PHYSICAL REVIEW E VOLUME 62, NUMBER 4 OCTOBER 2000

Reductive use of renormalization group

K. Nozaki,[1] Y. Oono,[2,3] and Y. Shiwa[4]
[1]*Department of Physics, Nagoya University, Nagoya, Japan*
[2]*Research Institute of Mathematical Sciences, Kyoto University, Kyoto, Japan*
[3]*Department of Physics, University of Illinois at Urbana-Champaign, 1110 W. Green Street, Urbana, Illinois 61801-3080**
[4]*Statistical Mechanics Laboratory, Kyoto Institute of Technology, Matsugasaki, Sakyo-ku, Kyoto 606-8585, Japan*
(Received 10 May 2000)

It has been recognized that singular perturbation and reductive perturbation can be unified from the renormalization group (RG) theoretical point of view. However, the recognition has been only formal in the sense that it has not given us any new insight nor provided any new technical advantage over the usual RG approach. With our approach, the proto RG method proposed here, we can clearly show that system reduction is the key to singular perturbation methods. The approach also makes the calculation of singular perturbation results more transparent than the conventional RG approach. Consequently, for example, a consistent and easy RG derivation of the rotational covariant Newell-Whitehead-Segel equation is possible.

PACS number(s): 64.60.Ak

Suppose we apply to a system a perturbation, e.g., we add to the equation governing the system some nonlinear terms, dissipative terms, a new boundary, etc. The system is usually not structurally stable against such perturbations, so the perturbation results are, if computed naively, plagued with singularities (secular terms). It has been recognized for some time [1] that these singularities in the naive perturbation theories can be renormalized away by the modification (renormalization) of the parameters in the unperturbed state (amplitude, phase, etc.). The renormalized results agree with or are sometimes (numerically at least) better than those traditionally computed with the aid of singular perturbation methods [2]. The modified parameters are governed by the renormalization group (RG) equations that turn out to be, e.g., large-scale slow-motion equations (reduced equations) [3] (often one of the famous named equations such as the Burgers equation, the Kuramoto-Sivashinsky equation, or the Boltzmann equation [4]).

These equations were traditionally derived with the aid of the reductive perturbation theory [5,6]. Let us call the method to obtain space-time large scale equations as RG equations the *RG theoretical reduction* (or the *reductive RG method*). Thus, a close relation between reductive perturbation and singular perturbation has been recognized. Also a more fundamental nature of reductive perturbation than singular perturbation has been suggested because the key features of singular perturbation results are governed by reduced equations = RG equations. However, the derivation of RG equations required singular perturbation results in the conventional RG approach [3,2], so the truly fundamental nature of reductive RG could not be demonstrated.

The main purpose of this Rapid Communication is to free the RG theoretical reduction from the necessity of explicit secular terms as much as possible. As a consequence, we can now clearly claim that extraction of global features of the systems through reduction is the key to singular perturbations. As an illustration of this transparent approach a consistent RG reduction of the Swift-Hohenberg equation to the rotationally covariant Newell-Whitehead-Segel (NWS) equation is presented; this resolves the dispute about its derivation by means of the RG method [7,8].

Our general strategy of the reductive RG explained in this paper may be summarized with the following five steps:

(1) Solve the zeroth order equation, and set up perturbation equations.
(2) Write down the general form of the secular terms.
(3) Find the equation governing resonant secular terms.
(4) Construct the proto RG equation.
(5) Reduce the proto RG equation to the RG equation.

We will explain the undefined terminology ''proto RG equation'' in due course. Here we discuss only the lowest nontrivial order, but the method works to higher orders. Although they require some information about the solutions to the perturbation equations, still the procedure is much simpler than the RG calculations given in, e.g., [2].

