq. (89), because they do not enter into the Taylor expansion for S_N and do not admix in renormalization to the terms of the Taylor expansion; see Sec. VI. Therefore, combining the RG representation (47) with the OPE representation (89) gives the desired asymptotic expression for the structure function in the inertial range:

$$S_N(\mathbf{r})=D_0^{-N/2}r^{N(1-\varepsilon/2)}\sum_{N'\leq N}\sum_P \{C_{N',p}(mr)^{\Delta[N',p]}+\cdots\}. \qquad (90)$$

The second summation runs over all values of p, allowed for a given N'; $C_{N',p}$ are numerical coefficients dependent on ε, d, $\rho_{1,2}$ and the angle ϑ between $\mathbf{r}$ and $\mathbf{n}$. The dots stand for the contributions of the operators other than $F[N,p]$, for example, $\partial^2\theta\partial^2\theta$; they give rise to the terms of order $(mr)^{2+O(\varepsilon)}$ and higher and will be neglected in what follows.

Some remarks are now in order.

(i) If the mixed correlator $\langle\mathbf{v}f\rangle$ is absent, the odd structure functions vanish, while the contributions to even functions are given only by the operators with even values of N'. In the isotropic case ($\rho_{1,2}=0$) only the contributions with $p=0$ survive; see [13]. In the presence of the anisotropy, $\rho_{1,2}\neq0$, the operators with $p\neq0$ acquire nonzero mean values, and their dimensions $\Delta[N',p]$ also appear on the right hand side of Eq. (90).

(ii) The leading term of the small mr behavior is obviously given by the contribution with the minimal possible value of $\Delta[N',p]$. Now we recall the hierarchy relations (81a), (81b), which hold for $\rho_{1,2}=0$ and therefore remain valid at least for $\rho_{1,2}\ll1$. This means that, if the anisotropy is weak enough, the leading term in Eq. (90) is given by the
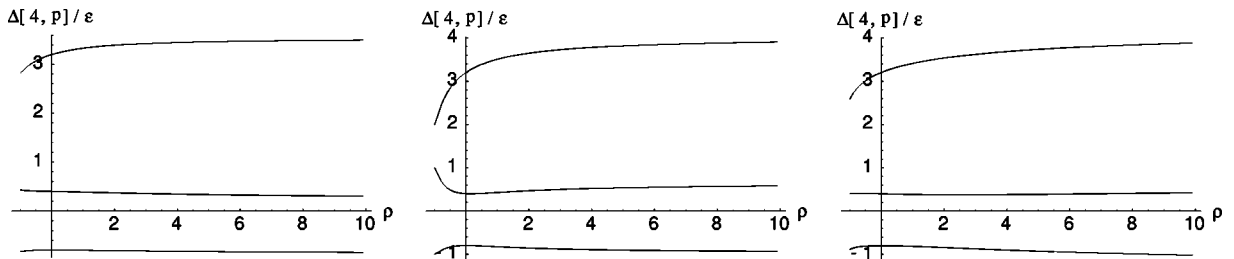

FIG. 3. Behavior of the critical dimension $\Delta[4,p]/\varepsilon$ for $d=3$ with $p=0,2,4$ (from below to above) vs $\rho\equiv\rho_1$ for $\rho_2=0$—left, vs $\rho\equiv\rho_1=\rho_2$—center, vs $\rho\equiv\rho_2$ for $\rho_1=0$—right.

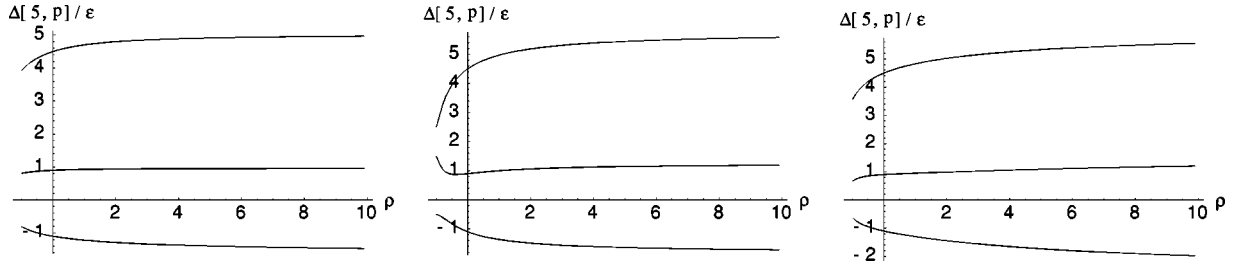FIG. 4. Behavior of the critical dimension $\Delta[5,p]/\varepsilon$ for $d=3$ with $p=1,3,5$ (from below to above) vs $\rho\equiv\rho_1$ for $\rho_2=0$—left, vs $\rho\equiv\rho_1=\rho_2$—center, vs $\rho\equiv\rho_2$ for $\rho_1=0$—right.

dimension $\Delta[N,0]$ for any S_N. For all the special cases studied in Sec. VI, this hierarchy persists also for finite values of the anisotropy parameters, and the contribution with $\Delta[N,0]$ remains the leading one for such N and $\rho_{1,2}$.

(iii) Of course, it is not impossible that the inequalities (81a), (81b) break down for some values of n, d and $\rho_{1,2}$, and the leading contribution to Eq. (90) is determined by a dimension with $N'\neq N$ and/or $p>0$.

Furthermore, it is not impossible that the matrix (79) for some $\rho_{1,2}$ had a pair of complex conjugate eigenvalues, Δ and Δ^*. Then the small mr behavior of the scaling function $\xi(mr)$ entering into Eq. (90) would involve oscillating terms of the form

$$(mr)^{\mathrm{Re}\,\Delta}\{C_1\cos[\mathrm{Im}\,\Delta(mr)]+C_2\sin[\mathrm{Im}\,\Delta(mr)]\},$$

with some constants C_i.

Another exotic situation emerges if the matrix (79) cannot be diagonalized and is only reduced to the Jordan form. In this case, the corresponding contribution to the scaling function would involve a logarithmic correction to the powerlike behavior, $(mr)^{\Delta}[C_1\ln(mr)+C_2]$, where Δ is the eigenvalue related to the Jordan cell. However, these interesting hypothetical possiblities are not actually realized for the special cases studied above in Sec. VI.

(iv) The inclusion of the mixed correlator $\langle\mathbf{v}f\rangle\propto\mathbf{n}\delta(t-t')C'(r/\ell)$ violates the evenness in $\mathbf{n}$ and gives rise to nonvanishing odd functions S_{2n+1} and to the contributions with odd N' to the expansion (90) for even functions. If the hierarchy relations (81a), (81b) hold, the leading term for the even functions will still be given by the contribution with $\Delta[N,0]$. If the relations (81c) hold, the leading term for the odd function S_{2n+1} will be given by the dimension $\Delta[2n,0]$ for $n<(d+2)/4$ and by $\Delta[2n+1,1]$ for $n>(d+2)/4$. Note that for the model with an imposed gradient, the leading

terms for S_{2n+1} are given by the dimensions $\Delta[2n+1,1]$ for all n; see [15]. This can be related to the observation of Ref. [41] that the odd structure functions of the velocity field appear more sensitive to the anisotropy than the even functions.

