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Analytical study of spatiotemporal chaos control by applying local injections

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Spatiotemporal chaos control by applying local feedback injections is investigated analytically. The influence of gradient force on the controllability is investigated. It is shown that as the gradient force of the system is larger than a critical value, local control can reach very high efficiency to drive the turbulent system of infinite size to a regular target state by using a single control signal. The complex Ginzburg-Landau equation is used as a model to confirm the above analysis, and a four-wave-mixing mode is revealed to determine the dynamical behavior of the controlled system at the onset of instability.

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Chaos control and synchronization have attracted much attention in the last decade [1]. We extend the idea of chaos control to spatiotemporal chaos. Local injection is one of the approaches most extensively used for spatiotemporal chaos control [2]. Recently, it has been found numerically that the existence of gradient force is of crucial importance for enhancing the control efficiency [3]. Nevertheless, no analytical results have been obtained in this aspect. In the present Rapid Communication, we will study this problem by an eigenvalue analysis of the controlled distributed systems. The mechanism underlying the influence of gradient force on the control efficiency will be generally clarified.

First, we consider a simple system of coupled oscillators with nearest couplings and periodic boundary condition

$$x_i=f(x_i)+\frac{\varepsilon}{2}(x_{i+1}+x_{i-1}-2x_i)+\frac{r}{2}(x_{i-1}-x_{i+1}),$$
$$x_{i+N}=x_i\,,\quad i=0,1,2,\ldots,N-1, \tag{1}$$

where ε is the diffusive coupling while r is the gradient coupling. Assuming the system is chaotic, we hope to control spatiotemporal chaos by driving the system to a wanted regular state, e.g., a homogeneous state of

$$x_i=0,\quad i=0,1,2,\ldots,N-1, \tag{2}$$

which is supposed to be the solution of the system and unstable without control. We apply negative feedback injection to the first oscillator for controlling the system

$$\dot{x}_i=f(x_i)+\frac{\varepsilon}{2}(x_{i+1}+x_{i-1}-2x_i)+\frac{r}{2}(x_{i-1}-x_{i+1}),$$
$$i=1,2,\ldots,N-1, \tag{3a}$$

$$\dot{x}_0=f(x_0)+\frac{\varepsilon}{2}(x_1+x_{N-1}-2x_0)+\frac{r}{2}(x_{N-1}-x_1)-\lambda x_0. \tag{3b}$$

Let us take a large feedback $\lambda\to\infty$, then the first site can be surely pinned to the target state $x_0=0$, this pinning reduces Eqs. (3b) to

$$x_0=x_N=0. \tag{4}$$

Linearizing Eqs. (1) against the target (2), we obtain

$$\dot{\mathbf{x}}=A\mathbf{x},\quad \mathbf{x}=(x_0,x_1,\ldots,x_{N-1})^T,$$

$$A=\begin{pmatrix} Df-\varepsilon & \frac{\varepsilon+r}{2} & 0 & 0 & \cdots & \frac{\varepsilon-r}{2}\\ \frac{\varepsilon-r}{2} & Df-\varepsilon & \frac{\varepsilon+r}{2} & 0 & \cdots & 0\\ 0 & \frac{\varepsilon-r}{2} & Df-\varepsilon & \frac{\varepsilon+r}{2} & & \vdots\\ \vdots & \vdots & \vdots & & \ddots & \frac{\varepsilon+r}{2}\\ \frac{\varepsilon+r}{2} & 0 & 0 & \cdots & \frac{\varepsilon-r}{2} & Df-\varepsilon \end{pmatrix},$$

$$Df=\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0}>0. \tag{5}$$

With control, Eqs. (3a) and (4) can be linearized to

$$\dot{\mathbf{x}}=B\mathbf{x},\quad \mathbf{x}=(x_1,x_2,\ldots,x_{N-1})^T,$$

$$B=\begin{pmatrix} Df-\varepsilon & \frac{\varepsilon+r}{2} & 0 & 0 & \cdots & 0\\ \frac{\varepsilon-r}{2} & Df-\varepsilon & \frac{\varepsilon+r}{2} & 0 & \cdots & 0\\ 0 & \frac{\varepsilon-r}{2} & Df-\varepsilon & \frac{\varepsilon+r}{2} & & \vdots\\ \vdots & \vdots & \vdots & & \ddots & \frac{\varepsilon+r}{2}\\ 0 & 0 & 0 & \cdots & \frac{\varepsilon-r}{2} & Df-\varepsilon \end{pmatrix}. \tag{6}$$

