Logistic map with a delayed feedback: Stability of a discrete time-delay control of chaos

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The logistic map with a delayed feedback is studied as a generic model. The stability of the model and its bifurcation scheme is analyzed as a function of the feedback amplitude and of the delay. Stability analysis is performed semianalytically. A relation between the delay and the periodicity of the orbit, which explains why some terms used in chaos control are ineffective, was found. The consequences for chaos control are discussed. The structure of bifurcations is found to depend strongly on the parity and on the length of the delay. Boundary crisis, the tangent, the Neimark, as well as the period-doubling bifurcations occur in this system. The effective dimension of the model is also discussed.

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I. INTRODUCTION

In recent years a growing interest in delayed equations: both differential [1–8] and difference [9–12], may be observed. They not only offer new modeling possibilities but also give efficient methods of chaos control requiring nearly no *a priori* knowledge about the controlled system (TDAS [13], ETDAS [14], as well as their various extensions [15–18]). However a number of open problems still exists in this area. One of them, the aim of this paper, is to show how the introduction of the delay alters the properties of the phase space of a simple discrete dynamical system. As the delay introduces new dimensions to the phase space of the system, the question arises does it affect also the effective dimensionality of the system? Note also that, as pointed out by Ott [19], for an irreversible discrete system introducing new dimensions to the system (as done by means of the delay term) enables reversibility. The boundary between the one and two-dimensional systems has already been studied in the Henon map [20], which may be considered a form of delayed logistic map (see below).

Another problem of interest is the theoretical description of the methods of chaos control by delayed feedback. In particular, in relation to the TDAS and ETDAS chaos control methods, how does the stability range depend on the delay length and the feedback amplitude and what may actually be the mechanism of the chaos control by delayed feedback [21–26]?

As the base map, we used the logistic map [27] but the model of the one-dimensional map with delay used here is generic.

The organization of the paper is as follows. Section II introduces the delayed systems, cites known results about them, and poses some questions. Section III describes the system and the way in which stability analysis was performed. The results of the numerical stability analysis are presented and discussed in Sec. IV. Section V shows the results interesting for chaos control. Section VI is devoted to the problem of the effective dimension of the system, whereas Sec. VII contains the conclusions.

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II. MEMORY IN DYNAMICAL SYSTEMS

Typically, in dynamical systems, we assume that the state of the system at the moment $t+dt$ depends only on its state at the moment t, i.e., that all causes have an instantaneous effect. This condition seems to be met in a large number of systems that may be described by differential equations. But problems arise if we have a system in which: (1) the information is fed back with a measurable delay, e.g., due to the spatial extensiveness of the system and the finite velocity of propagation of information, and (2) the characteristic timescale of the system is smaller than the delay time so that we cannot simply model all the changes as instantaneous. The delayed equations were used for modeling purposes in optics [1,2], chemistry [3], and biological systems [4,5]. Also in a number of problems of statistical physics, the so-called systems with memory emerge as important and yet directly unsolvable (see Ref. [9] and references therein).

Apart from modeling, time delays are used in different applications. A delayed feedback loop is used in digital filter design [28]. As negative feedback without a delay is able to stabilize the system in control engineering (e.g., Ref. [29]), the control of chaos by the delayed feedback [13–18] seems a natural extension. In the analysis of the stability of chaos, control discrete maps have been used as test cases for their simplicity [30].

In continuous-time systems, the delay formally introduces an infinite number of dimensions [31]. The dimension of the attractor [32] is, however, finite and rises linearly with the delay [31,33]. On the other hand, the metric entropy, being a measure of the complexity of the system, is nearly a constant function of the delay time [31,33]. Also the largest Lyapunov exponent is an exponentially decreasing function of the delay time [31]. So what actually is the effective dimension of this system? Is it rising linearly with the delay—as the dimension of the attractor does, or is it constant—as indicated by the constant entropy?

