Bounds on the convective heat transport in a rotating layer

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Previous bounds on the convective heat transport in a horizontal layer heated from below and rotating about a vertical axis have been improved through the use of separate energy balances for the poloidal and toroidal components of the velocity field. Because the additional constraint imposed for the solution of the variational problem for the extremalizing vector field leads to Euler-Lagrange equations which can no longer be solved analytically, numerical methods must be employed. A Galerkin scheme is introduced and the variational problem is solved in the case when stress-free conditions are assumed at the upper and lower boundaries. Results are presented as a function of the Rayleigh number and the rotation parameter for the Prandtl numbers $P=7,\ 0.7,\ 0.1$, and 0.025.

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I. INTRODUCTION

The theory of bounds on the convective heat transport in a layer heated from below was first formulated, to the best of our knowledge, in mathematical terms by Howard [1] following earlier ideas of Malkus [2]. Later the theory was extended to bounds on other transports in turbulent fluids by Busse [3] and general solutions were obtained in the case when the constraint of continuity is applied. For reviews of the theory of upper bounds or the optimum theory of turbulence we refer to [4] and [5]. Considerable efforts have been expanded to tighten the bounds through the imposition onto the extremalizing vector fields of additional constraints derived from the basic equations. Although the proposal for the use of the separate energy balances for poloidal and toroidal components of the velocity field was already made by [5], the solution of the corresponding variational problem was hindered by the lack of appropriate asymptotic techniques. Moreover, Kerswell and Soward [6] have demonstrated in the special case of plane Couette flow that the imposition of the separate energy balances for poloidal and toroidal components as constraints will not lead to improvements of the upper bound on the momentum transport.

In recent years the increasing available computer capacity has allowed to approach asymptotic regimes of high Rayleigh or Reynolds numbers through numerical solutions of the Euler-Lagrange equations of the variational problems [7]. The success in the numerical treatment of the simple variational problem has motivated us to use computational methods for the solution of the more complex problems which are obtained through the imposition of additional constraints. In order to avoid the negative result of [6] we have chosen the problem of convection in a rotating layer where poloidal and toroidal components of the velocity field are intimately coupled even in the linearized description of the problem. Since the Prandtl number P enters the variational functional in addition to the rotation parameter when poloidal and toroidal energy balances are introduced separately, the parameter space becomes much enlarged. Results have thus been obtained only for selected values of P and of the Taylor number.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

We consider a horizontal fluid layer of height h heated from below and rotating about a vertical axis with the angular velocity Ω. Using h as length scale, h^2/κ as time scale, where κ is the thermal diffusivity of the fluid, and $(T_2-T_1)/R$ as temperature scale, where T_1 and T_2 are the temperatures at top and bottom boundaries, we can write the basic equations of motion and the heat equation for the dimensionless deviation Θ from the temperature distribution of pure conduction in the form

$$P^{-1}\left(\frac{\partial}{\partial t}\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)+\tau\boldsymbol{k}\times\boldsymbol{u}=-\nabla\pi+\boldsymbol{k}\Theta+\nabla^2\boldsymbol{u}, \quad (1a)$$

$$\nabla\cdot\boldsymbol{u}=0, \quad (1b)$$

$$\frac{\partial}{\partial t}\Theta+\boldsymbol{u}\cdot\nabla\Theta=R\boldsymbol{k}\cdot\boldsymbol{u}+\nabla^2\Theta, \quad (1c)$$

where $\boldsymbol{k}$ is the vertical unit vector opposite to the direction of gravity and where the Rayleigh number R, the Prandtl number P and the rotation parameter τ are defined by

$$R=\frac{\gamma(T_2-T_1)gh^3}{\nu\kappa},\quad P=\frac{\nu}{\kappa},\quad \tau=\frac{2\Omega h^2}{\nu}. \quad (2)$$

Here γ denotes the coefficient of thermal expansion, g is the acceleration of gravity and ν is the kinematic viscosity of the fluid. The Boussinesq approximation has been assumed in that the variation of density is taken into account only in the gravity term. All terms that can be written as gradients in Eq. (1a) have been combined in $\nabla\pi$. Using a Cartesian system of coordinates with the z-coordinate in the direction of $\boldsymbol{k}$ we can prescribe the conditions at the upper and lower stress free boundaries in the form

$$\frac{\partial}{\partial z}\boldsymbol{u}\times\boldsymbol{k}=0,\quad \boldsymbol{u}\cdot\boldsymbol{k}=\Theta=0 \text{ at } z=\pm\frac{1}{2}. \quad (3)$$

In order to eliminate the constraint (1b) we use the general representation for solenoidal vector fields,

$$\boldsymbol{u} = \nabla\times(\nabla\,\phi\times\boldsymbol{k}) + \nabla\,\psi\times\boldsymbol{k} \equiv \boldsymbol{\delta}\phi + \boldsymbol{\epsilon}\psi \tag{4}$$

in the absence of a mean flow. Equations for the scalar functions ϕ and ψ are obtained in the form of the z-components of the $(\mathrm{curl})^2$ and of the curl of Eq. (1a),

$$P^{-1}\left(\frac{\partial}{\partial t}\nabla^2\Delta_2\phi + \boldsymbol{\delta}\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\right) + \tau\boldsymbol{k}\cdot\nabla\Delta_2\psi = -R\Delta_2\Theta + \nabla^4\Delta_2\phi, \tag{5a}$$

$$P^{-1}\left(\frac{\partial}{\partial t}\Delta_2\psi + \boldsymbol{\epsilon}\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\right) - \tau\boldsymbol{k}\cdot\nabla\Delta_2\phi = \nabla^2\Delta_2\psi, \tag{5b}$$