First, we illustrate our proposal with a very simple example, the van der Pol equation:

$$\frac{d^2y}{dt^2}+y=\epsilon(1-y^2)\frac{dy}{dt}. \tag{1}$$

We expand the solution naively as $y=y_0+\epsilon y_1+\epsilon^2 y_2+\cdots$. Step (1) gives just

$$y_0=Ae^{it}+A^*e^{-it}. \tag{2}$$

The term y_1 is governed by

$$\left(\frac{d^2}{dt^2}+1\right)y_1=(1-y_0^2)\frac{dy_0}{dt}, \tag{3}$$

so that y_1 must have the general form $y_1=P_1e^{it}+Q_1e^{3it}+$c.c., where c.c. implies complex conjugate terms [step (2)]. Equation (3) gives

$$L_tP_1=iA(1-|A|^2), \tag{4}$$

*Permanent address.

where

$$L_t \equiv \frac{d^2}{dt^2} + 2i\frac{d}{dt}. \tag{5}$$

This is the end of step (3). Let us define P_1 without the constant term. Then this is the secular term itself and must be renormalized into the modification of the integration constant of the zeroth order result (renormalization of A to A_R). Thus, the renormalized perturbation result can be written as

$$y(t) = A_R(\tau)e^{it} + \epsilon[P_1(t,A_R(\tau)) - P_1(\tau,A_R(\tau))]e^{it} + \cdots + \text{c.c.} \tag{6}$$

In the conventional RG approach [2] we say that y has nothing to do with the variable τ which we introduced, so $\partial y/\partial\tau = 0$. The resultant equation is the RG equation (in this simple case). In this conventional RG, we do not use the properties of the problem under study at all; the procedure is quite blind.

We should interpret the generic procedure of applying $\partial/\partial\tau$ as an attempt to characterize the function space spanned by singular solutions. Our crucial observation is that it is natural to use an operator governing the secular terms. The operator is that in Eq. (4). Let L_τ be the same operator as L_t defined in Eq. (5), with t being replaced by τ. Then, obviously $L_\tau y(t) = 0$, so with the aid of Eq. (4) we immediately obtain (to this order we may identify A_R in P_1 with A, a constant)

$$0 = L_\tau y = [L_\tau A_R - L_\tau P_1(\tau, A_R)]e^{it}, \tag{7}$$

or with the aid of Eq. (4) (we replace τ with t in the result)

$$\left(\frac{d^2}{dt^2} + 2i\frac{d}{dt}\right)A_R(t) = \epsilon i A_R(1 - |A_R|^2). \tag{8}$$

This is step (4). We call such an equation a *proto RG equation*. Notice that no explicit solution (other than the unperturbed solution) has ever been used. This turns out to be a crucial feature for partial differential equations (PDE).

The last step is to reduce this formal result to a consistent equation to the relevant order (the lowest nontrivial order in the present exposition). Equation (8) tells us that differentiation raises the power of ϵ, so to order ϵ we may discard the second order derivative. This is the final step (5):

$$\frac{dA_R}{dt} = \frac{1}{2}\epsilon A_R(1 - |A_R|^2), \tag{9}$$

which is the standard RG equation and is an amplitude equation. Higher order systematic calculation along this line is possible with slightly more calculations [9] (some of which are used below) [10,11]. Thus, obtaining the RG equation can be made largely independent from singular perturbation results, as conceptually expected: system reduction is the key to singular perturbation.

In the case of ODE, the secular terms due to the perturbation seem unambiguously identifiable, but for PDE the situation is usually complicated, because many different modes of singular behaviors are possible. Therefore, it is almost mandatory to avoid any particular perturbation result.

Let us illustrate the proto RG approach for PDE with the two-dimensional Swift-Hohenberg equation [12] that has caused some controversy [2,7,8]:

$$\frac{\partial u}{\partial t} = \epsilon(u - u^3) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)^2 u. \tag{10}$$

We consider this on the whole plane for all positive t. As a zeroth order solution, we choose a roll solution along the y axis: $Ae^{ikx} + \text{c.c.}$, where A is a complex numerical constant. We expand u around this solution as

$$u = Ae^{ikx} + \epsilon u_1 + \epsilon^2 u_2 + \cdots + \text{c.c.} \tag{11}$$

The first order correction obeys

$$\frac{\partial u_1}{\partial t} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)^2 u_1 = (1 - 3|A|^2)Ae^{ikx} - A^3 e^{3ikx}. \tag{12}$$

This is a linear PDE, so we may write the solution in the following form:

$$u_1 = P_1(t,\boldsymbol{r})e^{ikx} + Q_1(t,\boldsymbol{r})e^{3ikx}, \tag{13}$$

where $\boldsymbol{r} = (x,y)$. Since e^{ikx} is a null solution to the linear operator, P_1 must contain spatially secular (that is, unbounded or not integrable) terms. This is step (2).

