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Multistability and hysteresis phenomena in passive mode-locked lasers

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We present the analysis of laser passive mode-locking described by complex Ginzburg-Landau equation with a saturable gain and with a nonlinearity of losses decreasing as radiation intensity increases. The hysteresis dependence of the number of pulses in steady state on pump power has been found. The laser operation therewith is multistable: the number of pulses in the established regime depends on the initial state.

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The complex Ginzburg-Landau equation is extensively used in description and analysis of diversified nonlinear systems and phenomena: instability in Poiseuille flow [1], Rayleigh-Bénard instability in binary fluid mixtures [2], electroconvection in nematics [3], passive mode-locking in lasers [4], and so on. Varied states of these systems are described by plane-wave, turbulent [5], stable quasiperiodic, and stable pulse [6] solutions. Of special interest is the investigation of changes of nonlinear system states described by these solutions under changes of system parameters. In this Rapid Communication the bifurcations due to phase-modulation instability and connected with stable multiple pulse laser operation are investigated. The experimental system for observation of multiple pulse regimes, hysteresis phenomena, and multistability under study is a Kerr-lens mode-locked Ti:sapphire laser [7–9].

The complex Ginzburg-Landau equation describing the evolution of the field in a ring laser upon passive mode-locking in dimensionless variables has the form

$$\frac{\partial}{\partial t}E=(1+i\theta)\frac{\partial^2}{\partial z^2}E+(g+p|E|^2+iq|E|^2)E. \qquad (1)$$

The dimensionless field amplitude $E(z,t)$, time variable t, and coordinate z are determined through corresponding dimensional variables E', t', and z' by the following relations: $E=E'/\sqrt{I_a}$, $t=t'\sigma_0$, $z=z'\sqrt{\sigma_0/D_g}$, where I_a is the intensity of the field saturating the nonlinear losses, σ_0 is the net linear resonator losses including the linear field losses in the saturable absorber σ_1 (or the linear diffraction losses), $D_g \approx 0.5\sigma_0 v_{gr}^2/\Gamma^2$ is the frequency dispersion for the gain [13], v_{gr} is the group velocity of radiation, and Γ is the half-width of the gain frequency band expressed in radian per second. The parameter θ is ratio of the real and imaginary parts of the frequency dispersion of the permittivity for the intracavity distributed medium. The first term on the right-hand side of Eq. (1) is connected with the frequency dispersion of the gain and the linear refractive index [13]. The first term in the second parentheses describes the net gain involving the linear losses. The second term accounts for the nonlinear losses. The last term describes the nonlinear refractive index. The field amplitude $E(z,t)$ is subject to periodic boundary conditions.

Equation (1) is based on approximations used widely for the analysis of passive mode-locking of solid-state lasers [9–14]. The most principal ones of them are the approximation of the parabolic frequency dependence of the gain and of the refractive index and the approximation of inertialles nonlinear losses and refractive index. The former is correct if frequency width of radiation (inverse duration of pulse) is much less than spectral bandwidth of the gain. The latter is correct if a response-time of nonlinear losses and refractive index is much less than a pulse width. The detailed derivation of Eq. (1) for the case of passive mode-locking in lasers has been previously presented (see Ref. [13]).

Analysis of passive mode-locking is commonly performed for model with saturable gain

$$g=\frac{1+a}{1+b\int|E|^2\,dz}-1, \qquad (2)$$

where a is the relative pump excess above threshold, $b=I_a\sqrt{D_g/\sigma_0}/(LI_g)$, L is the cavity length, and I_g is the intensity of the monochromatic radiation saturating the gain. The saturation is determined by the total intracavity radiation energy and accordingly the integration in Eq. (2) is carried out over the whole cavity volume. The last term is related with the net linear cavity losses.