where Δ_2 denotes the horizontal Laplacian, $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The heat equation (1c) can be separated into its horizontally averaged part and its fluctuating part,

$$\frac{\partial}{\partial t}\overline{\Theta} - \frac{\partial}{\partial z}(\overline{\Theta\Delta_2\phi}) = \frac{\partial^2}{\partial z^2}\overline{\Theta}, \tag{5c}$$

$$\frac{\partial}{\partial t}\check{\Theta} + (\boldsymbol{\delta}\phi + \boldsymbol{\epsilon}\psi)\cdot\nabla(\overline{\Theta} + \check{\Theta}) + \frac{\partial}{\partial z}(\overline{\check{\Theta}\Delta_2\phi}) = -R\Delta_2\phi + \nabla^2\check{\Theta}, \tag{5d}$$

where the bar indicates the horizontal average and $\check{\Theta}$ is defined by $\check{\Theta} \equiv \Theta - \overline{\Theta}$.

Since we are interested in turbulent convection under stationary conditions we define this state by the property that horizontally averaged quantities are time independent. This property allows us to integrate (5c) with respect to z,

$$\frac{\partial}{\partial z}\overline{\Theta} = -\overline{\check{\Theta}\Delta_2\phi} + \langle\check{\Theta}\Delta_2\phi\rangle, \tag{6}$$

where the constant of integration has been determined through the use of the boundary condition (3) for Θ and where the angular brackets denote the average over the fluid layer. By multiplying Eqs. (5a), (5b), and (5d) by ϕ, ψ, and $\check{\Theta}$, respectively, and averaging the result over the fluid layer we obtain

$$\langle|\boldsymbol{k}\times\nabla\nabla^2\phi|\rangle + \tau\left\langle\Delta_2\phi\frac{\partial}{\partial z}\psi\right\rangle + P^{-1}\langle\boldsymbol{\delta}\phi\cdot[(\boldsymbol{\delta}\phi + \boldsymbol{\epsilon}\psi)\cdot\nabla]\boldsymbol{\epsilon}\psi\rangle = -\langle\check{\Theta}\Delta_2\phi\rangle \tag{7a}$$

$$\langle|\boldsymbol{k}\times\nabla\nabla\psi|^2\rangle - \tau\left\langle\Delta_2\phi\frac{\partial}{\partial z}\psi\right\rangle - P^{-1}\langle\boldsymbol{\delta}\phi\cdot[(\boldsymbol{\delta}\phi + \boldsymbol{\epsilon}\psi)\cdot\nabla]\boldsymbol{\epsilon}\psi\rangle = 0 \tag{7b}$$

$$\langle|\nabla\check{\Theta}|^2\rangle + \langle(\langle\check{\Theta}\Delta_2\phi\rangle - \overline{\check{\Theta}\Delta_2\phi})^2\rangle = -R\langle\check{\Theta}\Delta_2\phi\rangle. \tag{7c}$$

These energy relationships or power integrals are the same as those used by [1] and [8] except that the relationships for poloidal and toroidal components have been written separately. When Eqs. (7a) and (7b) are added, the parameters τ and P drop from the problem as has been the case in the earlier work. Here we wish to satisfy Eqs. (7a) and (7b) separately and therefore must introduce the additional constraint with the Lagrange multiplying factor λ into the variational problem. The variational problem to be considered in the following can thus be formulated:

For given values of the parameters P, τ and $\mu > 0$ find the minimum $R(\mu, P, \tau)$ of the variational functional

$$\mathcal{R}(\phi^*, \psi^*, \theta^*; \mu, P, \tau) \equiv \mathcal{R}_1 + \lambda\left(\mathcal{R}_2 + \frac{\sqrt{\mu}}{P}\mathcal{R}_3\right) \tag{8}$$

among all fields ϕ^*, ψ^*, θ^* satisfying the conditions $\phi^* = \partial^2\phi^*/\partial z^2 = \partial\psi^*/\partial z = \theta^* = 0$ at $z = \pm\frac{1}{2}$.

In expression (8) the definitions

$$\mathcal{R}_1 \equiv \frac{(\langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle + \langle|\boldsymbol{k}\times\nabla\nabla^2\phi^*|^2\rangle)\langle|\nabla\theta^*|^2\rangle + \mu\langle(\overline{\theta^*\Delta_2\phi^*} - \langle\theta^*\Delta_2\phi^*\rangle)^2\rangle}{\langle\theta^*\Delta_2\phi^*\rangle^2}, \tag{9a}$$

$$\mathcal{R}_2 \equiv \frac{\langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle - \tau\langle\Delta_2\phi^*\partial\psi^*/\partial z\rangle}{\langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle + \langle|\boldsymbol{k}\times\nabla\nabla^2\phi^*|^2\rangle}, \tag{9b}$$

$$\mathcal{R}_3 \equiv \frac{\langle\boldsymbol{\epsilon}\psi^*\cdot[(\boldsymbol{\epsilon}\psi^* + \boldsymbol{\delta}\phi^*)\cdot\nabla]\boldsymbol{\delta}\phi^*\rangle}{[\langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle + \langle|\boldsymbol{k}\times\nabla\nabla^2\phi^*|^2\rangle]^{3/2}} \tag{9c}$$

have been used. The variational functional is homogeneous in θ^* as well as in the fields ϕ^*, ψ^*. In the case $\psi^* \equiv 0$ the functional considered in [1] and [8] is recovered. After the minimum of the functional (8) has been determined one can fix the amplitude of the minimizing fields θ^*, ϕ^*, ψ^* by setting

$$\mu = -\langle\theta^*\Delta_2\phi^*\rangle = \langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle + \langle|\boldsymbol{k}\times\nabla\nabla^2\phi^*|^2\rangle. \tag{10}$$

Through this normalization one satisfies the sum of Eqs. (7a) and (7b) while the vanishing of the bracket multiplying λ in expression (8) corresponds to Eq. (7b) and relationship (8) yields (7c) with the Rayleigh number R as the minimizing value $R(\mu, P, \tau)$.

