Renormalization-group theoretical reduction of the Swift-Hohenberg model

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The Swift-Hohenberg model of the cellular pattern formation is exploited with a proto renormalization-group (RG) scheme. The method dispenses with the explicit perturbation solutions which are required in the standard RG approach. The RG equations obtained are the well-known reductive perturbation results such as a rotationally covariant amplitude equation and the nonlinear phase equations.

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I. INTRODUCTION

In spite of the enormous diversity of stable spatial structures in far-from-equilibrium systems of different origins, the unity of dynamical mechanisms of their birth and evolution allows us to formulate some fundamental models of the nonlinear theory of such structures. The Swift-Hohenberg (SH) equation [1] is one of the simplest and most canonical paradigms, and has been intensively studied in the past [2]. It is in the form of a partial differential equation (PDE) for a time-dependent function $\psi(\mathbf{x},t)$ and reads

$$\partial_t \psi = \varepsilon \psi - \psi^3 - (\partial_{\mathbf{x}}^2 + k_0^2)^2 \psi, \tag{1.1}$$

where $\partial_{\mathbf{x}}^2$ is a two-dimensional Laplacian with respect to the position vector $\mathbf{x}$. The ε is the bifurcation parameter, which controls the appearance of spatially periodic structures (''rolls'' or ''stripes'') of characteristic length of order k_0^{-1}.

Analytic solutions to the nonlinear PDE (1.1) have been calculated perturbatively in two limiting cases [2]: (i) Near threshold where $0 < \varepsilon \ll 1$. Here the nonlinearities are weak and the spatial and temporal modulations of the basic rolls become slow. The balance between these effects is described by the ''amplitude equation'' [3,4] for the envelope function of the basic state. (ii) Far from threshold but still in a range where the field is dominated locally by what would appear to be straight parallel rolls. Weak distortions of the regular patterns involving spatial modulations over distances large compared to k_0^{-1} can be treated perturbatively yielding the ''phase equation'' [5,6].

To derive the amplitude equation from (1.1), the method of multiple-scales approach (see, e.g., [2]) is used, while this method is combined with the nonlinear WKB technique [7] to construct the phase equation. We shall refer to these methods altogether as a reductive or singular perturbation theory.

Suppose we apply to a system a perturbation, e.g., we add to the equation governing the system some nonlinear terms, dissipative terms, etc. The system is usually not structurally stable against such perturbations, so the perturbation results are, if computed naively, plagued by singularities (secular terms). It has been recognized for some time [8] that these singularities in the naive perturbation theories can be renormalized away by the modification (renormalization) of the parameters in the unperturbed state (amplitude, phase, etc.). The renormalized results agree with or are sometimes better than those traditionally computed with the aid of singular perturbation methods [9]. The modified parameters are governed by the renormalization group (RG) equations that turn out to be, e.g., large-scale slow-motion equations (reduced equations) [10]. Let us call the method to obtain the space-time large scale equations as RG equations the RG theoretical reduction (or the reductive RG method). However, the RG reduction so far practiced [9,10] could be methodologically awkward, because we need explicit perturbation results.

Quite recently a new approach, the so-called proto RG operator scheme, has been proposed [11–13] to free as much as possible the RG theoretical reduction from the necessity of explicit secular terms. The RG procedure in this scheme is much simpler than the RG calculations given in Ref. [9]. The importance of this method is that to the lowest nontrivial order no explicit results are needed and yet it extracts observable global features of highly nonlinear systems. (It is true that the traditional reductive perturbation already has this feature, but the new reductive RG approach can give the same singular perturbation results as well.) This has been illustrated in Refs. [11,12] with many examples of asymptotic analysis of ordinary differential equations (ODEs) as well as PDEs, and the assertion that all the reductive perturbation equations are RG equations seems to hold. This paper will give a new example supporting this assertion.

The key step in the proto RG operator scheme to construct the RG equation is to find an appropriate differential operator that maps the set of its unbounded (secular) solutions to something tractable (to 1, say). For PDE, in contrast to ODE cases, this procedure becomes a bit non-trivial due to the fact that the secular terms are not unique for PDE. Although one can learn much of the essence from the published results Refs. [11,12], we feel we should reemphasize the step. Therefore, in this paper, we give non-trivial details about the proto RG operator scheme with the SH equation (1.1).

Section II is expository, and the proto RG operator scheme in its most abstract version is explained to derive the amplitude equation. The final result (which has already been published in Refs. [11–13]) turns out to be the reductive perturbation equation [4]. In Sec. III, the phase dynamics near the zigzag boundary is studied in the context of our scheme. The outcome is the well-known nonlinear phase equation [6,14,15]. The more general phase dynamics is considered in Sec. IV. The derivation is an RG version of the Cross-Newell phase equation [6]. Section V closes the paper with several remarks.

II. PROTO RENORMALIZATION-GROUP SCHEME

Since we treat the term $\varepsilon\psi-\psi^3$ in Eq. (1.1) as a perturbative one, we scale ψ as $\sqrt{\varepsilon}\psi$ and denote the new ψ with the same symbol. The SH equation (1.1) then reads

$$\partial_t\psi=\varepsilon(\psi-\psi^3)-(\partial_{\mathbf{x}}^2+k_0^2)^2\psi. \tag{2.1}$$

In order to explain the proto RG scheme in its most abstract version, we expound, in this section, details of the ε^2-RG calculation using Eq. (2.1).