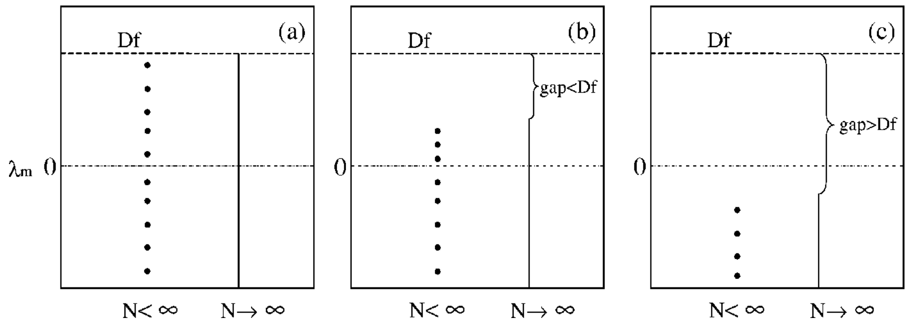

FIG. 1. Schematic figures of eigenvalue distribution of Eq. (8). (a) $r=0$. As $N<\infty$ the eigenvalue distribution is discrete and the largest eigenvalue is smaller than Df. As $N\rightarrow\infty$, the distribution becomes continuous and the largest eigenvalue $\lambda(m)$ approaches Df, and then the gap between Df and the eigenvalue distribution is zero. (b) $0<r<r_c$. As $N\rightarrow\infty$, there appears a gap between Df and $\lambda(m)$. However, the target state is still unstable because this gap is smaller than Df ($\lambda(m)>0$). (c) $r>r_c$. The gap $Df-\lambda(m)>Df$ and $\lambda(m)<0$, the target state is stable.

A is a $N\times N$ first order circulation matrix, of which the N eigenvalues can be easily worked out as

$$\lambda_j=Df-\varepsilon(1-\cos\theta_j)+ir\sin\theta_j,$$
$$\theta_j=\frac{2\pi j}{N},\quad j=0,1,2,\ldots,N-1. \tag{7}$$

The gradient coupling r influences the imaginary part of λ_j only, thus does not affect the stability of the system. The largest LE of the reference state is $\lambda(m)=\lambda_0=Df>0$. Therefore the homogeneous state (2) of the coupled system (1) is unstable if the local dynamics $\dot{\mathbf{x}}=f(\mathbf{x})$ is unstable.

With control, the matrix B is no longer circulated due to the vanishing of the up-right and down-left terms. The $N-1$ eigenvalues of B can be also computed,

$$\lambda_j=Df-\varepsilon+\sqrt{\varepsilon^2-r^2}\cos\frac{j\pi}{N},\quad j=1,2,\ldots,N-1. \tag{8}$$

Two essential differences between the eigenvalues of B [Eqs. (8)] and A [Eqs. (7)] are of crucial significance for the control efficiency. First, with control the gradient coupling r definitely changes the real part of the eigenvalues, and then influences the stability of the target state. Second, the largest LE, $\lambda(m)$, of Eq. (8), becomes smaller than Df. And larger ε and r and smaller N correspond to lower $\lambda(m)$. We are particularly interested in the possibility whether we can control spatiotemporal chaos with infinite size by injecting a single site $i=0$ only. Mathematically, we are interested whether $\lambda(m)$ in Eq. (8) can be smaller than zero when $Df>0$ and $N\rightarrow\infty$. For $r=0$, the answer is definitely no, because we have $\lambda^{N=\infty}(m,r=0)=\lim_{N\rightarrow\infty}\lambda_1=Df>0$. This situation is schematically shown in Fig. 1(a).