It may be illuminating to interpret the variational problem in a different way. λ can be considered as a parameter of the problem and the minimum $R_\lambda(\mu,P,\tau)$ can be determined for fixed values of λ. The Euler-Lagrange equations (11) derived below determine the maximum among the minima $R_\lambda(\mu,P,\tau)$. This maximum of R_λ obviously yields the best bound on the functional in the case of a finite value of $\langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle$. The difficulty of the formulation (8) of the variational problem is that the lowest value of R is obtained in the case $\langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle=0$ corresponding to the upper bound derived in [8]. But this solution is not possible as solution of the Euler-Lagrange equations given below for a given value $\tau\neq0$ unless $\lambda=0$ and it would not correspond to a maximum of R_λ. To avoid the mathematical difficulty with the singular case $\langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle=0$ it is convenient to disregard this possibility as unphysical since any physically realized convection will have $\langle|\boldsymbol{k}\times\nabla\nabla\psi^*|^2\rangle\neq0$ for $\tau\neq0$.

As in earlier formulations [1,8] of the variational problem, the minimum of (8) for a given value of μ is equivalent to the maximum μ for a given value of R as long as the relationship $R(\mu,P,\tau)$ is monotonous for fixed values of P,τ. In analyzing the variational problem (9) we shall restrict ourselves to those regions where such a monotony holds.

III. EULER-LAGRANGE EQUATIONS AND NUMERICAL METHODS

The Euler-Lagrange equations corresponding to a stationary value of the functional (8) can be written in the form

$$\nabla^4\Delta_2\phi\left(-\eta+\lambda R_2+3\frac{\lambda\sqrt{\mu}}{2P}R_3\right)+\Delta_2\theta(R_1+\overline{\theta\Delta_2\phi}-\langle\theta\Delta_2\phi\rangle)+\frac{\lambda}{2}\left[-\tau\Delta_2\frac{\partial}{\partial z}\psi+\frac{1}{P}\{\boldsymbol{\delta}\cdot(\boldsymbol{\epsilon}\psi\cdot\boldsymbol{\delta})\nabla\phi-\boldsymbol{\delta}\cdot[(\boldsymbol{\epsilon}\psi+\boldsymbol{\delta}\phi)\cdot\nabla]\boldsymbol{\epsilon}\psi\}\right]=0 \tag{11a}$$

$$\nabla^2\Delta_2\psi\left[\eta+\lambda\left(1-R_2-3\frac{\sqrt{\mu}}{2P}R_3\right)\right]+\frac{\lambda}{2}\left[\tau\frac{\partial}{\partial z}\Delta_2\phi-\frac{1}{P}\{\boldsymbol{\epsilon}\cdot(\boldsymbol{\epsilon}\psi\cdot\boldsymbol{\delta})\nabla\phi+\boldsymbol{\epsilon}\cdot[(\boldsymbol{\epsilon}\psi+\boldsymbol{\delta}\phi)\cdot\nabla]\boldsymbol{\delta}\phi\}\right]=0 \tag{11b}$$

$$\nabla^2\theta-\Delta_2\phi(R_1+\overline{\theta\Delta_2\phi}-\langle\theta\Delta_2\phi\rangle)=0, \tag{11c}$$

where the normalization conditions (10) have been used, where η denotes the ratio $\langle|\nabla\theta|^2\rangle/\mu$ and where R_1,R_2,R_3 refer to the values of the functionals (9a),(9b),(9c), respectively, for the fields that satisfy the Euler-Lagrange equations. To simplify the notation the superscript * on the variables ϕ, ψ, and θ has been dropped. From the variation of the functional with respect to λ one obtains

$$\langle|\boldsymbol{k}\times\nabla\nabla\psi|^2\rangle=\tau\langle\Delta_2\phi\partial\psi/\partial z\rangle-\frac{1}{P}\langle\boldsymbol{\epsilon}\psi\cdot[(\boldsymbol{\epsilon}\psi+\boldsymbol{\delta}\phi)\cdot\nabla]\boldsymbol{\delta}\phi\rangle. \tag{11d}$$

A consequence of Eqs. (11a) and (11b) is the simple integral balance

$$(\langle|\boldsymbol{k}\times\nabla\nabla\psi|^2\rangle+\langle|\boldsymbol{k}\times\nabla\nabla^2\phi|^2\rangle)\eta+\langle\theta\Delta_2\phi(R_1+\overline{\theta\Delta_2\phi}-\langle\theta\Delta_2\phi\rangle)\rangle=0 \tag{12}$$

which is equivalent to the power integrals corresponding to (7c) and the sum of (7a) and (7b).