In discrete-time systems, the number of dimensions is equal to the length of the delay [9] so the phase space of the system may be formally high dimensional but finite. This makes calculation of the Lyapunov exponents much easier. It seems that a delayed discrete system, being much simpler to analyze, provides a good model to resolve these contradictions.

As the delayed feedback may serve as a method of chaos control, does it really increase the effective dimension of the system? As pointed out by Chen [34], the delayed feedback introduced for the suppression of chaos indeed introduces unwanted new bifurcations, typical for multidimensional systems (e.g., Neimark bifurcation [35,11,36]). However, as we look at the delayed systems from the point of view of chaos control, the feedback seems to act rather as a stabilizing factor than as new degrees of freedom.

III. THE MODEL AND ITS STABILITY

A. Description of the model

We implement the memory term into the logistic map in the form of an additional ''echo type'' feedback loop:

$$x(n+1)=(1-K)\,rx(n)\,[1-x(n)]+Kx\,(n-k), \quad (1)$$

where K is the feedback amplitude and k is the delay length. This way of introduction of the delayed term seems the most general.

The map (1) may be formally expanded [9] to a set of $D=k+1$ coupled equations spanning D-dimensional phase space:

$$\begin{aligned} x_1(n+1)&=(1-K)\,rx_1(n)\,[1-x_1(n)]+Kx_2(n),\\ x_2(n+1)&=x_3(n),\\ &\dots\\ x_{k+1}(n+1)&=x_1(n). \end{aligned} \quad (2)$$

Note that only in one of D dimensions the variable x_1 undergoes nonlinear stretching and folding. The other variables x_2 to x_{k+1} are simply mapped onto one another. This is equivalent to a rotation around the axis x_1 of the coordinate system. The change of variables: $\xi=2x_1-1$ and $\eta=Kx_2$, for k equal to 1, transforms Eq. (2) to the Henon map.

As the delayed system is a multidimensional system, instead of the one initial condition, a vector should be introduced [31]. If the system were multistable, the choice of the initial condition would be important. The multistability of a certain type of delayed map was indeed reported [12] but for the equation studied here it has not been shown so far.

The fixed-point solutions of Eq. (2) are the same as for the nondelayed logistic map: a nontrivial one at $x_1^*=x_2^*=1-1/r$ and a trivial one at $x_1^*=x_2^*=0$. The Jacobi matrix has the simple form

$$J(r,K)=\begin{bmatrix} b(r) & K & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ 0 & 0 & 0 & .. & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & 1\\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (3)$$

As $\det(J)=K$ the system is dissipative for $|K|<1$. The interesting case of $|K|>1$ when the system expands in phase space will not be discussed in this paper [37]. Note that the dissipation rate of the system depends only on the feedback amplitude K—not on the parameters of the map itself. Note also that a matrix with zeros on the main diagonal and ones on the next diagonal is a rotation operator.

For $x_1=x_1^*$ the parameter $b(r)$ is given by

$$b(r)=\left.\frac{df_1}{dx_1}\right|=(2-r)(1-K). \quad (4)$$

If different maps as well as the TDAS control scheme were used only the function $b(r)$ would change.

The characteristic equation is of the order D:

$$(\lambda-b)\lambda^k-K=0. \quad (5)$$

Exact analytical solution of Eq. (5) exists only for $k=0$ and $k=1$, i.e., for the Henon map. For higher values of the delay k, Eq. (5) must be solved numerically. One special case of $b=(K-1)$, when for odd k, Eq. (5) is satisfied by $\lambda=-1$, will be mentioned below. As the coefficients b and K are real, the eigenvalues are real or complex-conjugate pairs. The free term of the characteristic equation is related to the product of the eigenvalues [38]:

$$(-1)^D\prod_{i=1} D\ \lambda_i=K. \quad (6)$$

Knowing that the Lyapunov exponents in the direction of the eigenvectors are defined as $\ln|\lambda_i|$ [39] and putting $\lambda=|\lambda|e^{i\varphi}$ into Eq. (6), we derive

$$\sum_{i=1}^{D}\ln|\lambda_i|+i\sum_{i=1}^{D}\varphi_i=\ln[(-1)^DK]. \quad (7)$$