The simplest form of nonlinearities p and q is

$$p=p_0,\quad q=q_0, \qquad (3)$$

where p_0, q_0 are numerical constants: $p_0=\sigma_1/\sigma_0$, $q_0=-2\omega n_2 I_a l/(n_0\sigma_0 L)$ (here ω is the carrier frequency of radiation, n_2 is the nonlinearity determining the nonlinear refractive index $\delta n'$ in absolute units for dimensional intensity $\delta n'=n_2|E'|^2$, n_0 is linear refractive index, and l is the length of the nonlinear medium). The model of passive mode-locking based on the approximations of Eqs. (1)–(3) with $\theta=0$, $q_0=0$ was investigated in Refs. [10,11]. The solution describing the light pulse in the hyperbolic secant form was found and the problem of its stability for small disturbances was solved. The stability of this solution is related to negative feedback due to the saturation of gain (2). There are also solutions with an infinite growth of the field intensity. However, these solutions are physically incorrect and related to imperfection of the model for nonlinear losses (3): what actually happens in real experimental systems is that the decrease in losses cannot be greater than the linear losses. In this regard the following form of nonlinearities,

$$p=\frac{p_0}{1+|E|^2}, \quad q=q_0, \tag{4}$$

is more realistic. The model of passive mode-locking describing Eqs. (1), (2), and (4) with $\theta=0$, $q_0=0$ was analyzed in Ref. [12]. It was shown with using of Lyapunov functional that in this case from any initial state the laser operation pass into the steady-state single pulse mode.

The passive mode-locking described by Eqs. (1)–(3) with regard to both frequency dispersion and nonlinearity of refractive index ($\theta\neq 0, q_0\neq 0$) was investigated in Refs. [13,14]. It was found that the solution describing the light pulse in the form of hyperbolic secant with a frequency chirp was

$$E_s=E_0\,\mathrm{sech}^{1+i\alpha}(\beta z)\mathrm{e}^{i\delta\omega t}. \tag{5}$$

The peak amplitude E_0, the frequency chirp parameter α, the inverse duration of the pulse β, and the frequency shift $\delta\omega$ are determined from algebraic equations obtained by substitution of Eq. (5) into Eq. (1) with regard to Eqs. (2) and (3). In the case of periodic boundary conditions, solution (5) is an approximate one. This approximation is good if the pulse duration β^{-1} is much less than the resonator length (length of period). Previously solution (5) of Eq. (1) was found in Ref. [1] in studies of hydrodynamics instability in Poiseuille flow.

The necessary condition for stability of solution (5) is $g<0$. Otherwise the net gain in the wings of the pulse is above zero, and small amplitude noise grows. The condition $g<0$ is obeyed under the following restriction on parameters of nonlinearities $\xi=q_0/p_0$ and dispersions θ [9]:

$$\frac{3\sqrt{1+\theta^2}-2\theta+2\theta^2(\sqrt{1+\theta^2}-\theta)}{1-\theta(\sqrt{1+\theta^2}-\theta)} > \xi > -\frac{3\sqrt{1+\theta^2}+2\theta+2\theta^2(\sqrt{1+\theta^2}+\theta)}{1+\theta(\sqrt{1+\theta^2}+\theta)}. \tag{6}$$

Under condition (6) the characteristic established regime is the regime with a single stationary pulse. Otherwise the turbulent regime for which the whole laser cavity is filled with radiation is realized [9]. The solutions with infinite growth of the field intensity have the same nature as in the above-mentioned case $\theta=0$, $q_0=0$.

An analysis of passive mode-locking with regard to both frequency dispersion and nonlinearity of refractive index ($\theta\neq 0$, $q_0\neq 0$) for the model of nonlinear losses (4) was performed in Ref. [9] by numerical simulation. It was shown that in the case of violation of condition (6) the same turbulent established regime is realized as in case (3). Otherwise the regime with several identical stationary pulses is established. With increasing of pumping, the number of these pulses increases by unit at the following values of pump power:

$$a_{th}^{(k)}\approx k a_{1cr}, \tag{7}$$

where $a_{th}^{(1)}=a_{1cr}$ is the threshold pumping at which the single pulse regime transforms to the two pulse one. Obtained results are in good agreement with the results of the experimental investigation of the multiple pulse operation of a Kerr-lens mode-locked Ti:sapphire laser [7,8].