Representations similar to Eqs. (49), (90) can easily be written for any equal-time pair correlator, provided its canonical and critical dimensions are known. In particular, for the operators $F[N,p]$ in the IR region ($\Lambda r\to\infty$, mr fixed) one obtains

$$\langle F[N_1,p_1]F[N_2,p_2]\rangle=\nu_0^{-(N_1+N_2)/2}\sum_{N,p}\sum_{N',p'}$$
$$\times(\Lambda r)^{-\Delta_{[N,p]}-\Delta_{[N',p']}}$$
$$\times\xi_{N,p;N',p'}(mr),\qquad(91)$$

where the summation indices N, N' satisfy the inequalities $N\leq N_1$, $N'\leq N_2$, and the indices p, p' take on all possible values allowed for given N, N'. The small mr behavior of the scaling functions $\xi_{N,p;N',p'}(mr)$ has the form

$$\xi_{N,p;N',p'}(mr)=\sum_{N'',p''}C_{N'',p''}(mr)^{\Delta[N'',p'']},\qquad(92)$$

with the restriction $N''\leq N+N'$ and corresponding values of p''; $C_{N'',p''}$ are some numerical coefficients.

So far, we have discussed the special case of the velocity correlator given by Eqs. (3), (14). Let us conclude this section with a brief discussion of the general case (11). The RG analysis given above in Secs. III–VII can be extended directly to this case; no serious alterations are required. From the expressions (21), (71) it immediately follows that only even polynomials in the expansion (12) can give contribu-
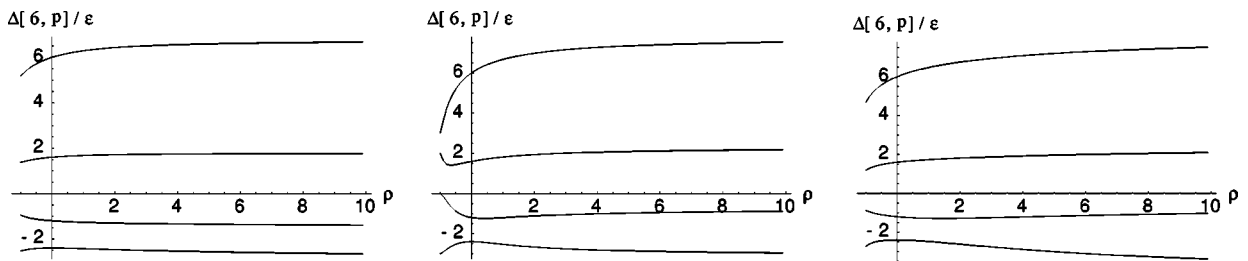

FIG. 5. Behavior of the critical dimension $\Delta[6,p]/\varepsilon$ for $d=3$ with $p=0,2,4,6$ (from below to above) vs $\rho\equiv\rho_1$ for $\rho_2=0$—left, vs $\rho\equiv\rho_1=\rho_2$—center, vs $\rho\equiv\rho_2$ for $\rho_1=0$—right.
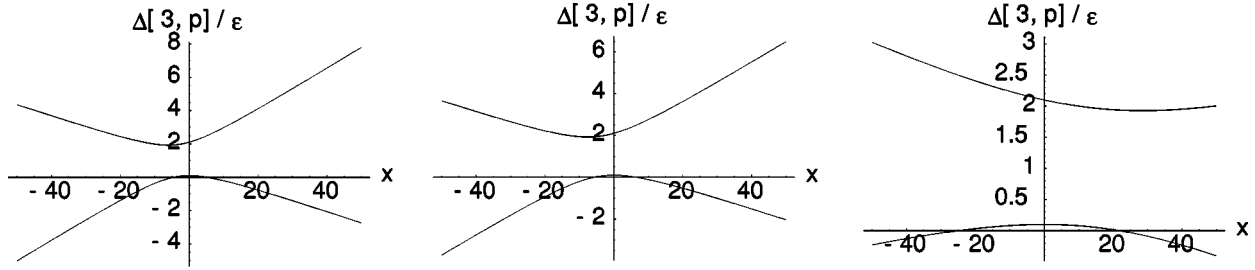
FIG. 6. Behavior of the critical dimension $\Delta[3,p]/\varepsilon$ for $d=3$ with $p=1,3$ (from below to above) vs $x \equiv a_2$ for $b_3=0$—left, vs $x \equiv a_2 = b_3$—center, vs $x \equiv b_3$ for $a_2=0$—right.

tions to the renormalization constants, and consequently, to the coordinates of the fixed point and the anomalous dimensions. For this reason, the odd polynomials were omitted in Eq. (12) from the very beginning. Moreover, it is clear from Eq. (21) that only the coefficients a_l with $l=0,1$ and b_l with $l=0,1,2$ contribute to the constants $Z_{1,2}$ in Eq. (28) and therefore to the basic RG functions (33) and to the coordinates of the fixed point in Eq. (42). Therefore, the fixed point in the general model (11) is parametrized completely by these five coefficients; the higher coefficients enter only via the positivity conditions (13).

Furthermore, it is clear from Eqs. (71) that for $\alpha=0$, only coefficients a_l with $l \leq 2$ and b_l with $l \leq 3$ can contribute to the integrals H_n and, consequently, to the one-loop critical dimensions (79). Therefore, the calculation of the latter essentially simplifies for the special case $a_0=1$, $a_1=0$ and $b_l=0$ for $l \leq 2$ in Eq. (11). Then the coordinates of the fixed point (42) are the same as in the isotropic model, in particular, $\alpha_*=0$, and the anomalous exponents will depend on the only two parameters a_2 and b_3. We have performed a few sample calculations for this situation; the results are presented in Figs. 6–9 for $\Delta[n,p]$ with $n=3,4,5,6$ vs a_2 for $b_3=0$, vs $a_2=b_3$, and vs b_3 for $a_2=0$. In all cases studied, the general picture has appeared similar to that outlined above for the case (14). In particular, the hierarchy of the critical dimensions, expressed by the inequalities (81), persists also for this case. We may conclude that the special case (14) case represents nicely all the main features of the general model (11).

VIII. EXACT SOLUTION FOR THE SECOND-ORDER STRUCTURE FUNCTION AND CALCULATION OF THE AMPLITUDES

In renormalized variables, dimensionality considerations give, see Eq. (44),
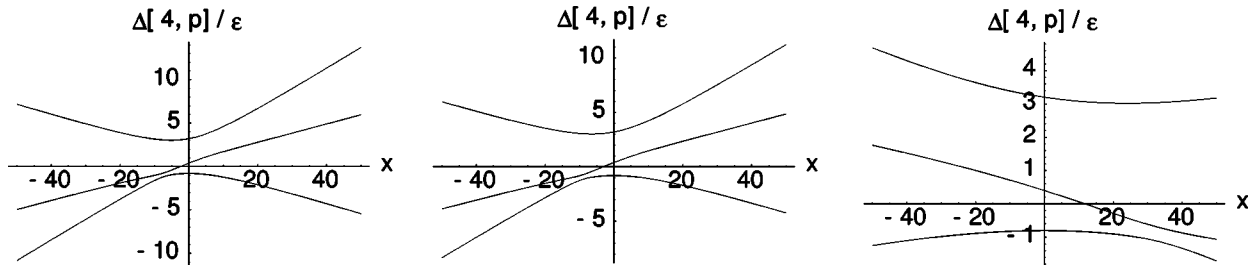
$$S_2(\mathbf{r}) = (r^2/\nu)R(\mu r, mr, g, \alpha), \tag{93}$$