Note that K in the right-hand side of Eq. (7) may also be negative, which may be put into Eq. (7) as $K=|K|e^{i\pi}$; then $\mathrm{Im}(\ln K)=\pi$. From the real part of the left-hand side of Eq. (7) we see that the sum of Lyapunov exponents is equal to $\ln|K|$ and is not dependent on the delay k. From the imaginary part of Eq. (7) we conclude that

$$\sum_{i=1}^{D}\varphi_i=\begin{cases}0=\Rightarrow(-1)^D K\geqslant 0\\ \pi\Rightarrow(-1)^D K<0.\end{cases} \quad (8)$$

This implies the existence of two types of solutions of Eq. (5). Note that the parity of the delay has emerged as an important parameter of the system. The change of the parity of the delay is in some sense equivalent to the change of the sign of the feedback.

B. Stability conditions

For $k>4$ the logistic map undergoes a global bifurcation, namely, boundary crisis as $f(0.5)>1$. By assuming that Eq. (1) for $x(n)=0.5$ should be less than or equal to 1, and neglecting $x(n-k)$ we obtain the following approximate global stability constraint:

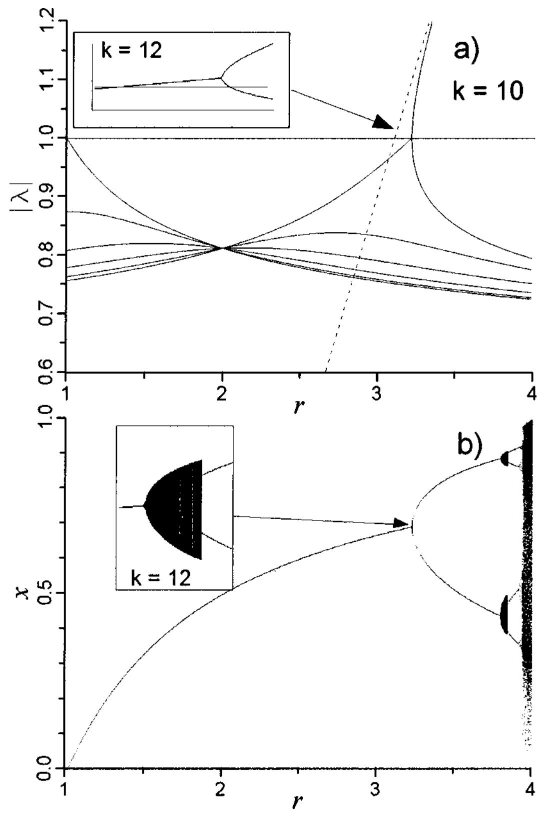

FIG. 1. The modules of eigenvalues (a) and bifurcation diagram (b) as function of the parameter r for even delay $k=10$ and $k=12$ (inlay), $K=0.1$.

$$r \leqslant \frac{4}{1-K}. \tag{9}$$

Better approximations may be obtained by setting $x(n-k)$ to $x_{\min}$ (the minimum x for a given r). Requiring that Eq. (1) should be greater than or equal to 0, we find a rather unexpected constraint that the system is stable for all positive feedbacks $K \geqslant 0$ [40]. For $K<0$, for all values of r, there exists a region of x for which the image of x is negative and the boundary crisis occurs.

IV. RESULTS OF THE STABILITY ANALYSIS

We have verified numerically that the structure of the numerical solutions of Eq. (5) is different for even and for odd values of the delay length k. The behavior of the largest eigenvalue for even delays and for $K>0$ is the same as that for $K<0$ and odd delays, as implied by Eq. (8). The same holds also for odd delays with $K>0$ and even delays with $K<0$.

We repeated our calculations also for the TDAS method of chaos control and found them qualitatively identical to those presented above.