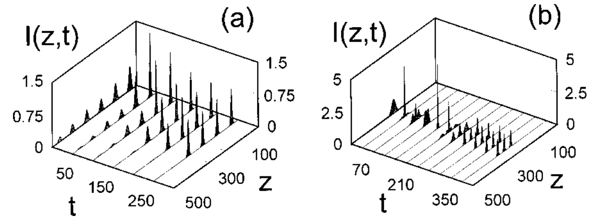

FIG. 1. Illustration of the dependence of the number of pulses in established multiple pulse operation on initial condition. The initial pulses are Gaussian. $\theta=0$, $p_0=0.2$, $q_0=0.3$, $b=0.1$, and $a=3.3$.

In this work we investigate the dependence of number of pulses in established operation on initial conditions and pump power in the frame of nonlinearities (4). The typical transient evolution for passive mode-locking is shown in Fig. 1. The spatial distribution of radiation in cavity $I(z)$ is presented at successive instants of time t. The shown length equal to 512 corresponds to the volume of the cavity in which the generated radiation is concentrated. Figure 1(a) demonstrates the competition between pulses of different amplitudes in the transient process and their coexistence in the steady state. The multiple pulse initial condition with various amplitudes of pulses models the variance of amplitudes of initial noise pulses. It might be well to point out that characteristics of individual pulses in the steady state are identical. Distances between these pulses depend on initial conditions. Figure 1(b) demonstrates the realization of multiple pulses operation as a result of splitting of an initial single pulse. Notice that parameters of the laser for both cases presented in Fig. 1 are the same. This clearly demonstrates the dependence of number of pulses in the steady state on initial conditions.

The detailed information on the dependence of number of pulses N in steady state on pump power a is presented in Fig. 2. The procedure of the construction of this dependence consists of the following. The initial field was chosen in the form of several pulses of various amplitudes. After a transient process the steady state was realized. The number of pulses in the steady state was marked off in Fig. 2. Then pump power a was changed slightly and after the transient

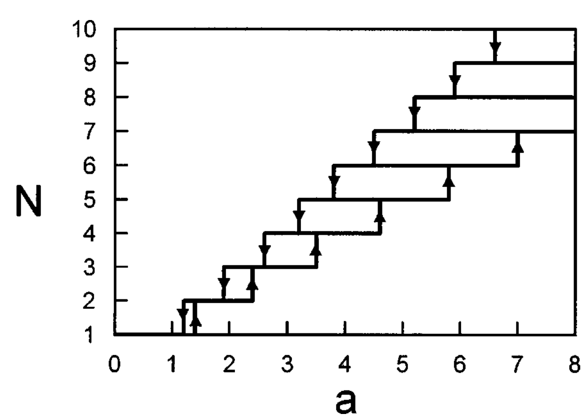

FIG. 2. Multistability and the hysteresis dependence of the number of pulses N in steady state on pump power a. The parameters θ, p_0, q_0, and b are the same as in Fig. 1.

process the number of pulses was newly marked off, and so on. By this means the dependence shown in Fig. 2 was determined. The variance of amplitudes of pulses due to spontaneous noise radiation was modeled by the addition of the initial multiple pulse field lowered by a factor of 100 to formed radiation at the instant the pump power was changed. The increase of the number of pulses is possible only in accordance with the lower stepwise curve marked by the corresponding arrows [see Eq. (7)]. The decrease of the number of pulses is realized in accordance with the upper stepwise curve. The corresponding threshold values for pump power at which the $(k+1)$-pulse regime transforms to the k-pulse one are determined by the formula

$$\tilde{a}_{th}^{(k)} \approx (k+1) a_{2cr}, \tag{8}$$

where a_{2cr} is the step value of pumping for decrease of the pulse number. The horizontal lines show the constancy of number of pulses under changes of pumping a. These horizontal lines in combination with stepwise curves form closed hysteresis loops. As can be seen from Fig. 2, the dependence of pulse number N on pumping a is a many-valued function; that is, the generation is multistable. The number of pulses in an established regime depends on initial conditions. The number of possible steady states increases with increasing pump power.