The step (3) is as follows. Since

$$\left[\frac{\partial}{\partial t} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)^2\right]P_1 e^{ikx} = (1 - 3|A|^2)Ae^{ikx}, \tag{14}$$

we have

$$\left[\frac{\partial}{\partial t} + \frac{\partial^4}{\partial x^4} + 4ik\frac{\partial^3}{\partial x^3} + 2\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2} + \left(-4k^2\frac{\partial^2}{\partial x^2} + 4ik\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x} + \frac{\partial^4}{\partial y^4}\right)\right]P_1 \equiv LP_1 = (1 - 3|A|^2)A. \tag{15}$$

From this, we clearly see that P_1 can never be a constant. We can read $Q_1 = -A^3/64k^4$ off from the equation for Q_1.

The renormalized perturbation series reads

$$u = A_R(\tau,\boldsymbol{\rho})e^{ikx} + \epsilon[P_1(t,\boldsymbol{r}) - P_1(\tau,\boldsymbol{\rho})]e^{ikx} + \cdots + \text{c.c.} \tag{16}$$

Hence, the proto RG equation can be obtained with the aid of Eq. (15) as

$$L_{\tau,\boldsymbol{\rho}}A_R(\tau,\boldsymbol{\rho}) = \epsilon(1 - 3|A_R|^2)A_R, \tag{17}$$

where $L_{\tau,\boldsymbol{\rho}}$ is L with the replacement $t \to \tau$, $\boldsymbol{r} \to \boldsymbol{\rho}$. Notice that we have never used any explicit solutions of the perturbative equation. This is the end of step (4). The reader may wonder how unique the choice of the operator L is; to map the un-

wanted secular term, any operator mapping it to zero should do. We do not have a general answer to this question. However, since we are interested in the global features of the system, the operator must be as low order differential operator as possible. With this requirement L is unique in practice, and is given by Eq. (15).

L contains much more terms than the standard result. To reduce the equation further, we must choose the way we observe the system. If we wish to derive an equation governing isotropic evolution of the pattern, we must assume x and y (or $\partial/\partial x$ and $\partial/\partial y$) to scale in the same way, so $\partial/\partial t \sim \partial^2/\partial x^2 \sim \partial^2/\partial y^2 \sim \epsilon$. That is, to order ϵ with isotropy requirement we obtain the following RG equation:

$$\left(\frac{\partial}{\partial t} - 4k^2\frac{\partial^2}{\partial x^2}\right)A_R = \epsilon(1-3|A_R|^2)A_R . \tag{18}$$

This is step 5.

To obtain the next order, we need the equation for the second order term:

$$\left[\frac{\partial}{\partial t} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)^2\right]u_2 = \{(1-6|A|^2)P_1 - 3A^2P_1^* - 3A^{*2}Q_1\}e^{ikx} + \{(1-6|A|^2)Q_1 - 3A^2P_1\}e^{3ikx} - 3A^2Q_1e^{5ikx}. \tag{19}$$

The general form of the solution is given by [step (2)] (we assume all the solutions to the corresponding homogeneous equation are taken care of by the zeroth order consideration)

$$u_2 = P_2(t,\boldsymbol{r})e^{ikx} + Q_2(t,\boldsymbol{r})e^{3ikx} + R_2e^{5ikx}. \tag{20}$$

Here, P_2 is obviously secular. Q_2 is also secular because the inhomogeneous term with e^{3ikx} is already secular in Eq. (19). LP_2 is given by [step (3)]

$$LP_2 = (1-6|A|^2)P_1 - 3A^2P_1^* - 3A^{*2}Q_1 . \tag{21}$$

However, to use this equation to remove the secular term from the second order result, we may ignore all the secular terms on the right-hand side of Eq. (21) that will be gone anyway [13]. Thus, we need only $Q_1 = -A^3/64k^4$ obtained already. Hence, the proto RG equation reads

$$LA_R = \epsilon(1-3|A_R|^2)A_R + \epsilon^2\frac{3}{64k^4}|A_R|^4A_R . \tag{22}$$

To order ϵ^2 this is actually the RG equation, and is the rotationally covariant NWS equation first obtained by Gunaratne *et al.* [14]. Note that Eq. (22) includes all possible terms up to order ϵ^2, while the result in [7] did not contain the last term on the left-hand side of Eq. (22).