A. Properties of eigenvalues for even delays

A typical dependence of the eigenvalues on the control parameter r for an even delay $k=10$ (note that in this case D is odd) is shown in Fig. 1(a) with the corresponding bifurcation diagram [Fig. 1(b)]. The insets present a blowup of the vicinity of the point of the first period doubling for $k=12$. The results, presented here for $K=0.1$, are generic for all $K>0$.

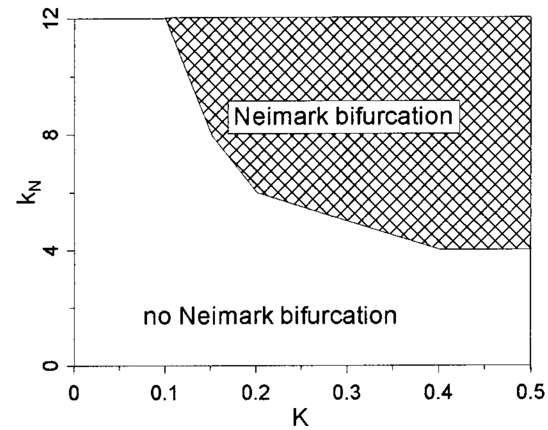

FIG. 2. The threshold delay for Neimark instability k_N as a function of K for even delay.

For $r<3.2$ there exists a real eigenvalue and $k/2$ complex-conjugate pairs. For $r \approx 3.2$ a tangent bifurcation takes place in which a pair of complex-conjugate eigenvalues is converted into a pair of real eigenvalues. When the parameter r increases further, the modulus of one of these real eigenvalues exceeds 1, which is a period-doubling bifurcation, and, as $r \rightarrow 4$, asymptotically reaches $|b(r)|$ [marked as the dotted line in Fig. 1(a)]. For the nondelayed map, i.e., for $K=0$, the regime of stability of the fixed-point solution ends at $r=3.0$. Thus the stability region of that solution has been extended by the delayed feedback.

The above described bifurcation scenario is valid for all even $k \leqslant 10$.

For $k>10$ and even, the pair of the complex-conjugate eigenvalues crosses the unit circle before the tangent bifurcation occurs and a Neimark bifurcation takes place, as shown for $k=12$ in the inset in Fig. 1(a). Then, as r increases further, the quasiperiodic state resulting from Neimark bifurcation [broadening dark region in inset of Fig. 1(b)] looses stability and a period 2 orbit is found.

For different values of K the Neimark bifurcation at the point of the first period-doubling bifurcation occurs at different values of the threshold delay. This threshold delay, denoted as k_N, may be determined from the observation of the bifurcation diagrams. Figure 2 shows the dependence of k_N on K for even delays and $K>0$. Using the terms with $k>k_N$ with the feedback amplitude K too large may introduce instability in chaos control. Fortunately in the ETDAS scheme [14] the terms with larger delays have smaller amplitude K so the impact of the above-mentioned effect on the stability is diminished.

For k slightly below k_N and r in the neighborhood of the point of the first bifurcation, an extremely long quasiperiodic transient of the order of 10 000 iterations appears.

B. Properties of eigenvalues for odd delays

A typical dependence of the eigenvalues on the control parameter r for odd values of k is shown in Fig. 3(a) with the