At a zeroth order in ε, we have a roll solution: $Ae^{i\mathbf{k}\cdot\mathbf{x}}+\text{c.c.}$ with $k\equiv|\mathbf{k}|=k_0$, where c.c. denotes the complex conjugate and A is a complex numerical constant. We expand the solution to (2.1) as

$$\psi(t,\mathbf{x})=Ae^{i\mathbf{k}\cdot\mathbf{x}}+\varepsilon\psi_1(t,\mathbf{x},A)+\varepsilon^2\psi_2(t,\mathbf{x},A)+\cdots+\text{c.c.} \tag{2.2}$$

Here the dependence of the perturbed solutions on A is explicitly denoted for clarity of the subsequent discussion. The first and second order corrections obey the equations

$$L^{(0)}(\partial_t,\partial_{\mathbf{x}})\psi_1=A(1-3|A|^2)e^{i\mathbf{k}\cdot\mathbf{x}}-A^3e^{3i\mathbf{k}\cdot\mathbf{x}}+\text{c.c.}, \tag{2.3}$$

$$L^{(0)}(\partial_t,\partial_{\mathbf{x}})\psi_2=(1-6|A|^2)\psi_1-3(A^2e^{2i\mathbf{k}\cdot\mathbf{x}}+A^{*2}e^{-2i\mathbf{k}\cdot\mathbf{x}})\psi_1+\text{c.c.}, \tag{2.4}$$

where

$$L^{(0)}(\partial_t,\partial_{\mathbf{x}})=\partial_t+(\partial_{\mathbf{x}}^2+k_0^2)^2. \tag{2.5}$$

Notice that $\Phi_0\equiv e^{i\mathbf{k}\cdot\mathbf{x}}$ is the eigenfunction of the zero eigenvalue of $L^{(0)}$; $L^{(0)}\Phi_0=0$. Hence Φ_0 can be the source of the secular terms in ψ_1 and ψ_2.

Let $\hat\psi_1(\tau,\boldsymbol{\rho},t,\mathbf{x},A)$ and $\hat\psi_2(\tau,\boldsymbol{\rho},t,\mathbf{x},A)$, $\cdots$ be ψ_1 and ψ_2, $\cdots$ with variables $\{t,\mathbf{x}\}$ in the secular prefactors of Φ_0 replaced by $\{\tau,\boldsymbol{\rho}\}$, discarding constant terms. Let us also introduce the renormalized A:

$$A=A_RZ\equiv A_R(\tau,\boldsymbol{\rho})[1+\varepsilon Z_1(\tau,\boldsymbol{\rho})+\varepsilon Z_2(\tau,\boldsymbol{\rho})+\cdots]. \tag{2.6}$$

Then Eq. (2.2) can be written as

$$\psi=A_R\Phi_0+\varepsilon[A_RZ_1\Phi_0+\psi_1(\mathbf{r},A_R)]+\varepsilon^2[A_RZ_2\Phi_0+\psi_2(\mathbf{r},A_R)+A_RZ_1\psi_1'(\mathbf{r},A_R)]+\cdots+\text{c.c.}, \tag{2.7}$$

where $\mathbf{r}\equiv\{t,\mathbf{x}\}$ and $\psi_1'\equiv\partial\psi/\partial A_R$. The renormalization constant is determined order by order as

$$A_RZ_1\Phi_0+\hat\psi_1(\mathbf{r},\mathbf{R},A_R)=0,$$

$$A_RZ_2\Phi_0+\hat\psi_2(\mathbf{r},\mathbf{R},A_R)+A_RZ_1\hat\psi_1'(\mathbf{r},\mathbf{R},A_R)=0, \tag{2.8}$$

etc., where $\mathbf{R}\equiv\{\tau,\boldsymbol{\rho}\}$. Putting Eq. (2.8) into Eq. (2.7) we obtain the renormalized perturbation series

$$\psi=A_R\Phi_0+\varepsilon[\psi_1(\mathbf{r},A_R)-\hat\psi_1(\mathbf{r},\mathbf{R},A_R)]+\varepsilon^2\{\psi_2(\mathbf{r},A_R)-\hat\psi_2(\mathbf{r},\mathbf{R},A_R)+A_RZ_1[\psi_1'(\mathbf{r},A_R)-\hat\psi_1'(\mathbf{r},\mathbf{R},A_R)]\}+\cdots+\text{c.c.} \tag{2.9}$$

Notice that Eqs. (2.8) and (2.9) are the ε-expansion of

$$(A-A_R)\Phi_0+\varepsilon\hat\psi_1(\mathbf{r},\mathbf{R},A)+\varepsilon^2\hat\psi_2(\mathbf{r},\mathbf{R},A)+\cdots+\text{c.c.}=0, \tag{2.10}$$

and

$$\psi=A_R\Phi_0+\varepsilon[\psi_1(\mathbf{r},A)-\hat\psi_1(\mathbf{r},\mathbf{R},A)]+\varepsilon^2[\psi_2(\mathbf{r},A)-\hat\psi_2(\mathbf{r},\mathbf{R},A)]+\cdots+\text{c.c.}, \tag{2.11}$$

respectively. If we put $\mathbf{r}=\mathbf{R}$, then Eq. (2.11) reduces to $\psi=A_R\Phi_0$ as required.

Introduction of $\mathbf{R}$ is equivalent to splitting of the derivative ∂_t to $\partial_t+\partial_\tau$, and $\partial_{\mathbf{x}}$ to $\partial_{\mathbf{x}}+\partial_{\boldsymbol{\rho}}$;

$$L^{(0)}(\partial_t,\partial_{\mathbf{x}})\to L^{(0)}(\partial_t+\partial_\tau,\partial_{\mathbf{x}}+\partial_{\boldsymbol{\rho}}).$$

If we apply $L^{(0)}(\partial_t+\partial_\tau,\partial_{\mathbf{x}}+\partial_{\boldsymbol{\rho}})$ to $\hat\psi_1$ (or $\hat\psi_2$) and separate out the term containing Φ_0 from the outcome, then it must be identical to the coefficient of Φ_0 on the right-hand side of Eq. (2.3) [or (2.4)]. Namely, e.g.,

$$\mathcal{P}_{\mathbf{r}=\mathbf{R}}L^{(0)}(\partial_t+\partial_\tau,\partial_{\mathbf{x}}+\partial_{\boldsymbol{\rho}})\hat\psi_1=A(1-3|A|^2), \tag{2.12}$$

where $\mathcal{P}$ is the projection operator onto Φ_0 and the subscript implies the prescription to set $\{t,\mathbf{x}\}=\{\tau,\boldsymbol{\rho}\}$ after projection. Since

$$\mathcal{P}L^{(0)}(\partial_t,\partial_{\mathbf{x}})\hat\psi_1=0 \tag{2.13}$$

by the definition of $\hat\psi_1$, we rewrite the operators on the left-hand side of Eq. (2.12) as