where $R \equiv R_2$ is some function of dimensionless variables; the dependence on d, ε, $\rho_{1,2}$ and the angle ϑ between the vectors $\mathbf{r}$ and $\mathbf{n}$ is also implied. The asymptotic behavior of the function (93) in the IR region ($\Lambda r \gg 1$, mr fixed) is found from the solution of the RG equation, see Eqs. (47) and (48) in Sec. V:

$$S_2(\mathbf{r}) = D_0^{-1} r^{2-\varepsilon} g_* \xi(mr), \tag{94}$$

where the scaling function $\xi(u)$ is related to the function R from Eq. (93) as follows:

$$\xi(u) \equiv R(1,u,g_*,\alpha_*), \tag{95}$$

with g_* and α_* from Eq. (42). The function R in Eq. (93) can be calculated within the renormalized perturbation theory as a series in g,

$$R(g,\cdots) = \sum_{n=0}^{\infty} g^n R_n(\cdots), \tag{96}$$

where the dots stand for the arguments of R other than g. Making the substitutions $\mu r \to 1$, $g \to g_*$ and $\alpha \to \alpha_*$, expanding R_n in ε, and grouping together contributions of the same order, from Eq. (96) we obtain the ε expansion for the scaling function:

$$\xi(u) = \sum_{n=0}^{\infty} \varepsilon^n \xi_n(u). \tag{97}$$

[It is important here that the calculation of any finite order in ε requires only a finite number of terms in Eq. (96), because $g_* = O(\varepsilon)$, and the coefficients R_n for the renormalized
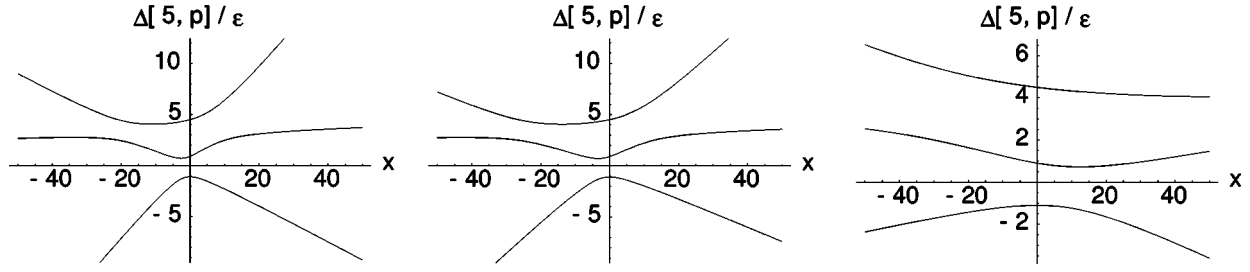

FIG. 7. Behavior of the critical dimension $\Delta[4,p]/\varepsilon$ for $d=3$ with $p=0,2,4$ (from below to above) vs $x \equiv a_2$ for $b_3=0$—left, vs $x \equiv a_2 = b_3$—center, vs $x \equiv b_3$ for $a_2=0$—right.

FIG. 8. Behavior of the critical dimension $\Delta[5,p]/\varepsilon$ for $d=3$ with $p=1,3,5$ (from below to above) vs $x \equiv a_2$ for $b_3=0$—left, vs $x \equiv a_2 = b_3$—center, vs $x \equiv b_3$ for $a_2=0$—right.

quantity do not contain poles in ε.] However, the expansion (97) is not suitable for the analysis of the small u behavior of $\xi(u)$: we shall see below that the coefficients ξ_n contain IR singularities of the form $u^p \ln^q u$, these "large IR logarithms" compensate for the smallness of ε, and the actual expansion parameter appears to be $\varepsilon \ln u$ rather than ε itself.

The formal statement of the problem is to sum up the expansion (97) assuming that ε is small with the additional condition that $\varepsilon \ln u = O(1)$. The desired solution is given by the OPE; it shows that the small u behavior of $\xi(u)$ for the second-order structure function has the form, see Eqs. (89) and (90) in Sec. VII:

$$\xi(u) = C_1 + C_2 u^{\Delta[2,2]} + \cdots. \tag{98}$$

The first term is the contribution of the operators 1 and $F[2,0]$ with the dimension $\Delta[2,0]=0$, see the text below Eq. (84), and $\Delta[2,2]$ is given to the order $O(\varepsilon)$ in Eq. (85); the dots in Eq. (98) stand for the contributions of order $u^{2+O(\varepsilon)}$ and higher. The coefficients C_i depend on ε, d, $\rho_{1,2}$, and the angle ϑ. They are sought in the form of series in ε:

$$C_n = \sum_{k=0}^{\infty} C_n^{(k)} \varepsilon^k. \tag{99}$$

Equation (98) can be expanded in ε and compared with Eq. (97). Such a comparison with known ε expansions for the coefficients ξ_n in Eq. (97) and exponents in Eq. (98) allows one to verify the representation Eq. (98) directly from the perturbation theory, and to determine the coefficients in Eq. (99).

In order to find the coefficients $C_{1,2}$, related to the exponents of order $O(\varepsilon)$ in Eq. (98), we do not need the entire

functions $\xi_n(u)$, we only need the terms of the form $\varepsilon^p \ln^q u$ in their small u asymptotic expansions. Knowledge of the terms 1 and $\varepsilon \ln u$ is sufficient for the calculation of the lowest (zero) order coefficients $C_{1,2}^{(0)}$. Calculation of the terms $(\varepsilon \ln u)^p$ with $p>1$ gives additional equations for the same quantities $C_{1,2}^{(0)}$, which must be satisfied automatically and can therefore be used to verify the representation (98). Calculation of the terms $\varepsilon^p \ln^q u$ with $p>q$ gives the equations for the higher terms $C_{1,2}^{(k)}$ with $k \geq 1$. Finally, the terms of the form $\varepsilon^p u^k \ln^q u$ with $k>0$ are related to the contributions of the form $u^{k+O(\varepsilon)}$ in Eq. (98).