Let us concentrate on physical mechanism of competition and coexistence of pulses with frequency chirp due to nonlinear refractive index. The nonlinear saturable losses (3) and (4) play the role of a positive feedback. The positive feedback selects the most intensive pulse and suppresses pulses with lesser amplitudes. However, there is a mechanism of negative feedback which in contrast equalizes amplitudes of pulses. The nature of this mechanism consists of the following. In view of the nonlinear refractive index the frequency chirp arises, and the spectrum of the formed pulse is broadened. As a consequence, the efficiency of the gain in the active medium having the finite amplification frequency band drops, with the result that the greater amplitude of the pulse entails its less amplification. Notice that the response time of this negative feedback is equal to the time of formation of the equilibrium frequency chirp for the given amplitude of the pulse. This equilibrium frequency chirp is determined by the balance between mechanisms inducing the pulse phase-modulation and causing its degradation.

Playing with the parameters that determine positive and negative feedback determines the type of established regime. Let us estimate quantitatively the net feedback. In our numerical simulation the transient process consists of three stages: (i) the establishment of equilibrium between the gain g and the total radiation energy $\int |E|^2\, dz$ (the fast stage); (ii) the establishment of equilibrium pulse duration and of equilibrium frequency chirp (more slower stage); and (iii) the competition between pulses with various amplitudes and with equilibrium chirp and duration (very slow stage in vicinity of bifurcation point). We analyze the third stage. The solution of Eq. (1) with nonlinearities (3) is searched in the form of several pulses with various amplitudes and with equilibrium duration and frequency chirp

$$E(z,t) = \sum_k E_{ok} \frac{e^{(\lambda_k + i\delta\omega_k)t}}{\cosh^{1+i\alpha_k}(\beta_k z)}, \tag{9}$$

where E_{ok}, α_k, β_k, $\delta\omega_k$, and λ_k are the peak amplitude, the equilibrium frequency chirp, the equilibrium inverse duration, the frequency shift, and the parameter of temporal increment for kth pulse. Substituting Eq. (9) in Eq. (1) gives

$$\lambda_k + i\delta\omega_k \approx \beta_k^2 (1+i\theta)(1+i\alpha_k)^2 + g, \tag{10}$$

$$(p+iq)|E_{0k}|^2 \approx \beta_k^2 (1+i\theta)(1+i\alpha_k)(2+i\alpha_k). \tag{11}$$

Solution (9) with parameters determined from Eqs. (10) and (11) is correct with $\lambda_k t \ll 1$ when in Eq. (11) $\exp(2\lambda_k t) \approx 1$.

From Eq. (11) the equilibrium frequency chirp and the equilibrium inverse duration are determined as functions of the peak amplitude of pulse $\alpha_k = \alpha_k(|E_{ok}|^2)$, $\beta_k = \beta_k(|E_{ok}|^2)$. For the temporal increment of the kth pulse we have the following expression:

$$\lambda_k = \frac{p}{2-\alpha_k^2} |E_{ok}|^2 (1-\alpha_k^2) + g. \tag{12}$$

Hereinafter, for simplicity, assume that $\theta = 0$. The temporal increment λ_k can be treated as the net coefficient of amplification for the kth pulse. The last term in Eq. (12) connected with the gain in the active medium and playing the role of negative feedback [see Eq. (2)] is the same for all pulses and has no influence on competition between them. The term connected with the unit in the parentheses due to nonlinear losses (for definiteness let us assume that $\alpha_k^2 \ll 1$) describes the positive feedback. The term connected with α_k^2 in the parentheses describes the negative feedback due to the frequency dispersion of the gain [the term $(\partial^2/\partial z^2)E$ on the right-hand side of Eq. (1)]. With nonlinearities (3) the equation for the frequency chirp α_k does not depend on the peak amplitude E_{ok} and is determined by parameters of the laser,

$$\frac{\alpha_k}{2-\alpha_k^2} = \frac{q}{3p} = \frac{q_0}{3p_0}. \tag{13}$$

If condition (6) is fulfilled then $\alpha_k^2 < 1$ and the positive feedback prevails: the transient process passes into the regime with a single stationary pulse. If condition (6) breaks down then $\alpha_k^2 > 1$ and, on the contrary, the negative feedback exceeds the positive feedback. As a result, small-amplitude pulses that arise from perturbations grow in the wings of the powerful pulses. The transient evolution lasts until the whole cavity of laser is filled with the generated radiation.