$$\begin{aligned}\mathcal{P}L^{(0)}(\partial_t+\partial_\tau,\partial_{\mathbf{x}}+\partial_{\boldsymbol{\rho}})\hat\psi_1&=\mathcal{P}[L^{(0)}(\partial_t+\partial_\tau,\partial_{\mathbf{x}}+\partial_{\boldsymbol{\rho}})-L^{(0)}(\partial_t,\partial_{\mathbf{x}})]\hat\psi_1\\&\quad-\mathcal{P}L^{(0)}(\partial_t,\partial_{\mathbf{x}})\hat\psi_1\\&=\mathcal{P}F\hat\psi_1,\end{aligned} \tag{2.14}$$

with

$$\begin{aligned}\mathcal{F}&\equiv L^{(0)}(\partial_t+\partial_\tau,\partial_{\mathbf{x}}+\partial_{\boldsymbol{\rho}})-L^{(0)}(\partial_t,\partial_{\mathbf{x}})\\&=\partial_\tau+\partial_{\boldsymbol{\rho}}^4+2(\partial_{\mathbf{x}}^2+k_0^2)\partial_{\boldsymbol{\rho}}^2\\&\quad+4(\partial_{\boldsymbol{\rho}}^2+\partial_{\mathbf{x}}\cdot\partial_{\boldsymbol{\rho}}+\partial_{\mathbf{x}}^2+k_0^2)\partial_{\mathbf{x}}\cdot\partial_{\boldsymbol{\rho}}.\end{aligned} \tag{2.15}$$

We now apply the operator $\mathcal{S}\equiv\mathcal{P}F$ to Eq. (2.11). Then we obtain

$$\mathcal{S}(A_R\Phi_0)=\varepsilon\mathcal{S}\hat{\psi}_1+\varepsilon^2\mathcal{S}\hat{\psi}_2+\cdots$$
$$=\varepsilon A(1-3|A|^2)+\varepsilon^2\mathcal{P}[(1-6|A|^2)\psi_1$$
$$-3(A^2\mathrm{e}^{2i\mathbf{k}\cdot\mathbf{x}}+A^{*2}\mathrm{e}^{-2i\mathbf{k}\cdot\mathbf{x}})\psi_1]+\cdots. \quad (2.16)$$

However,

$$\mathcal{F}(A_R\Phi_0)=\Phi_0 L_{\tau,\boldsymbol{\rho}}A_R, \quad (2.17)$$

where

$$L_{\tau,\boldsymbol{\rho}}=\partial_\tau+\partial_{\boldsymbol{\rho}}^4+4i\mathbf{k}\cdot\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\rho}}^2-4k_\ell k_m\frac{\partial^2}{\partial\rho_\ell\partial\rho_m}. \quad (2.18)$$

Here and hereafter, the summation convention always applies to subscripts occurring twice in vector expressions. Therefore Eq. (2.16) becomes

$$L_{\tau,\boldsymbol{\rho}}A_R=\varepsilon A(1-3|A|^2)+\varepsilon^2\mathcal{P}[(1-6|A|^2)\psi_1$$
$$-3(A^2\mathrm{e}^{2i\mathbf{k}\cdot\mathbf{x}}+A^{*2}\mathrm{e}^{-2i\mathbf{k}\cdot\mathbf{x}})\psi_1] \quad (2.19)$$

to $O(\varepsilon^2)$. On the right-hand side one may ignore all the terms explicitly dependent on space and time. From Eq. (2.3) we find that in ψ_1 the term proportional to $e^{\pm i\mathbf{k}\cdot\mathbf{x}}$ is secular (hence it can never be a constant), while the term $\propto e^{\pm 3i\mathbf{k}\cdot\mathbf{x}}$ is nonsecular. Thus we look for the term of the form $B\mathrm{e}^{\pm 3i\mathbf{k}\cdot\mathbf{x}}$ in ψ_1 with B being constant. We find $B=-A^3/(64k^4)$.

In this way Eq. (2.19) is reduced to

$$L_{\tau,\boldsymbol{\rho}}A_R=\varepsilon A(1-3|A|^2)+\varepsilon^2\frac{3}{64k^4}A|A|^4 \quad (2.20)$$

to $O(\varepsilon^2)$. Here we replace A on the right-hand side with A_RZ. We may then set $\{\tau,\boldsymbol{\rho}\}=0$ so that $Z=1$ since the right-hand side should not depend on space-time explicitly. Therefore we finally obtain the proto RG equation. (Actually this is the RG equation to $O(\varepsilon^2)$; see [11] on this point.) It reads

$$(\partial_\tau-4k_0^2\Box^2)A_R=\varepsilon A_R(1-3|A_R|^2)+\varepsilon^2\frac{3}{64k^4}A_R|A_R|^4, \quad (2.21)$$

where $\Box$ is the rotationally covariant operator [4]

$$\Box\equiv\hat{\mathbf{k}}\cdot\partial_{\boldsymbol{\rho}}-\frac{i}{2k_0}\partial_{\boldsymbol{\rho}}^2, \quad (2.22)$$

$\hat{\mathbf{k}}$ being the unit vector along $\mathbf{k}$. The result (2.21) is exactly of the same form as the amplitude equation that Gunaratne *et al.* [4] obtained first with a use of the multiple-scale analysis.

III. PHASE EQUATION NEAR ZIGZAG BOUNDARY

We split the right-hand side of Eq. (1.1) into two parts as

$$\partial_t\psi=[\varepsilon\psi-\psi^3-(\partial_x^2+k_0^2)^2\psi]-[2\partial_y^2(\partial_x+k_0^2)+\partial_y^4]\psi. \quad (3.1)$$

The last term is regarded as a perturbation in this section, and we will not write explicitly the small parameter since we study only the lowest nontrivial order. This is the situation in the neighborhood of zigzag instabilities [2] in which the resistance of the rolls to perturbation with variation along their axes weakens. The unperturbed equation has a stationary solution:

$$\psi_0(\mathbf{x})=\psi_0(\theta(\mathbf{x})\equiv k_0x+\phi(y)),\quad \mathbf{x}=(x,y), \quad (3.2)$$

where $\psi_0\propto e^{i\theta}$.