In what follows, we confine ourselves with the calculation of the coefficients $C_{1,2}^{(0)}$. To this end, we have to find the function $\xi(u)$ with the accuracy of

$$\xi(u) = A_0 + A_1 \varepsilon \ln u, \tag{100}$$

where $A_i = A_i(d, \rho_{1,2}, \vartheta)$. From the definition of the structure functions (5) and the representations (94), (95) it follows that $\xi(u) = \langle D(\omega, k) \rangle_{RG}$, where by the brackets we have denoted the following operation:

$$\langle D(\omega, k) \rangle_{RG} \equiv \frac{2\nu}{r^2} \int \frac{d\omega}{(2\pi)} \int \frac{d\mathbf{k}}{(2\pi)^d} D(\omega, k)$$

$$\times [1 - \exp(i\mathbf{k}\mathbf{r})]|_{\mu=1/r, g=g_*, \alpha=\alpha_*} \tag{101}$$

and $D(\omega, k) = \langle \theta \theta \rangle$ is the exact pair correlator of the field θ, which we need to know here up to the one-loop approximation:
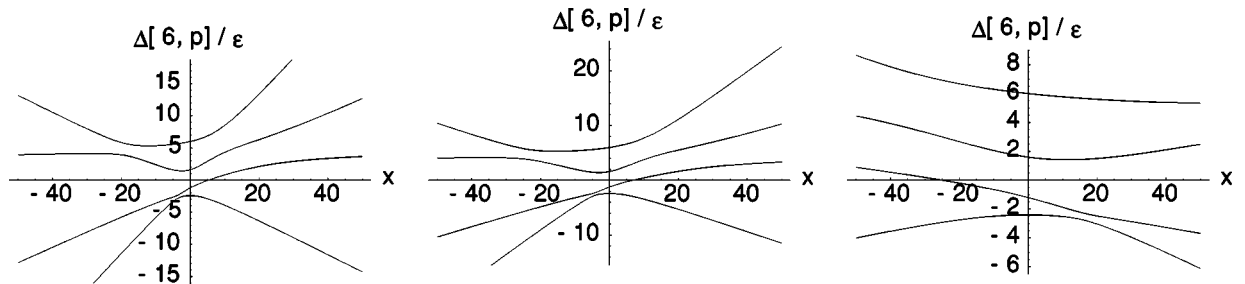

FIG. 9. Behavior of the critical dimension $\Delta[6,p]/\varepsilon$ for $d=3$ with $p=0,2,4$ (from below to above) vs $x \equiv a_2$ for $b_3=0$—left, vs $x \equiv a_2 = b_3$—center, vs $x \equiv b_3$ for $a_2=0$—right.

$$D(\omega, k) = \quad \text{------} + \text{------} + \text{------} + \text{------} + \cdots \qquad (102)$$

In what follows, we set $\ell = \infty$ in Eq. (2); the IR regularization is provided by the ''mass'' m in the correlator (3). In the momentum-frequency representation, the correlator (2) takes on the form $C(k) = (2\pi)^d \delta(\mathbf{k})$, and the bare propagator in the renormalized perturbation theory is obtained from $\langle \theta\theta \rangle_0$ in Eq. (18) with the substitution $\nu_0 k^2 \rightarrow \epsilon(k) \equiv \nu k^2 + \nu\alpha(\mathbf{kn})^2$.

Therefore, the contribution of the first (loopless) diagram in Eq. (102) to the function (101) has the form

$$\left\langle \frac{(2\pi)^d \delta(\mathbf{k})}{\omega^2 + \epsilon^2(k)} \right\rangle_{RG} = \frac{1}{2d}\left[1 - \frac{\alpha_*(1+2\cos^2\vartheta)}{(d+2)} + O(\rho_{1,2}^2) \right]. \qquad (103)$$

For the sake of simplicity, here and below we confine ourselves with the first order in the anisotropy parameters $\rho_{1,2}$ [we recall that $\alpha_* = O(\rho_{1,2})$]; only the final results will be given for general ρ. Note also that all the integrals entering into the left hand side of Eq. (103) are well-defined for $\ell = \infty$, for example,

$$\int d\mathbf{k}\frac{\delta(\mathbf{k})}{k^2}[1-\exp(i\mathbf{kr})] = \frac{r^2}{2d},$$

$$\int d\mathbf{k}\frac{\delta(\mathbf{k})}{k^2}\frac{k_i k_j}{k^2}[1-\exp(i\mathbf{kr})]$$

$$= \frac{r^2}{2d(d+2)}\left[\delta_{ij} + 2\frac{r_i r_j}{r^2} \right]. \qquad (104)$$

It is also not too difficult to take into account the contributions from the second and third (asymmetric) diagrams in Eq. (102). It follows from Eqs. (34) and (35) that one simply has to replace $\epsilon(k)$ in the left hand side of Eq. (103) with the expression $\nu_0 k^2 + \alpha_0 \nu_0(\mathbf{nk})^2 - \Sigma_{\theta'\theta}$ and change to the renormalized variables using the relations (29) and (38). The pole part of the integral $J(m)$ in Eq. (35) is then subtracted; the ε expansion of the difference gives rise to the logarithm of m. The substitution $g \rightarrow g_*$, $\alpha \rightarrow \alpha_*$ leads to drastic simplification [this is due to the fact that the expressions (42) for the fixed point originated from the same Eq. (35)], and the total contribution of the first three diagrams in Eq. (102) reduces to the loopless contribution (103) multiplied with the factor $(1 - \varepsilon \ln u)$.

Now let us turn to the last diagram in Eq. (102). The quantity to be ''averaged'' in Eq. (101) has the form

$$\frac{1}{\omega^2 + \epsilon^2(k)} \int \frac{d\omega'}{(2\pi)}$$

$$\times \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{g\nu\mu^\varepsilon T_{ij}(\mathbf{q}')}{((q')^2 + m^2)^{d/2+\varepsilon/2}} \frac{C(q)q_i q_j}{(\omega')^2 + \epsilon^2(q)},$$

where $\mathbf{q}' = \mathbf{k} - \mathbf{q}$. We perform the integrations over ω' and ω [the latter is implicit in Eq. (101)] and replace $\mathbf{q}'$ with $\mathbf{k}$ in the integral over $\mathbf{q}$ [we recall that $C(k) \propto \delta(\mathbf{k})$]; this gives

$$\frac{g\nu\mu^\varepsilon T_{ij}(\mathbf{k})}{4\epsilon(k)(k^2 + m^2)^{d/2+\varepsilon/2}} \int d\mathbf{q}\,\delta(\mathbf{q})\frac{q_i q_j}{\epsilon(q)}. \qquad (105)$$

Using Eq. (14) and the relation

$$\int d\mathbf{q}\,\delta(\mathbf{q})\frac{q_i q_j}{\epsilon(q)} = \frac{\delta_{ij}}{\nu d} - \frac{\alpha}{\nu d(d+2)}[\delta_{ij}+2n_i n_j] + O(\alpha^2)$$

we obtain for the quantity (105)

$$\frac{g\mu^\varepsilon}{4\,d\,\nu\,k^2(k^2+m^2)^{d/2+\varepsilon/2}}[c_0 + c_1\cos^2\psi] \qquad (106)$$

with the coefficients

$$c_0 = d-1+\rho_2 - \alpha(d+1)/(d+2),$$

$$c_1 = (d-1)\rho_1 - \rho_2 - \alpha(d^2+d-4)/(d+2)$$

(we also recall that ψ is the angle between $\mathbf{k}$ and $\mathbf{n}$). Substituting Eq. (105) into Eq. (101) gives

$$\frac{g_*}{2d}[c_0 I_0(u) + c_1 I_1(u)] \qquad (107)$$

with the dimensionless convergent integrals

$$I_n(u) = \frac{1}{r^2}\int \frac{d\mathbf{k}}{(2\pi)^d}\frac{(\cos^2\psi)^n}{k^2(k^2+m^2)^{d/2}}[1-\exp(i\mathbf{kr})] \qquad (108)$$

[within our accuracy, we have set $\varepsilon = 0$ in all exponents in Eqs. (107), (108)]. The leading terms of their small u behavior are found as follows: one differentiates $I_n(u)$ with respect to u, performs the change of variable $\mathbf{k} \rightarrow u\mathbf{k}$, expands in u the quantity in the square brackets, and selects the leading terms of order $1/u$. This gives

$$I_0(u) \simeq -\frac{C_d}{2d}\ln u, \quad I_1(u) \simeq -\frac{C_d \ln u}{2d(d+2)}[1+2\cos^2\vartheta], \qquad (109)$$

with C_d from Eq. (38).