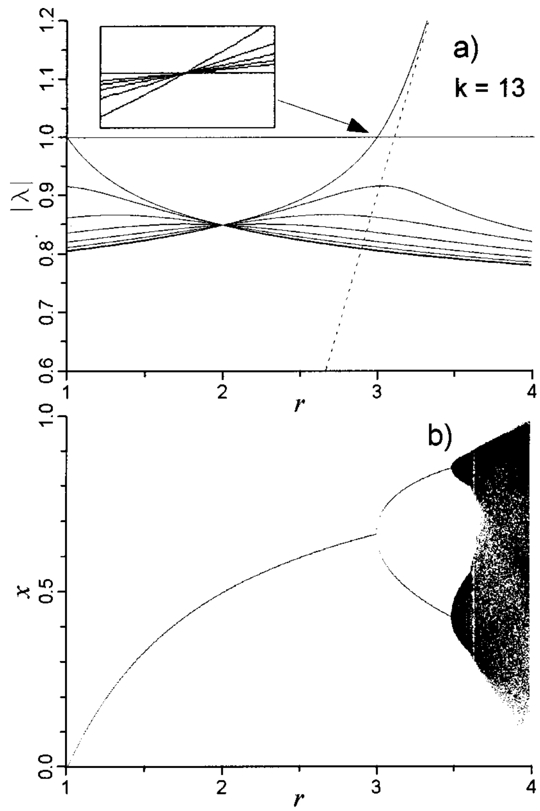

FIG. 3. The modules of eigenvalues (a) and bifurcation diagram (b) as functions of the parameter r for odd delay $k=13$, $K=0.1$. Inlay presents the largest eigenvalue in the vicinity of the critical point for $K=0.1$, 0.3, 0.4, and $K=0.5$.

corresponding bifurcation diagram [Fig. 3(b)]. For all r there now exists a pair of real eigenvalues and $(D-2)/2$ complex-conjugate pairs. At a critical value of $r=3.0$, the modulus of one of the real eigenvalues exceeds 1 and tends to infinity along the skew asymptote $|b(r)|$. The locus of the critical point does not depend on the value of K, as shown in the inset in Fig. 3(a).

Presence of the Neimark bifurcation at the points of higher-order period-doubling bifurcations [Figs. 1(b) and 3(b)] for some values of the delay length seems to be governed by some simple rules that may be deduced from the observation of bifurcation diagram. For instance the Neimark bifurcation is inhibited in the nth period doubling if D is a multiple of $2n$.

C. Mechanisms of instability and structure of the phase space

We have found the results of the linear stability analysis of a fixed-point solution in good agreement with the phase diagrams of the delayed map. Let us look at the following example.

Figure 4 depicts the phase diagram of the delayed map (part a) and the eigenvalue of the largest modulus, denoted as λ, (part b)—in the variables (K,r) for $k=1$. The fill patterns in Fig. 4(a) distinguish the regions of: instability (white), the fixed point, the period 2, 4, 8, and 16 solutions (different patterns), and chaos or solutions with a period larger than 128 (black). The vertical line at $K=0$ marks the case of no delay. As the parameter K increases, if the upper range of stability of any solution moves towards larger values of r (as for period 2 solution), the stability region of that solution is extended. The white regions for negative K are the regions of the boundary crisis. The diamonds mark the approximate boundary of one of these regions given by Eq. (9). It may be seen that the stability boundary in Fig. 4(a) exceeds the rough approximation given by Eq. (9). In Fig. 4(b) the solid lines are the contours of constant λ.

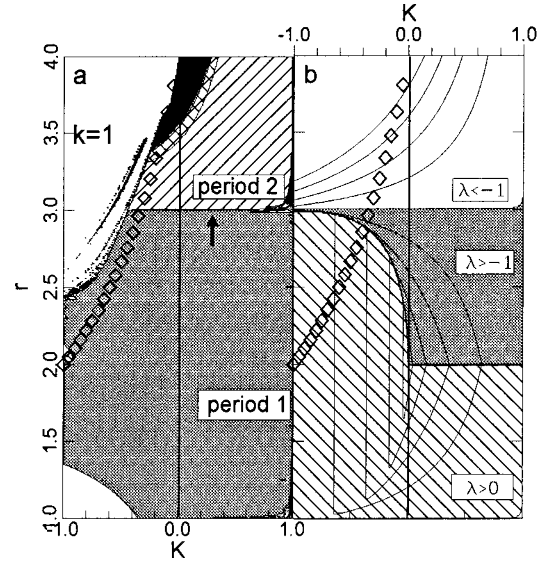

FIG. 4. The phase diagram of the delayed map (a) and the eigenvalue of the largest modulus λ in variables (K,r) (b) for the delay $k=1$. In (a) gray denotes the period 1 solution while slashed and crosshatched denote period 2 and period 4 solutions, respectively. White color means instability whereas black is for high-periodic or chaotic solutions. In (b) in the white region $\lambda>1$ or $\lambda<-1$. In the back-slashed regions $\lambda\in[0,1]$. In the gray regions $\lambda\in[-1,0]$. The boundary of tangency crisis is marked as diamonds in both parts.