Writing the deviation of the true solution from ψ_0 as ψ_1:

$$\psi=\psi_0+\psi_1, \quad (3.3)$$

we get to the lowest order

$$[\partial_t-L^{(0)}(\partial_x)]\psi_1=-[\partial_y^4+2\partial_y^2(\partial_x^2+k_0^2)]\psi_0, \quad (3.4)$$

where

$$L^{(0)}(\partial_x)\equiv\varepsilon-3\psi_0^2-(\partial_x^2+k_0^2)^2. \quad (3.5)$$

The straightforward calculation of the right-hand side of Eq. (3.4) yields

$$(\partial_t-L^{(0)})\psi_1=P_1\psi_0'+P_2\psi_0''+P_3\psi_0'''+P_4\psi_0'''', \quad (3.6)$$

where the $'$ denotes the differentiation with respect to the phase θ, and

$$P_1=-[\partial_y^4\phi+2k_0^2\partial_y^2\phi],$$
$$P_2=-[4\partial_y^4\phi\partial_y\phi+3(\partial_y^2\phi)^2+2k_0^2(\partial_y\phi)^2],$$
$$P_3=-[6\partial_y^2\phi(\partial_y\phi)^2+2k_0^2\partial_y^2\phi],$$
$$P_4=-[(\partial_y\phi)^4+2k_0^2(\partial_y\phi)^2]. \quad (3.7)$$

We note here that ψ_0' is the zero-eigenvalue eigenfunction of $L^{(0)}$; $L^{(0)}\psi_0'=0$. Therefore this can be the source of the secular terms of ψ_1.

Let us renormalize the phase ϕ as

$$\psi(t,\mathbf{x})=\psi_0(k_0x+\phi_R(\tau,\xi,y))+\psi_1(t,\mathbf{x})-\hat{\psi}_1(\tau,\xi,t,\mathbf{x}), \quad (3.8)$$

where $\hat{\psi}_1$ is obtained from ψ_1 by replacing t and x in the secular term prefactors of ψ_0' with τ and ξ and discarding the constant prefactors. Introduction of these variables $\{\tau,\xi\}$ corresponds to splitting of ∂_t and ∂_x to $\partial_t+\partial_\tau$ and $\partial_x+\partial_\xi$, respectively. Namely, it is equivalent to replacing the differential operator $\mathcal{L}(\partial_t,\partial_x)\equiv\partial_t-L^{(0)}(\partial_x)$ according to

$$\mathcal{L}(\partial_t,\partial_x)\to\hat{\mathcal{L}}\equiv\mathcal{L}(\partial_t+\partial_\tau,\partial_x+\partial_\xi).$$

If we apply $\hat{\mathcal{L}}$ to $\hat{\psi}_1$ and separate out the term containing ψ_0' from the outcome, it must be identical to the coefficient of ψ_0' on the right-hand side of Eq. (3.6). Notice here that ψ_0''

and ψ_0'''' are orthogonal to ψ_0' but $\psi_0'''\propto\psi_0'$. Introducing the projection operator $\mathcal{P}$ onto the null space of $L^{(0)}$, we thus find

$$\mathcal{P}\hat{\mathcal{L}}\hat{\psi}_1=\mathcal{P}(P_1\psi_0'+P_3\psi_0'''). \tag{3.9}$$

However, $\mathcal{P}\mathcal{L}(\partial_t,\partial_x)\hat{\psi}_1=0$ by the very definition of $\hat{\psi}_1$. Thus Eq. (3.9) is rewritten as

$$\mathcal{P}F\hat{\psi}_1=\mathcal{P}(P_1\psi_0'+P_3\psi_0'''), \tag{3.10}$$

where

$$\begin{aligned}\mathcal{F}&\equiv\mathcal{L}(\partial_t+\partial_\tau,\partial_x+\partial_\xi)-\mathcal{L}(\partial_t,\partial_x)\\&=\partial_\tau+[2(\partial_x^2+\partial_y^2+k_0^2)+\partial_\xi^2+2\partial_x\partial_\xi](\partial_\xi^2+2\partial_x\partial_\xi).\end{aligned} \tag{3.11}$$

Operating $\mathcal{F}$ on Eq. (3.8), we therefore obtain

$$\mathcal{P}F\psi_0(k_0x+\phi_R(\tau,\xi,y))=\mathcal{P}(P_1\psi_0'+P_3\psi_0'''). \tag{3.12}$$

Computing both sides of Eq. (3.12) explicitly and with the subsequent replacement of the variables $\{\tau,\xi\}\to\{t,x\}$, we arrive at

$$\begin{aligned}&\partial_t\phi+2\partial_y^2\partial_x^2\phi+\partial_x^4\phi+\partial_y^4\phi-4k_0^2\partial_x^2\phi-[8k_0\partial_y\phi\partial_{xy}^2\phi\\&\quad+4k_0\partial_y^2\phi\partial_x\phi+12k_0\partial_x\phi\partial_x^2\phi]-[2(\partial_y\phi)^2\partial_x^2\phi\\&\quad+2\partial_y^2\phi(\partial_x\phi)^2+8\partial_y\phi\partial_x\phi\partial_{xy}\phi+6(\partial_x\phi)^2\partial_x^2\phi\\&\quad+6\partial_y^2\phi(\partial_y\phi)^2]=0.\end{aligned} \tag{3.13}$$

We have used the relation $\psi_0'''=-\psi_0'$. This is the proto RG equation.

To reduce the equation further, we must choose the way we observe the system. If we choose

$$\partial_x\sim\eta^2,\quad\partial_y\sim\eta,\quad\partial_t\sim\eta^4\quad(\eta\ll1), \tag{3.14}$$

and keeping the leading terms, which are $O(\eta^4)$, then we find

$$\begin{aligned}&\partial_t\phi-6\partial_y^2\phi(\partial_y\phi)^2+\partial_y^4\phi-8k_0\partial_y\phi\partial_{xy}^2\phi-4k_0\partial_x\phi\partial_y^2\phi\\&\quad-4k_0^2\partial_x^2\phi=0.\end{aligned} \tag{3.15}$$

This is the phase equation which Cross and Newell [6] derived first by the multiple-scales analysis (see also [14,15]). We note that the small parameter in their analysis is η^2 = size of roll/size of system, and the result (3.15) is obtained by the expansion of (1.1) up to $O(\eta^8)$.