Collecting all the contributions of the form Eq. (100) in Eq. (101) within our accuracy gives

$$A_0 = \frac{1}{2d}\left[1 - \frac{\alpha_*(1+2\cos^2\vartheta)}{d+2}\right],$$

$$A_1 = -\frac{(d+1)[(d^2+d-4)\rho_1-d\rho_2]}{(d-1)^2(d+2)^3}P_2(\cos\vartheta),$$

$$(110)$$

where $P_2(z) = z^2 - 1/d$ is the second Gegenbauer polynomial.

It is sufficient to take $\Delta \equiv \Delta[2,2]$ in the isotropic approximation, $\Delta = \varepsilon d(d+1)/(d-1)(d+2)$, see Eq. (80) [it is important here that $A_1 = O(\rho_{1,2})$].

Comparing Eq. (98) with Eq. (100) gives $C_2 = A_1\varepsilon/\Delta$ and $C_1 = A_0 - C_2$. Substituting these expressions into Eq. (98) and then into Eq. (94) and using Eq. (42) we obtain the final expression for S_2:

$$S_2(\mathbf{r}) = \frac{\varepsilon D_0^{-1} r^{2-\varepsilon}}{(d-1)C_d}\left\{1 - (\rho_1+\rho_2)/d + 2\frac{(\rho_1-\rho_2)}{(d+2)}P_2(\cos\vartheta)\right.$$

$$\left. -(mr)^\Delta\frac{2[(d^2+d-4)\rho_1-d\rho_2]}{(d-1)(d+2)^2}P_2(\cos\vartheta)\right\}.$$

$$(111)$$

Similar (but much more cumbersome) calculations can also be performed if the anisotropy parameters $\rho_{1,2}$ are not supposed to be small. The result has the form

$$S_2(\mathbf{r}) = \frac{d\varepsilon D_0^{-1} r^{2-\varepsilon}}{(d-1)(d+\rho_1+\rho_2)C_d}$$

$$\times\left\{1 + \frac{g_*(d+\alpha_*)(d+1)(\rho_2-\rho_1)}{2(d-1)(d+2)\Delta}\widetilde{P}_2(\cos\vartheta)\right.$$

$$\times\left[1 - \frac{3(d+\alpha_*)}{(d+2)}F_2^*\right] - (mr)^\Delta\frac{(d+\alpha_*)}{(d-1)d(d+2)}\widetilde{P}_2$$

$$\times(\cos\vartheta)\left[\alpha_*(3dF_2^* - (d+2)F_1^*)\right.$$

$$\left.\left.+ \frac{g_*(d+1)(\rho_2-\rho_1)}{2\Delta}\left(1 - \frac{3(d+\alpha_*)}{(d+2)}F_2^*\right)\right]\right\},$$

$$(112)$$

where the quantities g_*, α_* and $\Delta \equiv \Delta[2,2]$ are given in Eqs. (42) and (85), $\widetilde{P}_2(z) \equiv z^2 - (1+\alpha_*)/(d+\alpha_*)$ and $F_n^* = F(1, 1/2+n; d/2+n; -\alpha_*)$ [as in Eq. (85)].

Now let us turn to the exact equation satisfied by the second-order structure function $S_2(\mathbf{r})$. From the definition of the latter and Eq. (23) we obtain

$$2\nu_0\Delta S_2(\mathbf{r}) + D_0 S_{ij}(\mathbf{r})\partial_i\partial_j S_2(\mathbf{r}) = 2C(r/\ell). \quad (113)$$

In the inertial range we neglect the viscosity and set $C(r/\ell) \simeq C(0) = 1$; this gives

$$S_{ij}(\mathbf{r})\partial_i\partial_j S_2(\mathbf{r}) = 2D_0^{-1}. \quad (114)$$

We should also set $mr \ll 1$. The integral (24) has a finite limit for $m=0$; the explicit calculation with T_{ij} from Eq. (14) gives

$$S_{ij}(\mathbf{r}) = \frac{2r^\varepsilon}{C(\varepsilon)}\left\{(d+\varepsilon-1)\delta_{ij} - \varepsilon r_i r_j/r^2 + \rho_1(1+\varepsilon z^2)\right.$$

$$+ \rho_2[(d+\varepsilon-2)n_i n_j - \varepsilon z(n_i r_j + n_j r_i)/r]$$

$$+ \frac{(\rho_2-\rho_1)}{(d+\varepsilon+2)}[\delta_{ij} + 2n_i n_j + \varepsilon(\delta_{ij}z^2 + r_i r_j/r^2$$

$$\left. + 2z(n_i r_j + n_j r_i)/r) + \varepsilon(\varepsilon-2)z^2 r_i r_j/r^2]\right\},$$

$$(115)$$

where $z \equiv (\mathbf{nr})/r$ and

$$C(\varepsilon) \equiv -2^{2+\varepsilon}(4\pi)^{d/2}\Gamma(d/2+\varepsilon/2+1)/\Gamma(-\varepsilon/2)$$

$$= 2\varepsilon d/C_d + O(\varepsilon^2). \quad (116)$$

For what follows, we need to know the action of the differential operator in the left hand side of Eq. (114) onto a function of the form $r^{2-\varepsilon+\gamma}\psi(z)$, where γ is an arbitrary exponent (below we use γ to denote different exponents; the precise meaning in each case will be clear from the context):

$$S_{ij}(\mathbf{r})\partial_i\partial_j[r^{2-\varepsilon+\gamma}\psi(z)] = \frac{r^\gamma}{C(\varepsilon)}\mathcal{L}(\gamma)\psi(z). \quad (117)$$

Here $\mathcal{L}(\gamma)$ is a second-order differential operator with respect to z, whose explicit form is

$$\mathcal{L}(\gamma) = (2+\gamma-\varepsilon)\{(d-1)(d+\gamma)(d+2+\varepsilon) + (d-1)(d+2)(\rho_1+\rho_2) + 2\varepsilon(d\rho_2-\rho_1)$$

$$+ z^2\varepsilon d(d+1)(\rho_1-\rho_2) + \gamma[(d+1)\rho_1 + (\varepsilon+1)\rho_2 + z^2(\rho_1(\varepsilon d-\varepsilon-2)) + \rho_2(d^2-2-\varepsilon)]\} + \{(1-d)(d-1+\varepsilon)(d$$

$$+2+\varepsilon) - (d^2+1)\rho_1 + (d^2-d-1)\rho_2 - \varepsilon(d-1)\rho_1 - \varepsilon(d^2-2)\rho_2 + 4\varepsilon^2\rho_1 - (2d+1)\varepsilon^2\rho_2 + z^2[2\rho_1-(d^2-2)\rho_2$$