In order to solve equations (11) numerically we introduce the general Galerkin ansatz suggested by the form of the equations

$$\phi=\sum_{p=1}^{N}\varphi_p(x,y)A_p(z),\quad \psi=\sum_{p=1}^{N}\varphi_p(x,y)B_p(z),$$
$$\theta=\sum_{p=1}^{N}\varphi_p(x,y)T_p(z), \tag{13a}$$

where the boundary conditions for ϕ, ψ and θ can be satisfied through the choice

$$A_p=\sum_{q=1}^{M}a_{pq}\sin q\pi\left(z+\frac{1}{2}\right),$$

$$B_p=\sum_{q=0}^{M}b_{pq}\cos q\pi\left(z+\frac{1}{2}\right),\quad T_p=\sum_{q=1}^{M}t_{pq}\sin q\pi\left(z+\frac{1}{2}\right) \tag{13b}$$

for the z-dependence. The summation limits N,M should tend to infinity in order to approximate the exact solution of Eqs. (11). In reality finite values are required by the finite capacity of computers. But through a comparison of results obtained for different M and N the quality of approximation can readily be checked. The Euler-Lagrange equations (11) determine those fields for which the variational functional (8) assumes a stationary value with respect to small variations. There will be an infinite manifold of these solutions in general, but only those corresponding to lowest minimum $R(\mu,P,\tau)$ for given values of μ,P,τ are of interest. Since it appears to be impossible to scan the entire infinite manifold of solutions for the absolute minimum it seems justified to choose certain types of solutions which appear to be most likely to correspond to the extremal value of the functional. One of these is the two-dimensional type of solution which is distinguished by its simplicity. The two-dimensional solution corresponds closely to the solution in the case without separate toroidal and poloidal energy balances in which case the horizontal structure of the extremalizing vector field remains unspecified. The horizontal pattern remains unspecified even in the special limit $P\rightarrow\infty$ where the equation of motion becomes linear and can be included as constraint for the extremalizing vector field. In this way asymptotic solutions of the variational problem in which the rotation parameter has entered have been produced in [9], [10], and [11].

For the two-dimensional solution the terms proportional to P^{-1} in Eqs. (11) vanish and thus the P-dependence of extremum of the functional (8) disappears. It is sufficient in this case to consider the case $N=1$ with $\varphi_1(x,y)$ satisfying

$\Delta_2\varphi_1 = -\alpha^2\varphi_1$ and the main task is to find the minimum of R as function of the wave number α for various values of τ. These single-α-solutions are known to provide the upper bound at low values of R. At higher values of R multi-α-solutions must also be admitted for the competition [7], but we shall not consider those in this paper.

In order to include the P-dependence three-dimensional solutions must be considered. Because of the horizontal isotropy of the problem it is reasonable to restrict the attention to those solutions which do not exhibit a single preferred direction in the x,y-plane. Among these solutions those with a hexagonal symmetry will give maximal contributions from the quadratic terms characterized by the factor P^{-1}. We are thus led to the ansatz

$$\varphi_p(x,y) = d_p \sum_{j,l,m} \cos(\boldsymbol{K}_{jlm}\cdot\boldsymbol{r}) \tag{14}$$

$$\text{with } |\boldsymbol{K}_{jlm}| = a(p),$$

where the three basic wave vectors

$$\boldsymbol{k}_1 = (\alpha,0,0), \quad \boldsymbol{k}_2 = (-\alpha/2,\beta,0), \quad \boldsymbol{k}_3 = (-\alpha/2,-\beta,0) \tag{15}$$

of the hexagonal lattice have been introduced with $\beta = \alpha\sqrt{3}/2$ and where $\boldsymbol{K}_{jlm}$ and $a(p)$ are defined by

$$\boldsymbol{K}_{jlm} = j\boldsymbol{k}_1 + l\boldsymbol{k}_2 + m\boldsymbol{k}_3, \tag{16}$$

$$a(p) = \frac{1}{2}\{(p+1)^2\alpha^2 + [1+(-1)^p]^2\beta^2\}^{1/2}. \tag{17}$$

The subscripts j,l,m run through positive and negative integers, but cases for which $\boldsymbol{K}_{jlm} = \pm\boldsymbol{K}_{qrs}$ with either $j\neq q$ or $r\neq l$ or $m\neq s$ are excluded in the summation of expression (14). With the expression $a(p)$ all possible vectors $\boldsymbol{K}_{jlm}$ with $\boldsymbol{K}_{jlm}\leqslant a(5)$ can be collected. This property will be sufficient for the numerical analysis since the summation in expressions (13) will be truncated at $N\leqslant 5$.

After the Galerkin representation (13) has been introduced into the Euler-Lagrange equations (11) and these equations have been projected onto the set of expansion functions, a system of nonlinear algebraic equations for the unknown coefficients a_{pq}, b_{pq} and t_{pq} is obtained which can be solved by a Newton-Raphson iteration method once the parameters R, P, τ and α have been prescribed. The preferred wave number α is determined through a maximization of the convective heat transport, $\mu = -\langle\theta\Delta_2\phi\rangle$, as a function of α.

IV. EXTREMALIZING SOLUTIONS OF THE EULER-LAGRANGE EQUATIONS

The structure of the solution space shows some similarity with the Rayleigh-Bénard problem of physically realized convection in a layer heated from below. In this case flows in the form of rolls or hexagonal cells also compete with the difference that the relevant triple integrals arise from deviations from the Boussinesq approximation [12]. In both, the non-Boussinesq and the upper-bound problem, the heat transport of rolls always exceeds the heat transport by hexagonal flows for sufficiently high values of R.

There is another interesting relationship between the variational problem (8) and the physically realized convection. In the limit $\mu\to 0$ equations (11) reduce to the linear set of equations which determine the onset of convection in a rotating layer and the minimum value of R is given by the relationship

$$R_c = R(0,P,\tau) = [\tau^2\pi^2 + (\pi^2+\alpha_c^2)^3]/\alpha_c^2 \tag{18a}$$

where the minimizing value α_c^2 is determined as the real solution of the cubic equation

$$(2\alpha^2 - \pi^2)(\pi^2+\alpha^2)^2 = \tau^2\pi^2. \tag{18b}$$

The corresponding extremalizing fields ϕ^*,ψ^*,θ^* in the two-dimensional case are given by

$$\phi = \cos\alpha x \cos\pi z, \quad \psi = \frac{-\tau\pi}{\pi^2+\alpha^2}\cos\alpha x \sin\pi z,$$

$$\theta = \frac{\alpha^2}{\pi^2+\alpha^2}\cos\alpha x\cos\pi z. \tag{19}$$

Tables for R_c, α_c as a function of τ^2 can be found in Chandrasekhar's book [13].