IV. CROSS-NEWELL PHASE EQUATION

For ε not small but at a late stage of the roll formation, $\partial\psi/\partial t$ becomes the only small perturbation that enters the SH equation (1.1). In this section we consider such a situation.

To that end we introduce the small parameter η in the SH equation as

$$\eta\partial_t\psi=\varepsilon\psi-\psi^3-(\partial_{\mathbf{x}}^2+k_0^2)^2\psi. \tag{4.1}$$

The control parameter ε is not a small parameter in this section. At late stages in the pattern formation process, roll platforms change slowly from place to place. We then simply guess that the only important order parameter is the pattern wave vector, $\mathbf{k}$. The amplitude of the ψ field will be slaved to $\mathbf{k}$ everywhere. Hence η corresponds to the ratio of the amplitude relaxation time to a time scale we are concerned with. Then the zeroth order solution ψ_0 of Eq. (4.1) satisfies

$$\varepsilon\psi_0-\psi_0^3-(\partial_{\mathbf{x}}^2+k_0^2)^2\psi_0=0, \tag{4.2}$$

and represents the stationary state. Equation (4.2) has a solution of the form

$$\psi_0(\mathbf{x},t)=f(\theta(\mathbf{x},t)\equiv\mathbf{k}(t)\cdot\mathbf{x}+\phi(t),\mathbf{k}(t)), \tag{4.3}$$

where the function f is 2π-periodic in θ, and $\mathbf{k}(t)$ and $\phi(t)$ are arbitrary functions of t. Specifically,

$$\mathbf{k}=\frac{\partial\theta}{\partial\mathbf{x}}. \tag{4.4}$$

Writing the deviation of the true solution from ψ_0 as

$$\psi=\psi_0+\eta\psi_1+\cdots, \tag{4.5}$$

we obtain at $O(\eta)$

$$L^{(0)}(\partial_{\mathbf{x}})\psi_1=\partial_t\psi_0=\frac{\partial\theta}{\partial t}\frac{\partial f}{\partial\theta}+\frac{\partial k_\ell}{\partial t}\frac{\partial f}{\partial k_\ell}. \tag{4.6}$$

Here

$$L^{(0)}(\partial_{\mathbf{x}})=\varepsilon-3\psi_0^2-(\partial_{\mathbf{x}}^2+k_0^2)^2. \tag{4.7}$$

Notice that this operator has a zero-eigenvalue eigenfunction $\Phi_0\equiv\partial f/\partial\theta$, i.e., $L^{(0)}\Phi_0=0$. Hence this Φ_0 appearing in Eq. (4.6) can be the source of the secular terms in ψ_1.

Let us renormalize Eq. (4.5) as

$$\psi(\mathbf{x})=f(\mathbf{k}\cdot\mathbf{x}+\phi^R(\boldsymbol{\rho}),\mathbf{k}^R(\boldsymbol{\rho}))+\eta\psi_1(\mathbf{x})-\eta\hat{\psi}_1(\boldsymbol{\rho},\mathbf{x}), \tag{4.8}$$

where we have not written out the t-dependence of $\mathbf{k},\phi^R,\mathbf{k}^R$, etc., for notational simplicity. In Eq. (4.8) the renormalized wave vector $\mathbf{k}^R$ is defined by

$$\mathbf{k}^R=\frac{\partial\theta^R}{\partial\boldsymbol{\rho}},\quad\theta^R(\boldsymbol{\rho})\equiv\mathbf{k}\cdot\boldsymbol{\rho}+\phi^R(\boldsymbol{\rho}), \tag{4.9}$$

being consistent with the definitions (4.3) and (4.4). The $\hat{\psi}_1(\boldsymbol{\rho},\mathbf{x})$ is ψ_1 with $\mathbf{x}$ in the secular prefactors of Φ_0 and $\partial f/\partial\mathbf{k}$ replaced by $\boldsymbol{\rho}$. Then proceeding as before, we find that

$$\mathcal{P}_{\mathbf{x}=\boldsymbol{\rho}}\mathcal{F}\hat{\psi}_1=\frac{\partial\theta}{\partial t}, \tag{4.10}$$

where

$$\mathcal{F}\equiv L^{(0)}(\partial_{\mathbf{x}}+\partial_{\boldsymbol{\rho}})-L^{(0)}(\partial_{\mathbf{x}})=-[2(\partial_{\mathbf{x}}^2+k_0^2)+\partial_{\boldsymbol{\rho}}^2+2\partial_{\mathbf{x}}\cdot\partial_{\boldsymbol{\rho}}]\times(\partial_{\boldsymbol{\rho}}^2+2\partial_{\mathbf{x}}\cdot\partial_{\boldsymbol{\rho}}). \tag{4.11}$$

Thus applying the operator $\mathcal{S}\equiv\mathcal{P}_{\mathbf{x}=\boldsymbol{\rho}}\mathcal{F}$ to Eq. (4.8), we obtain

$$\eta\frac{\partial\theta}{\partial t}=\mathcal{S}f(\mathbf{k}\cdot\mathbf{x}+\phi^R,\mathbf{k}^R). \tag{4.12}$$