$$+\varepsilon(d+1)((1-d)\rho_1 + (2d-3)\rho_2) + \varepsilon^2(-(d+3)\rho_1 + (2d+1)\rho_2)]$$

$$-\gamma(1-z^2)[4\rho_1 - 2(d^2-2)\rho_2 + \varepsilon(4\rho_1-2d\rho_2)]\}z\partial_z + \{(d+\varepsilon-1)[d+2+\varepsilon+\rho_1+(d+1+\varepsilon)\rho_2]$$

$$+ z^2[(2+\varepsilon(d+1+\varepsilon))\rho_1 - ((d+\varepsilon)^2-2-\varepsilon)\rho_2]\}(1-z^2)\partial_z^2.$$

$$(118)$$

We seek the solution to Eq. (114) in the form

$$S_2(\mathbf{r}) = r^{2-\varepsilon} 2 D_0^{-1} C(\varepsilon) \psi(z), \qquad (119)$$

which gives the following differential equation for $\psi(z)$:

$$\mathcal{L}\psi(z) = 1, \quad \mathcal{L} \equiv \mathcal{L}(\gamma=0). \qquad (120)$$

Here the operator $\mathcal{L}$ has the form

$$\mathcal{L} = \ell_2(z)\partial_z^2 + \ell_1(z)\partial_z + \ell_0(z), \qquad (121)$$

with some functions $\ell_i(z)$ whose explicit form is readily obtained from Eq. (118) and will not be given here for the sake of brevity. The operator $\mathcal{L}$ is self-adjoint on the interval $-1 \le z \le 1$ with respect to the scalar product

$$(f,g) \equiv \int_{-1}^{1} dz\, \rho(z) f(z) g(z) \qquad (122)$$

with the weight function

$$\rho(z) = \ell_2^{-1}(z) \exp \int^{z} dz'\, \ell_1(z')/\ell_2(z'). \qquad (123)$$

The integration gives

$$\rho(z) = (1-z^2)^{(d-3)/2} \Bigg\{ 1$$
$$+ \frac{[2+\varepsilon(d+1+\varepsilon)]\rho_1 - [(d+\varepsilon)^2 - 2 - \varepsilon]\rho_2}{(d+\varepsilon-1)[2+d+\varepsilon+\rho_1+(d+1+\varepsilon)\rho_2]} z^2 \Bigg\}^{\wp},$$
$$(124a)$$

where

$$\wp \equiv \frac{(2+d-\varepsilon)[-(1+\varepsilon)\rho_1 + (d(d+\varepsilon)/2)-1]\rho_2]}{[2+\varepsilon(d+1+\varepsilon)]\rho_1 + [2+\varepsilon-(d+\varepsilon)^2]\rho_2} \qquad (124b)$$

[the lower limit in Eq. (123 is chosen such that $\rho(z=0) = 1$].

The self-adjoint operator $\mathcal{L}$ possesses a complete set of eigenfunctions, but it is not possible to find them explicitly in the general case. However, Eq. (119) can be treated within regular perturbation expansions in ε, $\rho_{1,2}$ and $1/d$. In what follows, we shall discuss them separately.

Perturbation theory in ε. We write the operator $\mathcal{L} = \mathcal{L}(\gamma=0)$ in Eq. (118) and the desired solution $\psi(z)$ as series in ε:

$$\mathcal{L} = \mathcal{L}_0 + \varepsilon \mathcal{L}_\varepsilon + \cdots, \quad \psi(z) = \psi^{(0)}(z) + \varepsilon \psi^{(1)}(z) + \cdots.$$

The leading operator $\mathcal{L}_0$ has the form Eq. (121) with the functions

$$\ell_i(z) \equiv \tilde{\ell}_i(z)[(d-1)(d+2) + \rho_1(d+1) + \rho_2],$$

where

$$\tilde{\ell}_0 = 2(d+\alpha_*), \quad \tilde{\ell}_1 = -z[d-1-\alpha_*(1-z^2)],$$

$$\tilde{\ell}_2 = (1-z^2)[1 + \alpha_*(1-z^2)].$$

It is easy to see that the constant is one of the eigenfunctions of the operator $\mathcal{L}_0$ (a consequence of the fact that ℓ_0 is independent of z). The other eigenfunction (with the eigenvalue of zero) is $\tilde{P}_2(z)$ from Eq. (112):

$$\mathcal{L}_0 \tilde{P}_2(z) = 0.$$

Therefore, a solution to the equation $\mathcal{L}_0 \psi^{(0)}(z) = 1$ can be written in the form

$$\psi^{(0)}(z) = \ell_0^{-1} + C\tilde{P}_2(z). \qquad (125)$$

In order to find the constant C, consider the equation for the correction $\psi^{(1)}(z)$:

$$\mathcal{L}_\varepsilon \psi^{(0)} + \mathcal{L}_0 \psi^{(1)} = 0. \qquad (126)$$

Using the scalar product (122) with the weight (124) at $\varepsilon = 0$,

$$\rho^{(0)}(z) = (1-z^2)^{(d-3)/2} \left(1 - \frac{\alpha_*}{1+\alpha_*} z^2 \right)^{-1-d/2} \qquad (127)$$

(the operator $\mathcal{L}_0$ is self-adjoint with respect to it) we obtain $(\tilde{P}_2, \mathcal{L}_0 \psi^{(1)}) = (\mathcal{L}_0 \tilde{P}_2, \psi^{(1)}) = 0$. Then from Eq. (126) it follows that $(\tilde{P}_2, \mathcal{L}_\varepsilon \psi^{(0)}) = 0$. From the latter relation and Eq. (125) we find

$$C = -\frac{(\tilde{P}_2, \mathcal{L}_\varepsilon \ell_0^{-1})}{(\tilde{P}_2, \mathcal{L}_\varepsilon \tilde{P}_2)},$$

which gives

$$\psi^{(0)}(z) = \ell_0^{-1} \left[1 - \frac{(\tilde{P}_2, \mathcal{L}_\varepsilon \cdot 1)}{(\tilde{P}_2, \mathcal{L}_\varepsilon \tilde{P}_2)} \tilde{P}_2(z) \right]. \qquad (128)$$

Below we shall show that the solution (128) coincides with the RG result (112) for $m=0$. To this end, we shall show first that the last term in Eq. (112), proportional to $(mr)^\Delta$, is an eigenfunction of the operator in the left hand side of Eq. (113) with the eigenvalue of zero. We seek the zero mode of that operator in the form $r^{2-\varepsilon+\gamma}\Phi(z)$ with some exponent γ and function $\Phi(z)$. According to Eq. (117), we arrive at the equation $\mathcal{L}(\gamma)\Phi(z) = 0$ with $\mathcal{L}(\gamma)$ from (118). For $\varepsilon = 0$, the solution of the latter equation is given by $\gamma = 0$ and $\Phi(z) = \tilde{P}_2(z)$ from Eq. (112). The $O(\varepsilon)$ correction to the solution $\gamma = 0$ is sought from the equation

$$(\mathcal{L}_0 + \varepsilon \mathcal{L}_\varepsilon + \gamma \mathcal{L}_\gamma)\Phi(z) = 0,$$

where $\gamma \mathcal{L}_\gamma$ is the part of operator (118) linear in γ. We require that the correction operator $\varepsilon \mathcal{L}_\varepsilon + \gamma \mathcal{L}_\gamma$ do not shift the original (zero) eigenvalue. This leads to the equation $(\tilde{P}_2, (\varepsilon \mathcal{L}_\varepsilon + \gamma \mathcal{L}_\gamma)\tilde{P}_2) = 0$, from which it follows

$$\gamma = -\varepsilon \frac{(\tilde{P}_2, \mathcal{L}_\varepsilon \tilde{P}_2)}{(\tilde{P}_2, \mathcal{L}_\gamma \tilde{P}_2)}. \tag{129}$$

Using Eqs. (118) and (127) it is not difficult to check that the expression (129) coincides with the dimension (85), e.g., $\gamma = \Delta[2,2]$. Therefore, we have shown that the last term in Eq. (112) is indeed an eigenfunction of the operator from Eq. (114), and the exponent and function $\Phi(z)$ are found in the first and zeroth order in ε, respectively.

