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Soliton stability versus collapse

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Nonlinear Schrödinger equations with positive potentials are investigated. Stability of their stationary ground states is confronted with properties of nonlinear blow-up. The stability condition for these soliton modes is shown to involve restrictions on the potential slope, which do not forbid the collapse. We prove that stable ground states must have a power integral below their counterpart with no potential.

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The nonlinear Schrödinger (NLS) equation belongs to the class of universal models, which can be derived for describing the propagation of dispersive, nonlinear wave-packets [1]. It is often met in nonlinear optics [2], plasma physics [3], field theory [4], and physics of matter waves [5] as well. The fundamental nature of NLS becomes apparent in the universal methods for soliton stability [6]. Solitons are stationary nonlinear bound states, to which solutions to the NLS equation

$$i\partial_z\psi+\Delta\psi+|\psi|^{2\sigma}\psi-U(\vec{r},|\psi|^2)\psi=0, \quad (1)$$

may relax and evolve on long scales, when they are stable. In Eq. (1), ψ represents the slowly varying envelope of a high-frequency wave field propagating along the z axis within a diffraction/dispersion space of dimension number D. The nonlinearity $|\psi|^{2\sigma}\psi$ is a power-law nonlinearity with exponent $\sigma>0$, which reduces to cubic with $\sigma=+1$ for, e.g., an optical Kerr material [2]. The real function U is a potential, depending on the space variables and/or the wave field intensity, which can either stand for corrections to the nonlinear power-law response, or for some inhomogeneities in the medium. Soliton solutions, also termed as "ground states," are defined by the stationary solutions $\psi(\vec{r},z)=\chi(\vec{r},\lambda)\exp(i\lambda z)$, where χ is solution to the differential equation

$$-\lambda\chi+\nabla^2\chi+|\chi|^{2\sigma}\chi-U(\vec{r},|\chi|^2)\chi=0. \quad (2)$$

For $U=0$, it is well-known that no *stable* soliton solution exists when D satisfies $D\geqslant 2/\sigma$, and that any initial solution $\psi_0(\vec{r})\equiv\psi_0(\vec{r},0)$ is condemned to either collapse or spread out [1]. In contrast, for a bounded, space-dependent potential $U(\vec{r})$ vanishing at infinity, Rose and Weinstein [7] demonstrated that nonlinear ground states of Eq. (1) exist and can be stable, when $-\Delta+U$ already supports a bound state. In that case, the theorem for orbital stability, first established by Vakhitov and Kolokolov in [8] for saturable nonlinearities, applies: the ground state $\chi(.,\lambda)$ is orbitally stable if and only if

$$d_\lambda\int|\chi(\vec{r},\lambda)|^2 d\vec{r}>0 \quad (d_\lambda\equiv d/d\lambda). \quad (3)$$

Orbital stability means that, modulo the elementary symmetries of Eq. (1), the shape of ψ is preserved by perturbations having no growing modes, when they act on any solution close to the soliton orbit. Therefore, any initial datum ψ_0 nearby $\chi\exp(i\lambda z)$ does not collapse, but remains close to the ground state as z increases. In view of this, three pertinent questions arise: (i) Does Eq. (3) hold for potentials U including nonlinearities in $|\psi|^2$? (ii) What must the potential shape be to produce stable ground states? (iii) What is the role of the "free" NLS ground state defined for $U=0$ with respect to the bound states of Eq. (1)?

The goal of this Rapid Communication is to answer the above points. For technical convenience, we employ the notations introduced in [7]: Δ is the D-dimensional Laplace operator $\Sigma_i\partial^2/\partial x_i^2$ for vector $\vec{r}=(x_1,x_2,\ldots,x_D)$. H^1 is the Sobolev space $H^1=\{f:(1-\Delta)^{1/2}f\in L^2\}$ with norm $\|f\|_{H^1}^2=\|f\|_2^2+\|\vec{\nabla}f\|_2^2$, involving the L^p spaces $L^p=\{f:\|f\|_p\equiv(\int|f|^p d\vec{r})^{1/p}<+\infty\}$. We shall assume that $\psi_0(\vec{r})\in H^1$ and U is derived from $G\in L^1$:

$$G(\vec{r},|\psi|^2)\equiv\int_0^{|\psi|^2}U(\vec{r},s)\,ds. \quad (4)$$

First, we discuss sufficient conditions for collapse and consider potentials with a positive slope, for which blow-up can occur. Second, we determine some integral relations satisfied by the solitons and compare them with the requirements for collapse. We deduce that the L^2 norm of χ must be below that of free NLS ground states. Finally, we solve the stability problem for the solitons of Eq. (1) and show that the potential U, although appropriate for the collapse, can also assure their stability.