The result (18a) seems to be in contradiction with the fact that oscillatory convection can set in at Rayleigh numbers less than the critical value (18a) if only the Prandtl number P is low enough [13]. The resolution of this paradox arises from the fact that a potential coherent time dependence of the extremalizing fields of the variational problem has been neglected in the derivation of the Euler-Lagrange equations (11). In particular, the correlation between $\partial_z\psi$ and $\Delta_2\phi$ could be much less than determined by Eqs. (11) if a time dependence with phase shift between the two quantities is permitted. The analogy with the known solutions of the basic equations of convection suggests that the oscillatory form of convection carries only a low amount of heat and is superseded by subcritical finite amplitude convection at values of R slightly beyond its onset (see, for example, [14]). We thus feel satisfied to neglect extremalizing fields which exhibit a coherent time dependence. Instead we shall continue to use the time independent Euler-Lagrange equations (11) which provide the upper bound of the heat transport carried by stationary convection or by turbulent convection with an incoherent time dependence.

We start our discussion of the extremalizing solutions of the variational problem (8) by considering solutions of the roll type. In Figs 1(a) and 1(b), such solutions are plotted for various values of τ^2 for a given value of R. Since for $\tau^2 = 1500$ the critical value R_c of R is 2007, the extremalizing solution does not differ yet very much from the form (19) of the trigonometric functions assumed at $R=R_c$. For lower values of τ^2 the fixed value of R represents a higher multiple of the critical value R_c and a gradual evolution towards a boundary layering form of the solution can be noticed. Es-

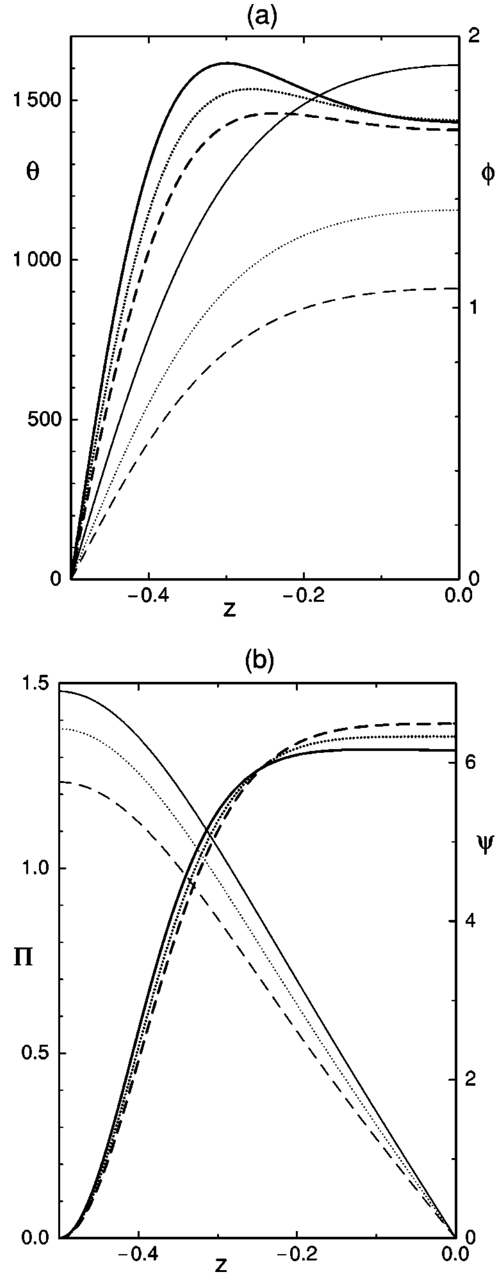

FIG. 1. (a) The z-dependences of the extremalizing fields θ (thick lines, left ordinate) and ϕ (thin lines, right ordinate) of the roll solutions for $R=5000$ in the cases $\tau^2=500$ (solid lines), $\tau^2=1000$ (dotted lines) and $\tau^2=1500$ (dashed lines). (b) The function $\Pi(z)\equiv\overline{\theta\Delta_2\phi}/\langle\theta\Delta_2\phi\rangle$ (thick lines) and the z-dependence of ψ for the same parameters as in Fig. 1(a).