Using Eq. (129), expression (128) can be rewritten in the form

$$\psi^{(0)}(z) = \ell_0^{-1} \left[1 + \frac{\varepsilon}{\Delta[2,2]} \frac{(\tilde{P}_2, \mathcal{L}_\varepsilon \cdot 1)}{(\tilde{P}_2, \mathcal{L}_\gamma \tilde{P}_2)} \tilde{P}_2(z) \right]. \tag{130}$$

Calculation of the scalar products using Eqs. (118) and (127) shows that Eq. (130) indeed coincides with the solution (112) for $m=0$.

Perturbation theory in the anisotropy parameters $\rho_{1,2}$. In this case it follows from Eq. (118) with $\rho_{1,2}=0$, that the leading operator $\mathcal{L}_0(\gamma)$ has the form Eq. (121) with

$$\ell_0 = (2+\gamma-\varepsilon)(d-1)(d+\gamma)(d+2+\varepsilon),$$

$$\ell_1 = -(d-1)(d-1+\varepsilon)(d+2+\varepsilon)z, \tag{131}$$

$$\ell_2 = (d-1+\varepsilon)(d+2+\varepsilon)(1-z^2).$$

It can be represented in the form

$$\mathcal{L}_0(\gamma) = -(d-1+\varepsilon)(d+2+\varepsilon)\tilde{\mathcal{L}} + \ell_0. \tag{132}$$

The eigenfunctions of the operator $\tilde{\mathcal{L}} = (z^2-1)\partial_z^2 + (d-1)z\partial_z$ are nothing other than the well known Gegenbauer polynomials $P_n(z)$, the corresponding eigenvalues being $n(n+d-2)$; see [39]. They form an orthogonal system on the interval $[-1,1]$ with the weight $(1-z^2)^{(d-3)/2}$ [see Eq. (124) with $\rho_{1,2}=0$]. Since the term ℓ_0 in Eq. (132) is independent of z, the eigenfunctions of the operator $\mathcal{L}_0(\gamma)$ are the same polynomials $P_n(z)$, and the eigenvalues have the form

$$-(d-1+\varepsilon)(d+2+\varepsilon)n(n+d-2) + \ell_0, \tag{133}$$

with ℓ_0 from Eq. (131).

The knowledge of the eigenfunctions and eigenvalues of the operator $\mathcal{L}_0 = \mathcal{L}_0(\gamma=0)$ in Eq. (120) allows one to easily construct the iterative solution of that equation. In particular, in the leading order one has

$$\psi^{(0)}(z) = 1/\ell_0 = 1/(2-\varepsilon)d(d-1)(d+\varepsilon).$$

For $m \neq 0$, the corrections to the solution of Eq. (114) are determined by the zero modes of the operator $\mathcal{L}(\gamma)$. In the leading (zeroth) order in $\rho_{1,2}$ they are simply found by equating the eigenvalues (133) to zero. The ''admissible'' (in the sense of [7–9]) solution is:

$$2\gamma_n^{(0)} = (\varepsilon-d-2)$$
$$+ \sqrt{(\varepsilon-d-2)^2 + 4n(n+d-2)(d+\varepsilon-1)/(d-1)}. \tag{134}$$

Therefore, in such approximation the zero modes of the operator from Eq. (114) have the form $r^{2-\varepsilon+\gamma_n^{(0)}} P_n(z)$, in agreement with Ref. [7] (owing to the $z \to -z$ symmetry of our model, only even n can contribute). From the RG viewpoints, the exponents in Eq. (134) are related to the n-th rank composite operators builtof two fields θ and n derivatives; in particular, γ_2 (which gives the leading correction for $mr \ll 1$) coincides with the dimension $\Delta[2,2]$.

In order to find the $O(\rho_{1,2})$ correction to $\gamma_2^{(0)}$, which we denote $\gamma_2^{(1)}$, we expand the operator $\mathcal{L}(\gamma)$ from Eq. (118) around the point $\rho_{1,2}=0$ and $\gamma=\gamma_2^{(0)}$ up to the terms linear in $\rho_{1,2}$ and $\gamma_2^{(1)}$:

$$\mathcal{L}(\gamma) \simeq \mathcal{L}_0 + \sum_{i=1}^2 \rho_i \mathcal{L}^i + \gamma_2^{(1)} \mathcal{L}_1,$$

which defines the operators $\mathcal{L}^i$ and $\mathcal{L}_1$. Proceeding as for the previous case, we obtain

$$\gamma_2^{(1)} = -\sum_{i=1}^2 \rho_i \frac{(P_2, \mathcal{L}^i P_2)}{(P_2, \mathcal{L}_1 P_2)},$$

where the scalar product is defined with the weight $(1-z^2)^{(d-3)/2}$. The explicit calculation gives the desired result:

$$\gamma_2^{(1)} = \frac{4(d-2)(d+1)\varepsilon[2\rho_1 + (d+\varepsilon)\rho_2]}{(d-1)(d+4)(d+2+\varepsilon)[4d(d^2-1)\varepsilon + (d-1)^2(d+2-\varepsilon)^2]^{1/2}}. \tag{135}$$

Perturbation theory in $1/d$. For $d \to \infty$, the weight (124) takes on the form

$$\rho(z) \simeq \left(\frac{1-z^2}{1-\rho_2 z^2/(1+\rho_2)} \right)^{d/2}. \tag{136}$$

From Eq. (136) it follows that, for $d \gg 1$, the function $\rho(z)$

differs remarkably from zero only in a small vicinity of the point $z \simeq 0$, more precisely, $z^2 \lesssim 1/d$. This becomes especially clear if one changes to the new variable $x^2 \equiv dz^2/2(1+\rho_2)$. In terms of x, the function (136) has a finite limit for $d \to \infty$:

$$\rho(x) \simeq e^{-x^2}. \tag{137}$$

The integration domain in the scalar product (122) becomes $-\sqrt{d} \leq x \leq \sqrt{d}$, which for $d \to \infty$ gives $-\infty \leq x \leq \infty$.