1. Conditions for blow-up. Equation (1) preserves the L^2 norm (so-called mass or power integral)

$$N\equiv\|\psi\|_2^2=\int|\psi|^2 d\vec{r}=\|\psi_0\|_2^2 \quad (5)$$

and the Hamiltonian

$$H=H[\psi]\equiv\|\vec{\nabla}\psi\|_2^2-\frac{1}{\sigma+1}\|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}+\int G(\vec{r},|\psi|^2)d\vec{r}. \quad (6)$$

Standard computations [9] furthermore yield the following identity for the integral $\langle r^2\rangle\equiv\int r^2|\psi|^2 d\vec{r}/N$:

$$d_z^2\langle r^2\rangle=\frac{4}{N}\left\{2H+\frac{2-\sigma D}{\sigma+1}\|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}-D\times\int[(1+2/D)G(\vec{r},|\psi|^2)-|\psi|^2U(\vec{r},|\psi|^2)]d\vec{r}-\int\left[\int_0^{|\psi|^2}r\partial_rU(\vec{r},s)ds\right]d\vec{r}\right\},\tag{7}$$

while the center of mass (centroid) of the solution, defined by $\langle\vec{r}\rangle\equiv\int\vec{r}|\psi|^2d\vec{r}/N$, evolves with the acceleration

$$d_z^2\langle\vec{r}\rangle=-\frac{2}{N}\int\left[\int_0^{|\psi|^2}\hat{x}_i\partial_{x_i}U(x_i,s)ds\right]d\vec{r}.\tag{8}$$

In these expressions, ∂_rU $(r\equiv|\vec{r}|)$ and $\partial_{x_i}U$ denote partial differentiations of U with respect to its explicit dependencies on the space variables only, whereas in what follows $\vec{\nabla}U$ will refer to the total derivative $\vec{\nabla}U=\hat{x}_i\partial_{x_i}U+U'\vec{\nabla}|\psi|^2$, where $U'\equiv\partial U/\partial|\psi|^2$. Equation (8) shows that, with a space-dependent potential U, the trajectory of a wave-packet governed by Eq. (1) is displaced along z through the D-dimensional space and translational invariance is lost. Besides, Eq. (7) indicates that the wavefunction ψ can undergo a collapse with $\langle r^2\rangle\to0$ at a finite propagation distance, z_c. *Sufficient conditions* for collapse are then given by the combination of the following four requirements:

(i) $D\geqslant2/\sigma$,
(ii) $H<0$,
(iii) $(1+2/D)G(\vec{r},s)\geqslant sU(\vec{r},s),\ \ \forall s\geqslant0$,
(iv) $\partial_rU(\vec{r},s)\geqslant0,\ \ \forall s\geqslant0$.

When the above conditions are all realized for a given initial datum ψ_0, the integral $\langle r^2\rangle$ inevitably tends to zero as $z\to z_c$. By applying the estimate $N\leqslant(4/D^2)\|\vec{\nabla}\psi\|_2^2\langle r^2\rangle$ [1], $\langle r^2\rangle\to0$ implies the divergence of the gradient norm $\|\vec{\nabla}\psi\|_2$ in the same limit. As a result, the solution blows up and cannot survive in H^1 [9]. Other sufficient conditions for blow-up can be established with positive Hamiltonians, provided that ψ_0 possesses an initial prefocusing $(d_z\langle r^2\rangle|_{z=0}<0)$ related to the gradient of its phase (see, e.g., [1]). However, we shall disregard them, as we are rather interested in the fate of initial states being close to stationary bound states with no spatially dependent phase. Wishing to confront the stability of these states with their possibility of blowing up, we shall moreover suppose that the constraint of multidimensionality (i) holds and that the potential already satisfies (iii). This precludes negative functions U and saturating nonlinearities $U\sim|\psi|^{2\gamma}$ with $\gamma>\sigma$ as well. So, for technical convenience, we consider

$$0<U'(\vec{r},s)\leqslant\sigma s^{\sigma-1},\quad\forall s\geqslant0,\quad\text{and}\ U(\vec{r},0)\geqslant0.\tag{9}$$