The shape of the stability region of a fixed-point solution, as seen in Fig. 4(a) is a result of two independent mechanisms:

(i) The first mechanism of instability is the local period-doubling bifurcation at $r=3$ [lower bound of the white region in Fig. 4(b)]. We may refer to this effect as a loss of local stability.

(ii) The second mechanism is such that the solution is locally stable but due to the boundary crisis the whole attractor becomes unstable [upper- and lower-left corner in Fig. 4(a)]. This may be called a loss of global stability.

Those two mechanisms are quite independent. Local stability is more sensitive to changes of the delay time k. The global stability depends mainly on the properties of the non-delayed map function.

Note that the only way in which the exact form of the delayed map (1) affects the eigenvalues is through the function $b(r)$. So for any one-dimensional map, the structure of the Jacobi matrix remains the same. Thus one may expect that the results obtained here are representative for a wider class of iterated maps and the effects found here may also be found for other systems. A convincing example is that, when

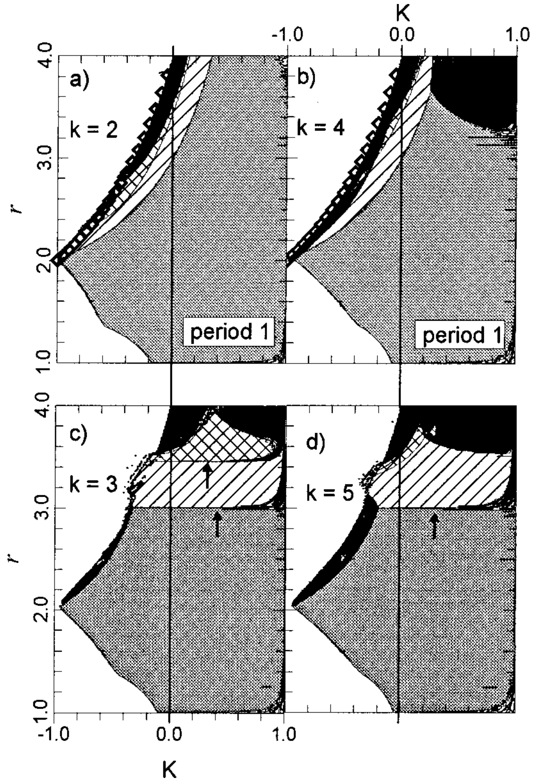

FIG. 5. The phase diagram of the delayed map in variables (K,r) for the delay $k=2$ (a), $k=4$ (b), $k=3$ (c), and $k=5$ (d). Arrows mark pinned boundaries of stability. All other markings as in Fig. 4(a).

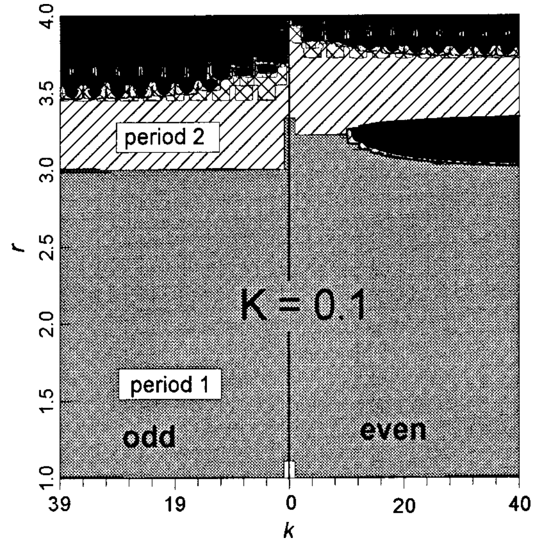

FIG. 6. The phase diagram of the delayed map in variables (k,r) for $K=0.1$. All markings as in Fig. 4(a).

the TDAS type control, which changes only the function $b(r)$, was compared with the echo type feedback, used here, no qualitative difference between the stability ranges was found.