According to Eq. (118), Eq. (120) in terms of the variable x and in the leading approximation in $1/d$ can be written in the form

$$\mathcal{L}_0 \psi(x) = 2/d^3, \tag{138}$$

where

$$\mathcal{L}_0 = \mathcal{L}_0(\gamma=0), \quad \mathcal{L}_0(\gamma) = \partial_x^2 - 2x\partial_x + 2(2+\gamma-\varepsilon). \tag{139}$$

The eigenfunctions of the operator $\partial_x^2 - 2x\partial_x$ are the well known Hermit polynomials $H_n(x)$ (see, e.g., [39]):

$$(\partial_x^2 - 2x\partial_x)H_n(x) = 2n\, H_n(x), \quad n=0,1,2,\ldots,$$

which form an orthogonal system on the axis $-\infty \leq x \leq \infty$ with the weight (137). It is clear from Eq. (139) that the eigenfunctions of the operator $\mathcal{L}_0(\gamma)$ are also the polynomials $H_n(x)$, the corresponding eigenvalues being $\lambda_n = 2(n+\varepsilon-2-\gamma)$.

The zero modes ($\lambda_n = 0$) correspond to the values

$$\gamma = n - 2 + \varepsilon, \tag{140}$$

where $n = 2$, 4, 6 and so on, owing to the symmetry $z \to -z$ of our model; the minimal eigenvalue is $\gamma = \varepsilon$ with $n=2$. Therefore, the operator $\mathcal{L}_0 = \mathcal{L}_0(\gamma=0)$ has no zero modes, and the solution to Eq. (138) is simply given by $\psi(x) = 1/d^3(2-\varepsilon)$.

The corrections to the solution $\gamma = \varepsilon$ in $1/d$ are found as above for the perturbation theories in ε and $\rho_{1,2}$. This gives

$$\gamma = \varepsilon + 2\varepsilon(1 + 2\rho_2 + \rho_1\rho_2)/d^2 + O(1/d^3)$$

[the $O(1/d)$ term is absent], and the corresponding zero mode is

$$(1-x^2) - \rho_2(2+\rho_1+\rho_2)/d(1+\rho_2) + O(1/d^2).$$

IX. CONCLUSION

Let us recall briefly the main points of the paper.

We have studied the anomalous scaling behavior of a passive scalar advected by the time-decorrelated strongly anisotropic Gaussian velocity field. The statistics of the latter is given by Eqs. (3) and (11). The general case (11) involves infinitely many parameters; most practical calculations have been performed for the truncated two-parametric model (14), which seems to represent nicely the main features of the general case (14).

The original stochastic problem (1)–(3) can be cast as a renormalizable field theoretic model [Eqs. (15)–(18)], which allows one to apply the RG and OPE techniques to it. The corresponding RG equations have an IR stable fixed point, Eq. (42), which leads to the asymptotic expressions of the type Eqs. (49), (52) for various correlation functions in the region $\Lambda r \gg 1$.

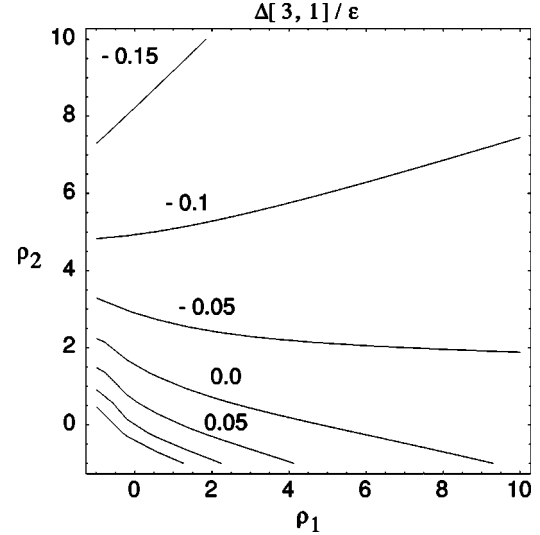Those expressions involve certain scaling functions of the

FIG. 10. Levels of the dimension $\Delta[3,1]/\varepsilon$ for $d=3$ on the plane $\rho_1-\rho_2$. Value changes from -0.15 (top) to 0.15 (bottom) with step 0.05.

variable mr, whose behavior at $mr \ll 1$ is determined by the OPE. It establishes the existence of the inertial-range anomalous scaling behavior. The structure functions are given by superpositions of power laws with nonuniversal (dependent on the anisotropy parameters $\rho_{1,2}$) exponents.

The exponents are determined by the critical dimensions of composite operators (58) built of the scalar gradients. In contrast with the isotropic velocity field, these operators in our model mix in renormalization such that the matrices of their critical dimensions are neither diagonal nor triangular. These matrices are calculated explicitly to the order $O(\varepsilon)$ [Eqs. (74), (76), (78), (79)], but their eigenvalues (anomalous exponents) can be found explicitly only as series in $\rho_{1,2}$ [Eqs. (86), (87)] or numerically (Figs. 1–9).

In the limit of vanishing anisotropy, the exponents can be associated with definite tensor composite operators built of the scalar gradients, and exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank, the less is the dimension and, consequently, the more important is the contribution to the inertial-range behavior [see Eqs. (81)].

The leading terms of the even (odd) structure functions are given by the scalar (vector) operators. For the finite anisotropy, the exponents cannot be associated with individual operators (which are essentially "mixed" in renormalization), but, surprising enough, the aforementioned hierarchy survives for all the cases studied, as is shown in Figs. 2–9.

The second-order structure function $S_2(\mathbf{r})$ is studied in more detail using the RG and zero-mode techniques; like in the isotropic case [6], its leading term has the form $S_2 \propto r^{2-\varepsilon}$, but the amplitude now depends on $\rho_{1,2}$ and the angle between the vectors $\mathbf{r}$ and $\mathbf{n}$ from Eq. (14). The first anisotropic correction has the form $(mr)^{\Delta[2,2]}$ with the exponent $\Delta[2,2] = O(\varepsilon)$ from Eq. (85). The function S_2 satisfies the exact equation (113); this allows for an alternative derivation of the perturbation theory in ε, and also gives a basis for perturbation theories in $1/d$ or the anisotropy parameters $\rho_{1,2}$; see the discussion in Sec. VIII.

It is well known that, for the isotropic velocity field, the anisotropy introduced at large scales by the external forcing or imposed mean gradient, persists in the inertial range and reveals itself in *odd* correlation functions: the skewness factor $S_3/S_2^{3/2}$ decreases for $mr \to 0$ but slowly (see Refs. [3,4,10,11]), while the higher-order ratios $S_{2n+1}/S_2^{n+1/2}$ increase (see, e.g., [15,17,19]).

In the case at hand, the inertial-range behavior of the skewness is given by $S_3/S_2^{3/2} \propto (mr)^{\Delta[3,1]}$. For $\rho_{1,2} \to 0$, the exponent $\Delta[3,1]$ is given by Eq. (80) with $n=3$ and $p=1$; it is positive and coincides with the result of Ref. [10]. The levels of the dimension $\Delta[3,1]$ on the $\rho_1 - \rho_2$ plane are shown in Fig. 10. One can see that, if the anisotropy becomes strong enough, $\Delta[3,1]$ becomes negative and the skewness factor *increases* going down towards to the depth

of the inertial range; the higher-order odd ratios increase already when the anisotropy is weak.

ACKNOWLEDGMENTS

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