2. Soliton properties versus conditions for collapse. For the "free" NLS equation $(U=0)$, the existence of ground states follows from minimizing appropriate functional, when $\lambda>0$ only [10]. The minimum of this functional is then realized on the positive solution of Eq. (2), denoted by R, with Hamiltonian $H[R]\equiv(1-2/\sigma D)\|\vec{\nabla}R\|_2^2$. Soliton solutions are functions of $r=|\vec{r}|$ alone and exist for $2/D\leqslant\sigma<2/(D-2)$ when $D\geqslant2$ [7,10]. They also possess the lowest L^2 norm $N_0=\|R_0\|_2^2$ computed from Eq. (2) with $\lambda=1$, as $R(r,\lambda)=\lambda^{1/2\sigma}R_0(\sqrt{\lambda}r)$. This constant norm N_0 intervenes in the Sobolev inequality

$$\|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}\leqslant C(N_0)\|\vec{\nabla}\psi\|_2^{\sigma D}\|\psi\|_2^{2+\sigma(2-D)},\tag{10}$$

where, e.g., $C(N_0)=(\sigma+1)/N_0^\sigma$ at the critical dimension $D=2/\sigma$. When U is space-dependent, the existence of ground states may proceed from analogous minimization principles performed on functionals such as

$$J^\lambda[f]=\frac{\|\vec{\nabla}f\|_2^2+\int G(\vec{r},|f|^2)d\vec{r}+\lambda\|f\|_2^2}{\|f\|_{2(\sigma+1)}^2}.\tag{11}$$

For example, with $U=U(\vec{r})$, the minimum of J^λ is attained at a positive solution of Eq. (2), which is nothing else but the ground state χ [7]. This solution also minimizes $H[f]$ under constraint, when it is stable. More general potentials in the form $U(\vec{r},\chi^2)$ have been investigated in [11], for which, with suitable growth conditions on U, Eq. (2) was shown to admit positive solutions that realize critical points for Hamiltonian-type functionals. Here, it is clear that $J^\lambda[f]$ has at least one critical point on the ground-state solution χ. In light of these results, we henceforth conjecture that $U(\vec{r},|\psi|^2)$ allows for the existence of ground states, defined as the unique, positive solutions of Eq. (2) realizing a *minimum* for $J^\lambda[f]$. From a practical point of view, these solutions can numerically be identified by means of classical shooting techniques.

Direct calculations yield the values of the gradient and $L^{2(\sigma+1)}$ norms of the soliton χ, namely,

$$\|\vec{\nabla}\chi\|_2^2=\frac{\sigma D}{2(\sigma+1)-\sigma D}\left[\lambda\|\chi\|_2^2+\int|\chi|^2U_sd\vec{r}\right]+\frac{(\sigma+1)}{2(\sigma+1)-\sigma D}\int|\chi|^2\vec{r}\cdot\vec{\nabla}U_sd\vec{r},\tag{12}$$

$$\frac{1}{\sigma+1}\|\chi\|_{2(\sigma+1)}^{2(\sigma+1)}=\frac{2}{2(\sigma+1)-\sigma D}\left[\lambda\|\chi\|_2^2+\int|\chi|^2U_sd\vec{r}+\tfrac{1}{2}\int|\chi|^2\vec{r}\cdot\vec{\nabla}U_sd\vec{r}\right].\tag{13}$$

For some positive potentials satisfying, e.g., $\partial_rU\geqslant0$, λ may not thus be restrained to $\lambda>0$ for insuring the positiveness of the norms (12) and (13), unlike the case $U=0$. Next, these expressions lead to the Hamiltonian

$$H_s=\frac{\sigma D-2}{2(\sigma+1)-\sigma D}\left[\lambda\|\chi\|_2^2+\int|\chi|^2U_sd\vec{r}\right]+\frac{\sigma}{2(\sigma+1)-\sigma D}\int|\chi|^2\vec{r}\cdot\vec{\nabla}U_sd\vec{r}+\int G(\vec{r},|\chi|^2)d\vec{r}.\tag{14}$$

Here, H_s and U_s are the Hamiltonian and potential evaluated on the soliton solution. By using the previous relations, the soliton is readily found to have a stationary mean-square radius, i.e., $d_z^2\langle r^2\rangle_s=0$, as expected. Solitonlike ground states thus correspond to boundary solutions lying in between the sufficient condition for collapse ($d_z^2\langle r^2\rangle<0$) and other spreading behaviors ($d_z^2\langle r^2\rangle>0$). By expanding $\vec{\nabla}U$ in Eq. (14), H_s is found to be always positive under the requirements (i), (iii), and (iv), and, therefore, it never fulfills (ii). Besides, the acceleration (8) of the centroid $\langle\vec{r}\rangle$ must satisfy $\int|\chi|^2\vec{\nabla}U_s d\vec{r}=\vec{0}$, which imposes a spatial distribution of U compatible with that of $|\chi|^2$: if $|\chi|^2$ is an even, bell-shaped function, then the gradient of U must be odd in the spatial variables.