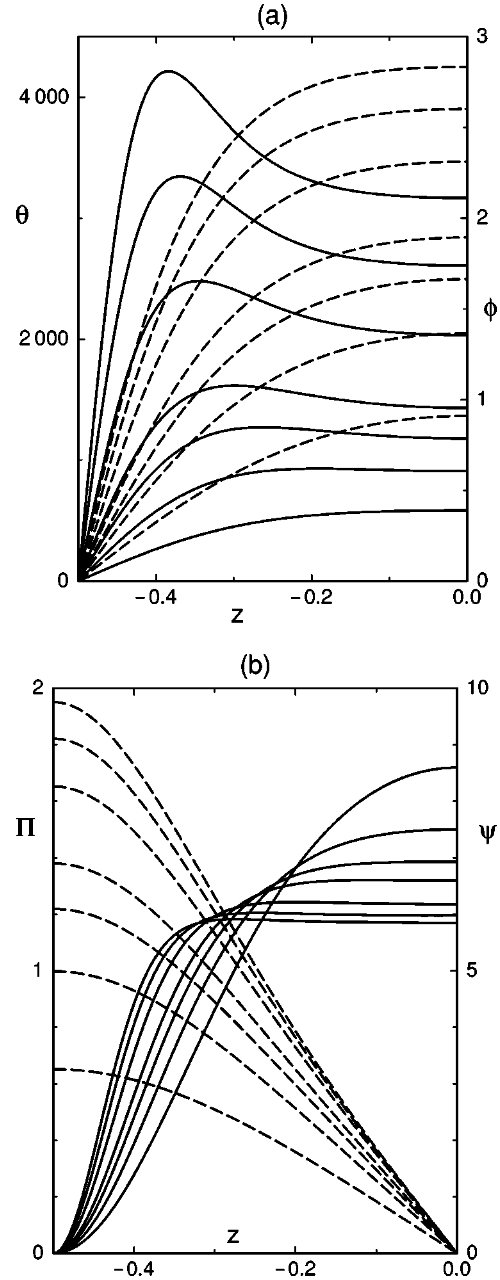

FIG. 2. (a) The z-dependences of the extremalizing fields θ (solid lines, left ordinate) and ψ (dashed lines, right ordinate) of the roll solution for $\tau^2=500$ in the cases $R=2\times10^3$, 3×10^3, 4×10^3, 5×10^3, 7.5×10^3, 10^4, 1.25×10^4 (from top to bottom). (b) The function $\Pi(z)\equiv\overline{\theta\Delta_2\phi}/\langle\theta\Delta_2\phi\rangle$ (solid lines) and the z-dependence of ψ for the same parameters as in Fig 2(a).

pecially the maximum of $T_1(z)$ which resides at $z=0$ for $\tau^2=1500$, splits into two maxima which move towards the boundaries as τ^2 decreases. This same feature is apparent at a fixed value of τ^2 when R increases as shown in Fig. 2(a).

In order to maximize the convective heat transport the function $\Pi(z)\equiv\overline{\theta\Delta_2\phi}/\langle\theta\Delta_2\phi\rangle$ must approach a constant value in the interior of the layer while keeping its rise from zero at the boundaries sufficiently smooth such that the dissipation of the fluctuating variables does not contribute too much in the functional (8). This tendency can already be noticed in Fig. 1(b), but is clearly exhibited in Fig. 2(b). As the boundary layers develop with increasing R the horizontal length scale of the roll-like solutions decreases as is evident from the increasing value of the wave number α shown in Fig 3. A power law dependence on R can be discerned at high values of R similar to the dependence $\alpha\sim R^{3/8}$ found in the nonrotating case [1,7].

The main results of the analysis is the upper bound μ for the heat transport by convection as a function of τ^2 and R. In Fig. 4 results for the upper bound obtained on the basis of the

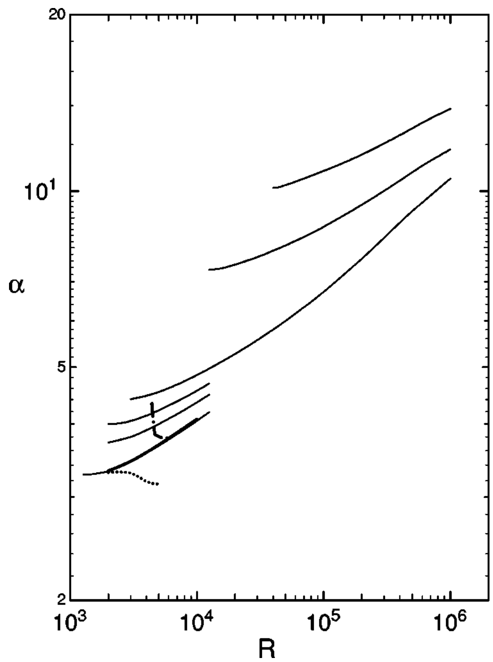

FIG. 3. The wave number α of the extremalizing roll solution in the cases $\tau^2=250\,000$, 40 000, 2500, 1500, 1000, and 500 (thin solid lines, from top to bottom) and the wave number of the extremalizing solution in the case τ^2 for $P=0.7$ (dotted line) and for $P=7$ (thick solid line) and in the case $\tau^2=40\,000$, $P=0.025$ (dashed-dotted line).

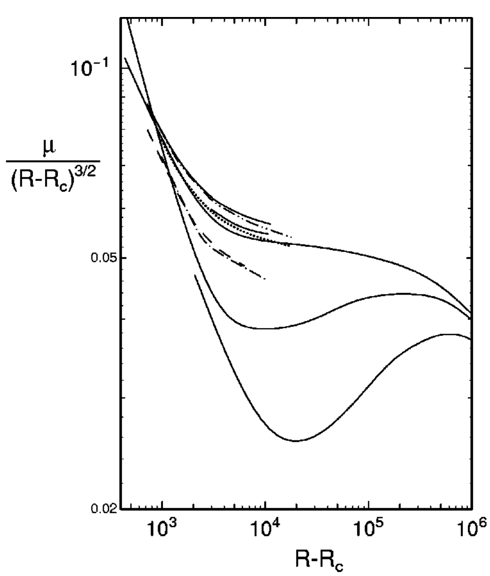

FIG. 4. The upper bound μ for the convective heat transport as a function of R in the case of the extremalizing roll solution (solid lines) for $\tau^2=500,1500,2500,4\times10^4,25\times10^4$ (from top to bottom). Also shown is μ for the extremalizing hexagon solution for $P=7$ (dash-double-dotted line for $\tau^2=500$ and dotted line for $\tau^2=1500$) and for $P=1$ (dashed line for $\tau^2=500$ and dash-dotted line for $\tau^2=1500$). The values of R_c are 1275, 2007, 2564, 12 135, 37 915 for $\tau^2=500,1500,2500,4\times10^4,25\times10^4$, respectively.