As may be seen in Fig. 4(a) the period-doubling cascade is present regardless of the value of k. The presence of the delay changes only the locus of the stability limits of certain solutions of Eq. (1). Also the shape of the boundary of the global bifurcation regions are approximately constant as the delay time k changes (see Fig. 5). In this sense we may say that the qualitative properties of the phase space are retained in the presence of the delay. A symbolic dynamics analysis of this effect is being developed [41].

V. STABILITY OF THE CHAOS CONTROL

Figure 5 depicts the phase diagrams of the delayed map in the variables (K,r) for $k=2$ (part a), $k=4$ (part b), $k=3$ (part c), and $k=5$ (part d). The most interesting effect visible in Fig. 5 is a pinning of the boundary of stability of some solutions. Namely, for odd delays k, if $D=k+1$ is a multiple of the period of the orbit of even period $2n$, the onset of the stability range for this orbit is constant regardless of the coupling strength K [arrows in Fig. 4(a) and Figs. 5(c) and 5(d)]. Thus the stability range of the periodic orbit of period n cannot be extended.

The reason for the pinning of the fixed-point solution is that, for $b(r)=(K-1)$ and odd delays, the characteristic equation (5) has the solution $\lambda=-1$. The constant limit of stability is the locus of points that satisfy the condition $b(r)=(K-1)$ [see Fig. 4(b) and inset in Fig. 3(a)]. For a fixed point this condition is fulfilled for $r=3$.

The consequence of pinning is such that the delays of odd length are not suitable for stabilizing the orbits of even periodicity. For even delays and orbits of a period-doubling cascade, for each r there exists such K for which the desired orbit is stable. Thus in the ETDAS scheme [14], if only the even delays were taken into account, the stability region of the orbits of even periodicity as well as the convergence rate could be improved. From this it may be conjectured that properly chosen odd delays extend the stability range of orbits of odd periodicity.

Figure 6 shows the phase diagram in the variables (k,r), for constant coupling $K=0.1$. It may be seen that for $r>3.5$ only the even delays extend the stability region of the period 2 solution. Note, that also the onset of the period 2 solution occurs at different values of the control parameter r for odd and for even delays. The black elliptic region between the period 1 and period 2 solutions for the even delays is the Neimark bifurcation region. The regular square-wave structure seen near the onset of chaos is the result of the Neimark bifurcations at the points of higher-order period doubling (as in Fig. 3) and of the low horizontal resolution of the plot.

VI. EFFECTIVE DIMENSION

There is no strict measure of the dimension of the attractor based on the Lyapunov exponents [32]. The Kaplan-Yorke dimension is proved to give the upper bound, while the lower bound is given by the number of positive Lyapunov exponents of the system (Lyapunov dimension) [32]. Figure 7 shows the Kaplan-Yorke dimension d_{KY} Lyapunov dimension d_L, standard deviation of the Lyapunov dimension d_L, and metric entropy h, calculated along the trajectory of the system for $r=4.0$ and $K=0.1$ as a function of the dimension of the phase space D. Linear fit