In order to compute the right-hand side of Eq. (4.12) we first calculate $\mathcal{F}f(\vartheta(\mathbf{x},\boldsymbol{\rho}),\mathbf{k}^R(\boldsymbol{\rho}))$, where $\vartheta(\mathbf{x},\boldsymbol{\rho})\equiv\mathbf{k}\cdot\mathbf{x}+\phi^R(\boldsymbol{\rho})$; note that "$\vartheta$" is not the same as θ in Eq. (4.3). We find

$$\begin{aligned}-\mathcal{F}f=&\left(\frac{\partial^4\vartheta}{\partial\rho_\ell^2\partial\rho_m^2}+2k_0^2\frac{\partial^2\vartheta}{\partial\rho_\ell^2}\right)\frac{\partial f}{\partial\vartheta}+\left[\frac{\partial^2\vartheta}{\partial\rho_\ell^2}\left(2\frac{\partial^2}{\partial x_m^2}+\dots\right)\right.\\&\left.+\frac{\partial^2\vartheta}{\partial\rho_\ell\partial\rho_m}\left(4\frac{\partial^2}{\partial x_\ell\partial x_m}+\dots\right)\right]\frac{\partial f}{\partial\vartheta}\\&+\left\{\frac{\partial^3k_n^R}{\partial\rho_\ell\partial\rho_m^2}\left(4\frac{\partial}{\partial x_\ell}+\dots\right)\right.\\&\left.+\frac{\partial k_n^R}{\partial\rho_\ell}\left[4\left(\frac{\partial^2}{\partial x_m^2}+k_0^2\right)\frac{\partial}{\partial x_\ell}+\dots\right]\right\}\frac{\partial f}{\partial k_n^R}\\&+\text{orthogonal terms.}\end{aligned} \tag{4.13}$$

Here the orthogonal terms denotes the terms orthogonal to the null space of the operator $L^{(0)}$. The ellipsis denotes the terms containing $\partial/\partial\rho_m$. This derivative gives rise to terms $\propto\partial\phi^R/\partial\rho_m$ and $\partial k_\ell^R/\partial\rho_m$, which we assume to be higher order than the terms explicitly written out in Eq. (4.13). Furthermore, the consistency condition (4.9) implies that

$$\frac{\partial^2\vartheta}{\partial\rho_\ell\partial\rho_m}=\frac{\partial^2\theta^R}{\partial\rho_\ell\partial\rho_m},\quad\frac{\partial k_\ell^R}{\partial\rho_m}=\frac{\partial^2\theta^R}{\partial\rho_\ell\partial\rho_m}. \tag{4.14}$$

Therefore

$$\begin{aligned}-\mathcal{F}f=&\left[\frac{\partial^4\theta^R}{\partial\rho_\ell^2\partial\rho_m^2}+2\frac{\partial^2\theta^R}{\partial\rho_\ell^2}\left(k_0^2+\frac{\partial^2}{\partial x_m^2}\right)\right.\\&\left.+4\frac{\partial^2\theta^R}{\partial\rho_\ell\partial\rho_m}\frac{\partial^2}{\partial x_\ell\partial x_m}\right]\frac{\partial f}{\partial\theta}+4\left[\frac{\partial^2\theta^R}{\partial\rho_\ell\partial\rho_m}\left(k_0^2+\frac{\partial^2}{\partial x_m^2}\right)\right.\\&\left.+\frac{\partial^4\theta^R}{\partial\rho_\ell\partial\rho_m^2\partial\rho_n}\right]\frac{\partial}{\partial x_\ell}\frac{\partial f}{\partial k_n^R}+\text{orthogonal terms.}\end{aligned} \tag{4.15}$$

Thus Eq. (4.12) becomes

$$\eta\frac{\partial\theta^R}{\partial t}=L_{\ell m}\frac{\partial^4\theta^R}{\partial\rho_\ell\partial\rho_m\partial\rho_n^2}+D_{\ell m}\frac{\partial^2\theta^R}{\partial\rho_\ell\partial\rho_m}, \tag{4.16}$$

where

$$L_{\ell m}=-\left\langle\Phi_0^\dagger,\delta_{\ell m}\Phi_0+4\frac{\partial}{\partial x_\ell}\frac{\partial f}{\partial k_m}\right\rangle,$$

$$\begin{aligned}D_{\ell m}=-\Big\langle&\Phi_0^\dagger,2\delta_{\ell m}(\partial_{\mathbf{x}}^2+k_0^2)\Phi_0+4\frac{\partial^2}{\partial x_\ell\partial x_m}\Phi_0\\&+4(\partial_{\mathbf{x}}^2+k_0^2)\frac{\partial}{\partial x_\ell}\frac{\partial f}{\partial k_m}\Big\rangle.\end{aligned} \tag{4.17}$$

The $\Phi_0^\dagger$ is the zero-eigenvalue eigenfunction of the adjoint to $L^{(0)}$, and the scalar product $\langle a,b\rangle$ is defined by

$$\langle a,b\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\theta ab.$$

This is the proto RG equation (within the approximation mentioned above). By inspection we realize that the differentiation with respect to $\boldsymbol{\rho}$ twice raises the power of η. Namely, we find that the $L_{\ell m}$-term is the $O(\eta^2)$ contribution. Consequently we may discard it to obtain the RG equation to $O(\eta)$ as

$$\eta\frac{\partial\theta^R}{\partial t}=D_{\ell m}\frac{\partial^2\theta^R}{\partial\rho_\ell\partial\rho_m}. \tag{4.18}$$

This agrees with the result of Sasa [17] who employed the standard RG perturbation method (due to Chen *et al.* [9]). In the one mode-approximation, $D_{\ell m}$ agrees with the diffusion tensor obtained by Cross and Newell [6]; see also Ref. [16]. It thus demonstrates that our RG equation is nothing but the Cross-Newell phase equation.

In closing a remark on the $L_{\ell m}$-term is in order. Proceeding to the next order calculation one may convince oneself that this term is in fact the dominant one among the $O(\eta^2)$ contributions. In particular, in the one-mode approximation with $k\approx k_0$, we find $L_{\ell m}=-\delta_{\ell m}$. With the new variable $\mathbf{X}\equiv\eta\boldsymbol{\rho}$, Eq. (4.16) is then reduced to

$$\frac{\partial\theta^R}{\partial t}=-\eta\frac{\partial^4\theta^R}{\partial X^4}+D_{\ell m}\frac{\partial\theta^R}{\partial X_\ell\partial X_m}. \tag{4.19}$$

Indeed Passot and Newell [15] showed earlier that the regularization is necessary of the Cross-Newell phase diffusion equation (4.18) near the pattern singularities and that the principal regularizing contribution to the right-hand side of Eq. (4.18) is $-\nabla\nabla^2\cdot\mathbf{k}$ when $k\approx k_0$, in agreement with our RG result (4.19).