The key element of singular perturbation is to separate out global systematic effects of perturbations. If we can characterize the function space spanned by the solutions, the most important results of singular perturbation can be captured. Our observation is that the proto RG operator is the operator characterizing the function space, although we still need clearer mathematical characterization of our procedure. In this way, reductive perturbation is the essence of singular perturbation [15].

In example (10) studied above, the smallness of ϵ represents perturbations whose collective effects we are concerned with. For ϵ not small but at a late stage of the roll formation, $\partial u/\partial t$ becomes the only small perturbation that enters the equation. The phase dynamics is now the relevant slow motion. In this case the proto RG method works again [16], and the reduced equation is the Cross-Newell phase equation [17].

In summary, we have considerably freed the RG theoretical reduction from explicit perturbation results with the aid of the proto RG equation, demonstrating the fundamental nature of system reduction in singular perturbation problems. Our procedure also simplifies system reduction. As an illustration we have resolved the controversy as to the RG reduction of the Swift-Hohenberg equation to the isotropic Newell-Whitehead-Segel equation.

Y.O. is grateful to Fred Furtado and Sin-ichi Sasa for useful discussions, corrections, and correspondence, and to Yoichiro Takahashi for his hospitality at Research Institute for Mathematical Sciences in Kyoto and for a partial financial aid. The present work is, in part, supported by the National Science Foundation Grant No. NSF-DMR 99-70690, and by the Japan Society for Promotion of Science Grant-in-Aid for Scientific Research (A) 09304022.

[1] N. D. Goldenfeld, O. Martin, and Y. Oono, J. Sci. Comput. **4**, 355 (1989).
[2] L.-Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996).
[3] L.-Y. Chen, N. Goldenfeld, Y. Oono, and G. Paquette, Physica A **204**, 111 (1994).
[4] O. Pashko and Y. Oono, Int. J. Mod. Phys. B **41**, 3290 (2000).
[5] T. Taniuti, Suppl. Prog. Theor. Phys. **55**, 1 (1974).
[6] H. Mori and Y. Kuramoto, *Dissipative Structures, and Chaos*, translated by G. C. Paquette (Springer, Berlin, 1998).
[7] R. Graham, Phys. Rev. Lett. **76**, 2185 (1996); **80**, 3888(E) (1998).
[8] K. Matsuba and K. Nozaki, Phys. Rev. Lett. **80**, 3886 (1998).
[9] K. Nozaki and Y. Oono (unpublished).
[10] If we wish to have a singular perturbation result $y(t) = A_R(t)e^{it} + Q_1e^{3it}$ + c.c. to order ϵ explicitly, then we need the solution $A_R(t)$ to this RG equation and $Q_1 = iA_R^3/8$ that can be obtained by inspection of Eq. (3). For linear problems, the proto RG method works to all orders (and actually beyond them [9,11]).
[11] Y. Oono, Int. J. Mod. Phys. B (to be published). (Gakushuin International Summer School of Mathematical Physics special issue, edited by H. Ezawa and H. Araki.)
[12] J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).
[13] Since the renormalization to order ϵ should have removed all the secularities from P_1, in the final result, this quantity must vanish. Thus, we may ignore all the nonconstant terms in the perturbation equations when we use it in the derivation of

proto RG equations. A general discussion is given in [9].

[14] G. H. Gunaratne, Qi Ouyang, and H. L. Swinney, Phys. Rev. E **50**, 2802 (1994).

[15] The reader may wonder whether there is any use of our RG reduction in boundary layer type problems. In the RG approach of [2] inner problems are solved and their global feature determines the boundary layer (without inner-outer matching), so these problems and long-time or global slow modulation problems have no fundamental difference. This is why system reduction is the universal key.

[16] Y. Shiwa (unpublished).

[17] M. C. Cross and A. C. Newell, Physica D **10**, 299 (1984).