We now discuss the compatibility between the characteristic integrals of stable χ and conditions for collapse. A necessary condition for collapse at critical dimension ($\sigma D=2$) can be obtained by bounding the Hamiltonian

$$H\geqslant\|\vec{\nabla}\psi\|_2^2(1-N^\sigma/N_0^\sigma)+\int G(\vec{r},|\psi|^2)d\vec{r},$$

with the inequality (10). As $G\in L^1$, the blow-up of the gradient norm can only occur for $D=2/\sigma$, whenever $N>N_0$. Conversely, the minimum of H is attained on the soliton solution if $H[\chi]>-\infty$, which is ensured for $N_s\equiv\|\chi\|_2^2<N_0=\|R\|_2^2$ only.

At supercritical dimensions $D>2/\sigma$, the same result holds: stable ground states have a L^2 norm below their counterpart for $U=0$. To show this, we argue on gradient norms and Hamiltonian domains. First, in the domain $H\leqslant H[R]$, all ground states with $H_s\leqslant H[R]$ immediately verify $\|\vec{\nabla}\chi\|_2^2\leqslant\|\vec{\nabla}R\|_2^2$, as inferred from relations (12) and (14). Second, let us rewrite Eq. (7) as

$$d_z^2\langle r^2\rangle=\frac{4\sigma D}{N}\{H-F(\psi)\},\tag{15}$$

$$F(\psi)\equiv\left(1-\frac{2}{\sigma D}\right)\|\vec{\nabla}\psi\|_2^2+\left(1+\frac{1}{\sigma}\right)\int G d\vec{r}-\frac{1}{\sigma}\int|\psi|^2 U d\vec{r}+\frac{1}{\sigma D}\int\left[\int_0^{|\psi|^2}r\partial_r U(\vec{r},s)ds\right]d\vec{r}.\tag{16}$$

Under the current assumptions (iii) and (iv), $F(\psi)$ satisfies $F(\psi)\geqslant(1-2/\sigma D)\|\vec{\nabla}\psi\|_2^2$ and it converges to H_s as ψ relaxes to $\psi=\chi\exp(i\lambda z)$. Thus, all initial data ψ_0 verifying $H<H_s$ will produce collapse at a finite propagation distance. In the complementary domain $H_s\geqslant H>H[R]$, it is then sufficient to choose initial data with $\|\vec{\nabla}\psi_0\|_2^2>\|\vec{\nabla}R\|_2^2$, for condemning any solution to blow up with $\langle r^2\rangle\to 0$ and $\|\vec{\nabla}\psi\|_2^2\to+\infty$. Indeed, from

$$\frac{N}{4\sigma D}d_z^2\langle r^2\rangle\leqslant H-\frac{\sigma D-2}{\sigma D}\|\vec{\nabla}\psi\|_2^2<H-H[R],$$

collapse systematically arises in this Hamiltonian range for gradient norms $\|\vec{\nabla}\psi\|_2^2>H/(1-2/\sigma D)>H[R]/(1-2/\sigma D)\equiv\|\vec{\nabla}R\|_2^2$. Consequently, to avoid the blow-up, the norm $\|\vec{\nabla}\chi\|_2^2$ must satisfy the opposite inequality,

$$\|\vec{\nabla}\chi\|_2^2\leqslant\|\vec{\nabla}R\|_2^2.\tag{17}$$

Since $\|\vec{\nabla}\chi\|_2^2/\|\vec{\nabla}R\|_2^2\geqslant\|\chi\|_2^2/\|R\|_2^2$ from Eq. (12), we finally deduce

$$N_s=\|\chi\|_2^2\leqslant\|R\|_2^2=N_0/\lambda^{D/2-1/\sigma}.\tag{18}$$

This result shows that, if stable ground states exist, their L^2 norm must be lower than $\|R\|_2^2$, except in the limit $\lambda\to+\infty$, for which Eq. (2) suggests that $\|\chi\|_2^2$ can attain $\|R\|_2^2$ from below. While N_s is bounded by a fixed constant at critical dimension, this norm always vanishes at large λ for supercritical dimensions, and may, for appropriate potentials, access high values when λ approaches zero.