In the case of nonlinearities (4) the picture of transient evolution changes cardinally. The approximate estimation of the frequency chirp can be obtained by substitution of the values p and q from Eq. (4) into Eq. (13). In this case the frequency chirp depends on the peak intensity,

$$\frac{\alpha_k}{2-\alpha_k^2} \approx \frac{q}{3p} \approx \frac{q_0(1+|E_{0k}|^2)}{3p_0}. \tag{14}$$

With p_0 and q_0 satisfying condition (6) the characteristic form of the dependence of $\delta\lambda_k = \lambda_k - g$ on the peak intensity

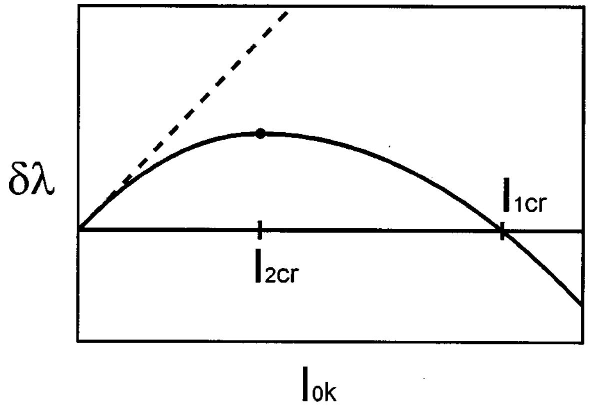

FIG. 3. Dependence of the amplification coefficient $\delta\lambda$ for the pulse with equilibrium duration and frequency chirp on the peak intensity I_{ok}. The condition (6) for parameters θ, p_0, and q_0 is fulfilled. The dashed line corresponds to nonlinearities (3). The solid curve corresponds to nonlinearities (4).

is presented in Fig. 3. At small peak intensity $I_{0k}=|E_{0k}|^2$ this dependence is linear. With increasing peak intensity the frequency chirp increases, and at $|E_{ok}|^2=3p_0/q_0-1$ it reaches the level $\alpha_k^2=1$. Therefore $\delta\lambda_k=0$.

The form of the dependence of the amplification coefficient for pulse $\delta\lambda_k$ on its peak intensity shown in Fig. 3 allows an understanding of multiple pulse operation, hysteresis phenomena, and multistability. At small pumping power for all peak intensities $|E_{ok}|^2<I_{cr2}$ and the greater intensity of pulse entails its greater amplification. As a result, the single pulse established operation is realized. With increasing pump power the peak intensity of this single pulse in the steady state increases. When it becomes greater than I_{cr1} then the amplification for small amplitude pulses becomes positive and the second pulse arises in generation. For peak intensities $|E_{ok}|^2>I_{cr2}$ the greater intensity of pulse entails its less amplification and amplitudes of these pulses are equalized. As this takes place, the peak intensities of these two pulses becomes less than I_{cr1} because of energy balance. With further increasing pump power the peak intensities of these two pulses increase and reach the level I_{cr1} and then a third pulse arises in generation, and so on. This process is indicated by the lower stepwise curve in Fig. 2.

In the case of multiple pulse operation with decreasing pump power the peak intensities of all pulses are the same and decrease. As long as their peak intensities remained greater than I_{cr2}, the number of these pulses does not change. This process is described by the horizontal lines in Fig. 2. When the peak intensities of pulses reach the value I_{cr2} then because of perturbations the peak intensity of one of them becomes less than I_{cr2} and this pulse is suppressed. Thereafter the peak intensities of the remained pulses becomes greater than I_{cr2} because of energy balance. With further decreasing pump power the peak intensities of these pulses decrease and reach again the level I_{cr2} and then the successive pulse is suppressed, and so on. This process is shown by the upper stepwise curve in Fig. 2.

It would appear reasonable that experimental conditions for realization of both hysteresis and multistability is the same as with a multiple pulse operation due to phase-modulation instability [7–9].

In conclusion, it has been found that in passive mode-locked lasers with a nonlinearity of losses decreasing as radiation intensity increases the phase modulation instability due to nonlinear refractive index results in multistable operation and hysteresis phenomena. It seems likely that similar peculiarities can be manifested themselves in other systems described by a similar complex Ginzburg-Landau equation.

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