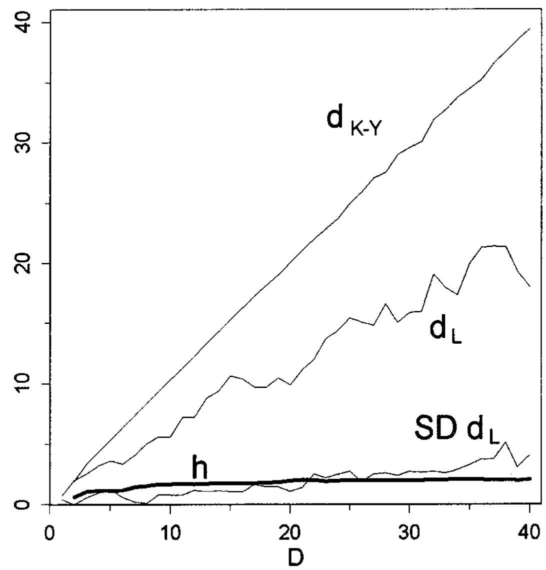

FIG. 7. The Kaplan-Yorke dimension d_{KY} (upper curve), Lyapunov dimension d_L (middle curve), standard deviation of d_L (lower curve), and the metric entropy h (bold curve) for $r=4.0$ and $K=0.1$ as the function of the dimension of the phase space D.

gave the slope equal to 0.972 ± 0.002 for d_{KY} and 0.51 ±0.01 for d_L. Thus the dimension of the attractor indeed grows linearly, however slower than the dimension of the phase space. Note also that the growth of d_L is nonmonotonic. The similar nonmonotonic growth of d_L was also reported for the Mackey-Glass equation [31]. This might suggest a nonmonotonic growth of the dimension of the attractor.

The standard deviation of d_L was calculated as the measure of the inhomogenity of the attractor. It grows linearly, although nonmonotonically, with slope equal to 0.094 ±0.006, slightly faster than the entropy. Thus for larger delays, in certain regions of the attractor, the dynamics becomes highly dimensional, with a number of unstable directions, whereas in other regions it remains relatively low dimensional.

The metric entropy h is nearly a constant function of the delay time, approximately equal to 2, with the rate of growth equal to 0.024 ± 0.002. Metric entropy is a measure of complexity or strength of nonlinearity of the system. It is defined as the sum of positive Lyapunov exponents: i.e., the total rate of expansion along all of the unstable directions [31]. According to Eq. (7), the sum of all Lyapunov exponents is constant, dependent only on K. Thus also the sum of negative Lyapunov exponents grows at the same rate as the entropy. The growing rate of expansion along the unstable directions is compensated by the growing rate of contraction along the stable directions.

One may argue if the system is characterized better by its dimension or by its complexity. But it seems reasonable to state that the effective dimension of the delayed equation is lower than the dimension of the attractor, as the rising dimension of the phase space does not enlarge the initial complexity of the underlying nondelayed system.

VII. CONCLUSIONS

Linear stability of the delayed logistic map was analyzed. The results are generic for one-dimensional delayed maps.

A pinning of the stability onset of periodic orbits for odd delays was found. It is a result of certain properties of the characteristic equation. Consequently the enhancement of the ETDAS method of chaos control was proposed.

The model studied exhibits a bifurcation structure that depends on a combination of the parity of the delay and the sign of the feedback. The Neimark bifurcations at the points of period doubling may take place for the lengths of the delay larger than a threshold value. This threshold value depends on the amplitude of the feedback. The Neimark bifurcation may also appear at the points of higher period doubling.

Global and local stability of the delayed system was studied. Global stability depends more on the properties of the phase space of the nondelayed map than on the properties of the delay itself. Stability conditions for global stability were found. The properties of the phase space, such as the existence of the period-doubling cascade and the shape of the region of the boundary crisis, are retained in the presence of the delay.

It was found that the dimension of the attractor grows linearly, however this growth may be nonmonotonic. Also the inhomogenity of the attractor rises as there appear regions of a large number of unstable directions. The introduction of the delay embeds the system into the multidimensional phase space and extends the region of stable solutions, as well as enlarges the number of unstable directions for the chaotic trajectories. However, the effective dimension of the system is lower than the dimension of the attractor as the introduction of the delay does not enlarge the strength of nonlinearity of the system.

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