V. CONCLUDING REMARKS

In summary, we have considered the evolution of a large-scale modulation of spatially periodic patterns. We have derived the amplitude equation and the nonlinear phase equations for the SH model, and demonstrated that those equations previously derived by the reductive perturbation method are all RG equations. We have done this without constructing the explicit secular solutions. This is called the proto RG approach, and the approach we have employed in this paper is the most abstract one. We remark that when a model equation has a limit cycle solution, the proto RG derivation of its phase equation is much simpler [12].

A consistent RG reduction of the SH equation to the rotationally covariant form is presented in Sec. II. This resolves the dispute about its derivation by Graham [18,19] by means of the RG method. Note that Eq. (2.21) includes all possible terms up to order ε^2, while the result in Ref. [18] did not contain the last term on the right-hand side of Eq. (2.21). Namely, the equation obtained by Graham is not consistent to order ε (as first realized by Ref. [19]). If we wish to retain all the differential operators in the equation as seen in Eq. (2.21), we need a higher order correction to the nonlinear term, which is the last term on the right-hand side of Eq. (2.21).

As stated above, Sasa [17] derived the same result as Eq. (4.18) by the combination of the explicit perturbative calculation of secular terms and the RG method of Ref. [9]. To contrast it with the proto RG approach developed in the present paper, we present in the Appendix the less abstract proto-RG calculation using the more explicit expression than Eq. (4.5). As the right-hand side of our target equation (4.6) is directly proportional to $\partial\theta/\partial t$, we can take a shortcut cutting off Eq. (4.12). Namely, writing down the following structure for the solution to Eq. (4.6)

$$\psi_1 = g\frac{\partial f}{\partial\theta} + h_\ell \frac{\partial f}{\partial k_\ell}, \tag{5.1}$$

where g and h_ℓ are polynomials of $\mathbf{x}$, we can obtain Eq. (4.18) from the straightforward calculation of Eq. (4.10) (but without calculating explicit expressions for g and h_ℓ as was done in Ref. [17]). This derivation is a proto RG version of Sasa's.

However, we point out that the renormalization procedure in Sasa's theory is contradictory. An internally consistent approach should give the same result if one employs Eq. (4.12) instead of Eq. (4.10). This is not the case with Sasa's renormalization. In place of Eq. (4.8) he renormalizes Eq. (4.5) as

$$\begin{aligned}\psi = {}& f(\mathbf{k}^R(\boldsymbol{\rho})\cdot\mathbf{x} + \phi^R(\boldsymbol{\rho}), \mathbf{k}^R(\boldsymbol{\rho})) + \eta[\delta\mathbf{k}(\boldsymbol{\rho})\cdot(\mathbf{x}-\boldsymbol{\rho})\Phi_0(\mathbf{x})\\ &+ \psi_1(\mathbf{x}) - \hat{\psi}_1(\boldsymbol{\rho},\mathbf{x})],\end{aligned} \tag{5.2}$$

where $\mathbf{k}^R$, $\delta\mathbf{k}$ and ϕ^R are defined by Eq. (A2). A little calculation shows that $\mathcal{S}[\eta[\delta\mathbf{k}\cdot(\mathbf{x}-\boldsymbol{\rho})\Phi_0] = 0$. Hence the counterpart of Eq. (4.12) in Sasa's theory reads

$$\mathcal{S}f(\mathbf{k}^R\cdot\mathbf{x} + \phi^R, \mathbf{k}^R) = \eta\frac{\partial\theta}{\partial t}. \tag{5.3}$$

The resultant phase equation from the straightforward calculation of the left-hand side following the same line of arguments as used in Sec. IV, however, is different from Eq. (4.18) [in fact, being the same expression but for different numerical factors from Eq. (4.18)]. This is inconsistent. The lack of the internal consistency can be attributed to the physically unreasonable definition of the renormalized phase in Eq. (5.2):

$$\mathbf{k}^R\cdot\mathbf{x} + \phi^R. \tag{5.4}$$

When the phase ϕ is renormalized, the renormalization should automatically and uniquely determine the wave vector renormalization, cf. Eqs. (3.8) and (4.9). The choice Eq. (5.4), on the other hand, leaves indeterminacy of the renormalized wave vector even after renormalization of ϕ.

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APPENDIX

In this appendix we take a less abstract approach than in Sec. IV to obtain the phase diffusion equation (4.18). Owing to the peculiar form of Eq. (4.10), the algebra is somewhat simpler with this method.

We start with the structure [cf. Eq. (4.6)]

$$\hat{\psi}_1(\boldsymbol{\rho},\mathbf{x}) = g(\boldsymbol{\rho})\frac{\partial f(\mathbf{x})}{\partial\theta} + h_\ell(\boldsymbol{\rho})\frac{\partial f(\mathbf{x})}{\partial k_\ell}. \tag{A1}$$

Here g and h_ℓ are polynomials of $\boldsymbol{\rho}$, and the $\mathbf{x}$-dependence of f arises through the variable θ [see Eq. (4.3)]. We define the renormalized $\mathbf{k}$ and ϕ by

$$\begin{aligned}\mathbf{k} &= \mathbf{k}^R(\boldsymbol{\rho}) + \eta\delta\mathbf{k}(\boldsymbol{\rho}),\\ \phi &= \phi^R(\boldsymbol{\rho}) + \eta\delta\phi(\boldsymbol{\rho}),\end{aligned} \tag{A2}$$

where $\delta\mathbf{k}$ and $\delta\phi$ are counter terms. Now that Eq. (4.5) is written as

$$\psi = f(\mathbf{k}\cdot\mathbf{x} + \phi^R, \mathbf{k}^R) + \eta\left(\delta\phi\frac{\partial f}{\partial\theta} + \delta k_\ell\frac{\partial f}{\partial k_\ell} + \psi_1\right) + \cdots, \tag{A3}$$

the secular part $\hat{\psi}_1$ satisfies

$$\delta\phi\frac{\partial f}{\partial\theta} + \delta k_\ell\frac{\partial f}{\partial k_\ell} + \hat{\psi}_1 = 0. \tag{A4}$$