3. Stability criterion. We now perturb the ground state χ and employ $\psi(\vec{r},z)=[\chi(\vec{r},\lambda)+v(\vec{r},z)+iw(\vec{r},z)]\exp(i\lambda z)$, where χ, v and w are real functions satisfying $v,w\ll\chi$. Inserting this solution into Eq. (1) and linearizing it lead to

$$v_z=L_0 w,\quad -w_z=L_1 v,\tag{19}$$

where the operators

$$L_0=\lambda-\Delta-\chi^{2\sigma}+U(\vec{r},\chi^2),\tag{20}$$

$$L_1=L_0-2\sigma\chi^{2\sigma}+2U'(\vec{r},\chi^2)\chi^2,\tag{21}$$

have the properties

$$L_0\chi=0,\quad L_1\frac{\partial\chi}{\partial\lambda}=-\chi,\quad L_1\vec{\nabla}\chi=-\chi\left(\frac{\vec{r}}{r}\right)\partial_r U,\tag{22}$$

with $\vec{\nabla}\chi\equiv(\vec{r}/r)\partial_r\chi$. Conditions (9) ensure that the infimum of the continuum spectrum of L_0 and L_1 is positive with $\lim_{r\to+\infty}\{\lambda+U+2U'\chi^2\}>0$. The operator $L_0\equiv-\chi^{-1}\nabla[\chi^2\nabla\chi^{-1}]$ is positive semidefinite with $L_0\chi=0$, where χ has no node since it is positive. Thus, L_0 is positive definite when operating on all functions v,w orthogonal to χ. By combining Eqs. (19) for functions $v\perp\chi$, a sufficient and necessary condition for orbital stability readily follows from the sign of $\mathrm{Inf}\langle v|L_1 v\rangle/\langle v|L_0^{-1}v\rangle$, which must be positive in the class of perturbations perpendicular to χ [12]. By convention, brackets denote the L^2 scalar product $\langle a|b\rangle=\int a^* b d\vec{r}$. Stability proof then results from determining the eigenvalue α entering the spectral problem $L_1 v=\alpha v+\beta\chi$, where $\beta\neq 0$ is a Lagrange multiplier related to the orthogonality constraint $\langle v|\chi\rangle=0$. Setting $v=(L_1-\alpha)^{-1}\beta\chi$ and applying $\langle v|\chi\rangle=0$, we have to analyze the variations of the function $g(\alpha)=\langle\chi|(L_1-\alpha)^{-1}\chi\rangle$. This function is well-defined and monotonically increasing for $\alpha\in[\lambda_0,\lambda_1]$, where $\lambda_0<\lambda_1$ are the first two discrete eigenvalues of L_1. With $\lambda_0<0$ and $\lambda_1>0$, solitons χ are then stable whenever $g(0)=\langle\chi|L_1^{-1}\chi\rangle$ is negative, which can be rewritten as Eq. (3) by means of relations (22).

Here, L_1 verifies $L_1<L_0$ together with

$$\langle\chi|L_1\chi\rangle=-2\sigma\|\chi\|_{2(\sigma+1)}^{2(\sigma+1)}+2\int U'(\vec{r},\chi^2)\chi^4 d\vec{r},\tag{23}$$

$$\langle\vec{\nabla}\chi|L_1\vec{\nabla}\chi\rangle=-\tfrac{1}{2}\int\vec{\nabla}\chi^2\left(\frac{\vec{r}}{r}\right)\partial_r U d\vec{r}. \tag{24}$$

From Eq. (23), L_1 has thus at least one negative eigenvalue, $\lambda_0<0$, such that $L_1\psi_0=\lambda_0\psi_0$, where ψ_0 is nodeless. As χ^2 decreases with r almost everywhere, relation (24) moreover shows that this eigenvalue may be unique if U satisfies the item (iv) for collapse, i.e., $\partial_r U\geqslant 0$. In the opposite case, $\partial_r U<0$, the integrals $\langle\chi|L_1\chi\rangle$ and $\langle\vec{\nabla}\chi|L_1\vec{\nabla}\chi\rangle$ both attain negative values, and since $\vec{\nabla}\chi$ and χ are orthogonal to each other, we can always construct an eigenvector of L_1 with negative eigenvalue, being orthogonal to ψ_0. With $\partial_r U<0$, L_1 has thus two distinct negative eigenvalues, which certainly leads to instability. Hence, $\partial_r U\geqslant 0$ guarantees soliton stability. Note that stability is not excluded if $\partial_r U$ is *locally* negative in some bounded regions of space, but still insures the positiveness of Eq. (24). By assuming U with a positive slope, uniqueness of $\lambda_0<0$ proceeds from the second variation of $J^\lambda[f]$ defined in Eq. (11). This second variation is computed from $d^2/d\epsilon^2|_{\epsilon=0}J^\lambda[\chi+\epsilon\eta]$ for any continuous real function η and it leads to