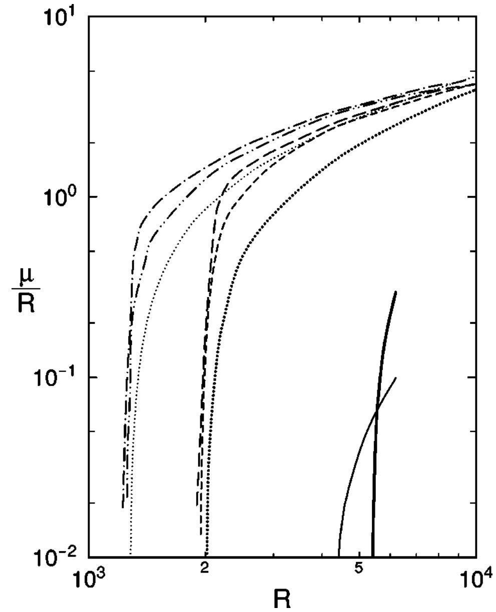

FIG. 5. The upper bound μ for the convective heat transport by the hexagon solution for $P=0.1(0.0247)$ indicated by a dash-double-dotted (dash-dotted) line for $\tau^2=500$ and a short-(long) dashed line for $\tau^2=1500$. The thin solid line indicates the upper bound for $P=0.0247$ and $\tau^2=10^4$. For comparison the upper bounds given by rolls for $\tau^2=500,1500,10^4$ are indicated by the thin dotted line, the thick dotted line and the thick solid line, respectively.

roll solutions have been plotted in the form of thin solid lines which from top to bottom correspond to increasing values of τ^2. It is evident from the figure that for an intermediate range of Rayleigh numbers the bound is significantly lowered in the case of strongly rotating layers. This result is remarkable in that it holds as a function of $R-R_c$ which implies that the inhibiting effect of the Coriolis force included in the linear theory has already been subtracted. For high values of R the upper bounds tend to converge as is clearly apparent in the figure. Since only single-α-solutions have been considered it must be expected that these upper bounds are replaced by those corresponding to two-α-solutions for R of the order 10^6 and higher [7].

Also included in Fig. 4 are upper bounds obtained on the basis of the hexagon solutions. For the Prandtl numbers used in the figure the hexagon bounds are always lower than the bounds obtained for the roll solution and they are thus not physically relevant. The upper bound for the heat transport is given by the hexagon solutions only for Prandtl number less than unity and only for a limited range of Rayleigh numbers. These properties are demonstrated in Fig. 5 where upper bounds provided by the hexagon solutions are shown for the Prandtl numbers 0.1 and 0.0247. The example of $\tau^2=10^4$ and $P=0.0247$ is particularly instructive since the hexagon upper bound extends to Rayleigh numbers much below the critical value R_c. The latter value is indicated by the vanishing of the maximum convective heat transport of the roll solution. At Rayleigh numbers not far beyond the critical value the latter transport already exceeds the hexagon value and thus provides the upper bound. For lower values of τ^2

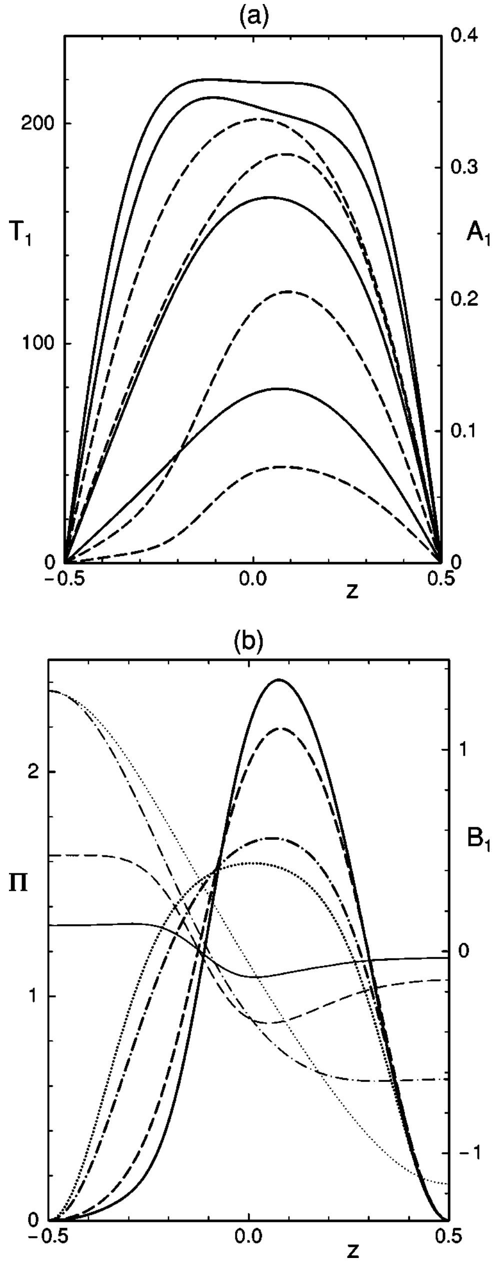

FIG. 6. (a) The z-dependences $T_1(z)$ (solid lines, left ordinate) and $A_1(z)$ (dashed lines, right ordinate) of the extremalizing hexagon solution in the case $R=2500$, $\tau^2=500$ for $P=0.0247,0.1,0.7,7$ (from bottom to top). (b) The functions $\Pi(z)\equiv\overline{\theta\Delta_2\phi}/\langle\theta\Delta_2\phi\rangle$ (thick lines, left ordinate) and $B_1(z)$ (thin lines, right ordinate) in the same cases as in Fig 6(a). Solid, dashed, dash-dotted and dotted lines correspond to $P=0.0247,0.1,0.7,7$, respectively.