Namely,

$$g = -\delta\phi, \quad h_\ell = -\delta k_\ell. \tag{A5}$$

Thus the renormalized perturbative result is written as

$$\psi = f(\mathbf{k}\cdot\mathbf{x} + \phi^R, \mathbf{k}^R) + \eta(\psi_1 - \hat{\psi}_1), \tag{A6}$$

to $O(\eta)$ in accordance with Eq. (4.8). Differentiating with respect to $\boldsymbol{\rho}$ the bare phase written as

$$\theta=\theta^R+\mathbf{k}^R\cdot\mathbf{x}^R-\eta(g+h_\ell x_\ell^R), \tag{A7}$$

where $\theta^R\equiv\mathbf{k}\cdot\boldsymbol{\rho}+\phi^R$ and $\mathbf{x}^R\equiv\mathbf{x}-\boldsymbol{\rho}$, we obtain

$$k_m^R=\frac{\partial\theta^R}{\partial\rho_m}+\eta\left(h_m-\frac{\partial g}{\partial\rho_m}\right). \tag{A8}$$

Here we have used the relation $\partial k_\ell^R/\partial\rho_m=\eta\partial h_\ell/\partial\rho_m$ which derives from $0=\partial k_\ell/\partial\rho_m=\partial(k_\ell^R-\eta h_\ell)/\partial\rho_m$. However, the translational invariance precludes any terms explicitly depending on $\boldsymbol{\rho}$ on the right-hand side of Eq. (A.8), which requires $h_m=\partial g/\partial\rho_m$. Hence

$$\mathbf{k}^R=\frac{\partial\theta^R}{\partial\boldsymbol{\rho}}, \tag{A9}$$

as in Eq. (4.9), and

$$\eta\frac{\partial h_m}{\partial\rho_\ell}=\eta\frac{\partial^2 g}{\partial\rho_\ell\partial\rho_m}=\frac{\partial^2\theta^R}{\partial\rho_\ell\partial\rho_m}. \tag{A10}$$

For the structure (A1),

$$L^{(0)}(\partial_{\mathbf{x}})\hat{\psi}_1=h_\ell L^{(0)}\frac{\partial f}{\partial k_\ell}=-h_\ell L_\ell^{(1)}\Phi_0, \tag{A11}$$

where the last equality follows from the fact that $L^{(0)}(\mathbf{x}\Phi_0+\partial f/\partial\mathbf{k})=0$, and $L_\ell^{(1)}=-4(\partial_{\mathbf{x}}^2+k_0^2)\partial_{x_\ell}$. However, $\mathcal{P}L^{(1)}\Phi_0=0$ by symmetry, so that

$$\mathcal{P}L^{(0)}\hat{\psi}_1=0. \tag{A12}$$

Thus Eq. (4.10) is confirmed as should be. Now the explicit form (A1) of $\hat{\psi}_1$ allows us to calculate the left-hand side of Eq. (4.10). Specifically we obtain

$$\begin{aligned}-\mathcal{F}\hat{\psi}_1=&\frac{\partial^4 g}{\partial\rho_\ell^2\partial\rho_m^2}\frac{\partial f}{\partial\theta}+4\frac{\partial^3 g}{\partial\rho_\ell\partial\rho_m^2}\frac{\partial}{\partial x_\ell}\frac{\partial f}{\partial\theta}+\frac{\partial^4}{\partial\rho_\ell^2\partial\rho_m^2}\frac{\partial f}{\partial k_\ell}\\&+4\frac{\partial^3 h_\ell}{\partial\rho_m\partial\rho_n^2}\frac{\partial}{\partial x_m}\frac{\partial f}{\partial k_\ell}+4\frac{\partial^2 g}{\partial\rho_\ell\partial\rho_m}\frac{\partial^2}{\partial x_\ell\partial x_m}\frac{\partial f}{\partial\theta}\\&+4\frac{\partial^2 h_\ell}{\partial\rho_m\partial\rho_n}\frac{\partial^2}{\partial x_m\partial x_n}\frac{\partial f}{\partial k_\ell}+2\frac{\partial^2 g}{\partial\rho_\ell^2}(\partial_{\mathbf{x}}^2+k_0^2)\frac{\partial f}{\partial\theta}\\&+4\frac{\partial g}{\partial\rho_\ell}(\partial_{\mathbf{x}}^2+k_0^2)\frac{\partial}{\partial x_\ell}\frac{\partial f}{\partial\theta}+2\frac{\partial^2 h_\ell}{\partial\rho_m^2}(\partial_{\mathbf{x}}^2+k_0^2)\frac{\partial f}{\partial k_\ell}\\&+4\frac{\partial h_\ell}{\partial\rho_m}(\partial_{\mathbf{x}}^2+k_0^2)\frac{\partial}{\partial x_m}\frac{\partial f}{\partial k_\ell}.\end{aligned} \tag{A13}$$

In comparison with Eq. (4.10) we see that g must be a third-degree polynomial in $\boldsymbol{\rho}$, whereas h_ℓ must be second-degree. Therefore (A13) is reduced to

$$\begin{aligned}-\mathcal{F}\hat{\psi}_1=&4\frac{\partial^2 g}{\partial\rho_\ell\partial\rho_m}\frac{\partial^2}{\partial x_\ell\partial x_m}\Phi_0+2\frac{\partial^2 g}{\partial\rho_\ell^2}(\partial_{\mathbf{x}}^2+k_0^2)\Phi_0\\&+4\frac{\partial h_\ell}{\partial\rho_m}(\partial_{\mathbf{x}}^2+k_0^2)\frac{\partial}{\partial x_m}\frac{\partial f}{\partial k_\ell}+\text{orthogonal terms}.\end{aligned} \tag{A14}$$

Finally, use of Eq. (A10) in Eq. (4.10) with the above reproduces our previous result (4.18).

An acute reader may have noticed that we end up with the result (4.18) without invoking the approximation that we used in Sec. IV posterior to Eq. (4.13). The reason is that Eq. (4.10) is the strictly $O(\eta)$ equation, whereas Eq. (4.12) contains higher order corrections because the renormalized f therein does so; this is most clearly evidenced by the appearance of the regularizing term in the proto RG equation (4.16), which is $O(\eta^2)$.

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