$$A\langle\eta|J''[\chi]\eta\rangle=\langle\eta|L_1\eta\rangle+\langle r_1|\eta\rangle, \tag{25}$$

with $A=\frac{1}{2}\|\chi\|_{2(\sigma+1)}^2>0$ and

$$r_1\equiv[2\sigma/\|\chi\|_{2(\sigma+1)}^{2(\sigma+1)}]\langle\eta|\chi^{2\sigma+1}\rangle\chi^{2\sigma+1}.$$

Since the minimum of $J^\lambda[f]$ is attained on the soliton solution, $J''[\chi]$ is non-negative under the constaints (9). $AJ''[\chi]$ moreover expresses as L_1+r_1, where r_1 is a rank one perturbation. Consequently, for any $v\in H^1$ orthogonal to $\chi^{2\sigma+1}$, L_1 reduces to $AJ''[\chi]$, which is positive. Suppose now that there exist two orthogonal eigenstates of L_1 with two distinct negative eigenvalues. We can always combine them linearly for constructing a vector v perpendicular to $\chi^{2\sigma+1}$. This would certainly lead to $\langle v|L_1 v\rangle<0$, which is impossible. Therefore, L_1 has exactly one negative eigenvalue, provided that $\partial_r U\geqslant 0$.

Finally, we need to clear up the condition for L_1 to have its second eigenvalue being positive. Let us suppose that there exists $\psi_1\neq 0$ such that $L_1\psi_1=\lambda_1\psi_1=0$. Then, necessarily, $\langle\psi_1|L_1\vec{\nabla}\chi\rangle=\vec{0}$ leads to $\langle\psi_1|\chi(\vec{r}/r)\partial_r U\rangle=\vec{0}$. Here, ψ_1 is perpendicular to χ [see Eq. (22)]. Also, because $L_1<L_0$ and χ is nodeless, ψ_1 has one node (e.g., ψ_1 is odd for radially symmetric ground states). Thus, the existence of $\lambda_1=0$ implies $\langle\psi_1|\chi(\vec{r}/r)\partial_r U\rangle=\vec{0}$, which has no reason to be realized. For χ even and $(\vec{r}/r)\partial_r U$ odd, this statement indeed holds for $D=1$ with ψ_1 being zero at center for symmetry reasons. At higher dimensions, ψ_1 must be decomposed into spherical harmonics as $\psi_1(r,\theta,\varphi)=\Sigma_{lm}\phi_{lm}(r)Y_{lm}(\theta,\varphi)$. It corresponds to the first spherical harmonics $\psi_1=\phi_1(r)\vec{r}/r$ with ψ_1 odd and $\phi_1(r)$ even in r, whenever $L_1+(D-1)/r^2$ is non-negative [12], which we admit. In that case, the former result holds with $\langle\phi_1|\chi\partial_r U\rangle\neq 0$, so that $\lambda_1>0$.

In summary, we have shown that the soliton solutions of Eq. (1) are orbitally stable when the potential U verifies Eq. (9) with $\partial_r U\geqslant 0$. Stable ground states of Eq. (1) are characterized by $N_s\leqslant\|R\|_2^2=N_0/\lambda^{D/2-1/\sigma}$, where R is the soliton mode of Eq. (2) with $U=0$. From a physical viewpoint, our result means that when a potential $U>0$ exhibits a positive curvature near the point where a stationary wave-packet lies with a mass below standard thresholds for self-focusing, it can arrest the natural spreading of free NLS solutions and produce a stable nonlinear waveguide. Reversely, with higher norms, the wave-packet is capable of collapsing *faster* than for $U=0$. As application examples, we emphasize potentials in the forms, e.g., $U=r^2$ for Bose-Einstein condensates [5,13] and other potentials such as $U=V(\vec{r})|\psi|^{2\gamma}$ with an increasing even function $V(\vec{r})>0$ and $\gamma<\sigma$.

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