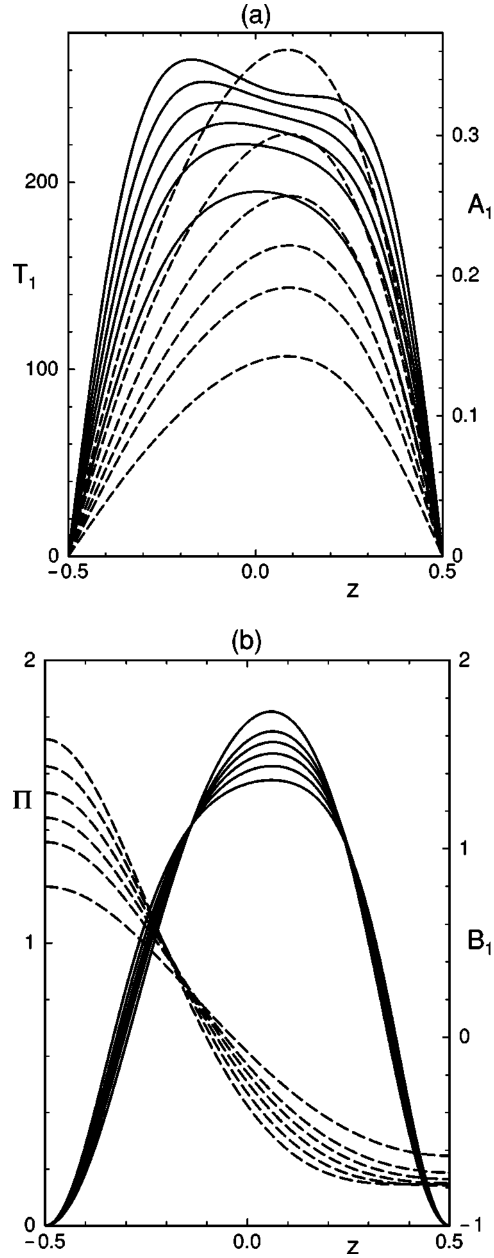

FIG. 7. (a) The z-dependences $T_1(z)$ (solid lines, left ordinate) and $A_1(z)$ (dashed lines, right ordinate) for the extremalizing hexagon solution in the case $R=3000$, $P=1$ for $\tau^2=500$, 750, 1000, 1250, 1500, 2000 (from top to bottom). (b) The functions $\Pi(z)\equiv\overline{\theta\Delta_2\phi}/\langle\theta\Delta_2\phi\rangle$ (solid lines, left ordinate) and $B_1(z)$ (thin lines, right ordinate) for the same parameters as in Fig. 7(a) (from bottom to top at $z=0$).

the subcritical extent of the hexagon upper bound is not quite as dramatic. But the range of Rayleigh numbers for which the hexagon upper bound exceeds the maximum convective heat transport of rolls is larger.

Because of its three-dimensional nature and its dependence on the Prandtl number P as additional parameter, the hexagon solutions of the Euler-Lagrange equations (11) are more difficult to investigate than the roll solutions and the range of R for which numerically converged solutions can be obtained is more limited. There are always two hexagon solutions corresponding to an extremum of the functional (8). They differ by the sign of their asymmetric components. The asymmetry is clearly evident in the z-dependences of $A_1(z)$, $T_1(z)$ and $\Pi(z)$ shown in Fig. 6. The function ψ which for rolls is antisymmetric with respect to $z=0$ acquires a symmetric part in the case of the hexagon solution as is demonstrated by the plots of its dominant contribution $B_1(z)$ in Fig. 6(b). The asymmetry increases with decreasing Prandtl number P and with increasing R, at least in the range where computations have been done [Fig. 7].

V. CONCLUDING REMARKS

The upper bound problem for the turbulent heat transport considered in this paper is unusual in several respects. For the first time, to the best of our knowledge, it has been shown that extremalizing vector fields with a three-dimensional structure are required to provide the upper bound. It has not been possible to prove in any rigorous way that the vector fields with hexagonal symmetry studied in this paper are indeed the extremalizing ones. But from the structure of the variational functional (8) it appears that this hypothesis is probably true since otherwise the triple integrals in functional (8) do not contribute in a most symmetric way. From the formulation of the upper-bound problem we had to except the possibility of coherent oscillation in time since otherwise the assumption of the time independence of spatially averaged quantities does not hold. It will be of interest to attempt in the future different formulations of the upper-bound problem which would include the coherent convection oscillations of a rotating fluid layer. Of course, the general upper bounds of Howard [1] and Busse [8] which hold independent of the rotation rate of the layer also hold in the case of coherent oscillations.

The analysis of the present paper should be clearly distinguished from energy bounds on the Rayleigh number for the onset of convection in a rotating layer ([15]; see also [16]). In this latter work bounds from below as a function of the amplitude of disturbances are found for the Rayleigh number beyond which growing disturbances become possible. As must be expected because of the possibility of subcritical onset of convection, the bounds decrease rapidly with growing amplitude of the disturbance.

The main result of the present paper is that for high rotation rate significantly lower bounds on the convective heat transport can be obtained through the consideration of separate energy balances for poloidal and toroidal components of the velocity field. For given values of the rotation parameter τ^2 the upper bound tends to approach the bound in the case $\tau^2=0$ when the Rayleigh number becomes sufficiently large. This feature agrees with the geophysical and astrophysical evidence that strongly convecting systems are not much influenced by the effects of rotation if only the Rayleigh number is high enough.

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