However, nonzero r dramatically changes the control efficiency. For $N\rightarrow\infty$, $r\neq 0$, we have

$$\lambda^{N=\infty}(m,r\neq 0)=\lim_{N\rightarrow\infty}\lambda_1=Df-(\varepsilon-\sqrt{\varepsilon^2-r^2}). \tag{9}$$

Now there appears an eigenvalue gap between Df and $\lambda^{N=\infty}(m)$ [see Figs. 1(b) and 1(c)],

$$\Delta=\varepsilon-\sqrt{\varepsilon^2-r^2}. \tag{10}$$

If this gap is larger than the local LE, Df, the target state can be stabilized by a single injection, though the system state has infinitely many positive LEs for $N\rightarrow\infty$ in uncontrolled situations. This is the case of Fig. 1(c).

The analysis from Eqs. (1) to (10) can be easily extended to coupled map lattices and nonlinear partial differential equations, and can be also extended to the cases with space-time-dependent target states. Now we apply the above idea to a model extensively investigated, the one-dimensional complex Ginzburg-Landau equation (CGLE) of system size L with periodic boundary condition [4–6]

$$\partial_t A=A+(1+\mathbf{i}c_1)\partial_x^2 A+r\partial_x A-(1+\mathbf{i}c_2)|A|^2A,$$
$$A(x+L)=A(x,t). \tag{11}$$

where $A(x,t)$ is a complex quantity, and the term of first spatial derivative $r\partial_x A$ represents gradient force. Equation (11) admits traveling wave solutions

$$\hat{A}(x,t)=A_0\exp[\mathbf{i}(kx-\omega t)],\quad k=\frac{2m\pi}{L},$$
$$A_0=\sqrt{1-k^2},\quad \omega=c_2+(c_1-c_2)k^2-rk. \tag{12}$$

As $1+c_1c_2<0$ and for $L\gg 1$, all the solutions (12) are unstable. Here, we fix $c_1=2.1$, $c_2=-1.5$, the system is then deeply in the defect turbulence region. Our task is to control this violent turbulence by injecting a feedback signal to the system at a single point $x=0$ and driving the system to one of the periodic states of Eq. (12). We will discuss the controllability based on the eigenvalue analysis. The imposed control input is given by

$$\partial_t A=A+r\partial_x A+(1+\mathbf{i}c_1)\partial_x^2 A-(1+\mathbf{i}c_2)|A|^2A$$
$$+\varepsilon\delta(x)[\hat{A}(x,t)-A], \tag{13}$$

where the final term is the negative feedback injection based on the deviation from the target state.

First we perform the linear stability analysis of the target state. Inserting

$$A(x,t)=A_0[1-a(x,t)]\exp\{i[kx-\omega t+\phi(x,t)]\}, \tag{14}$$

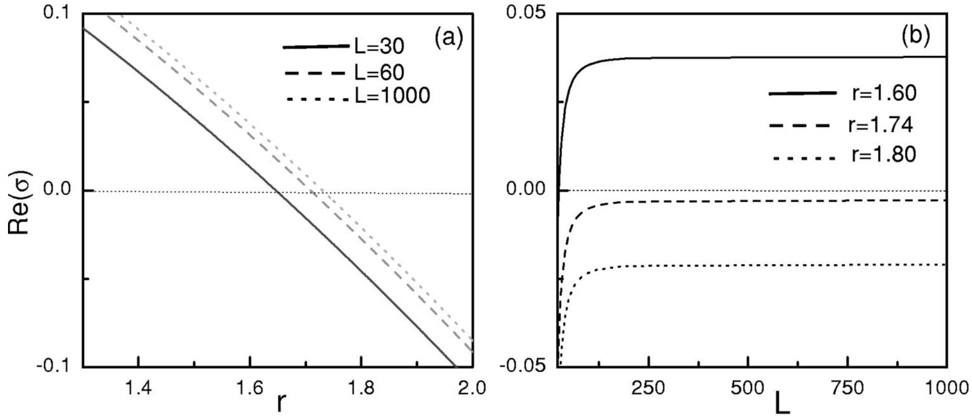

FIG. 2. $c_1=2.1$, $c_2=-1.5$, $k=0$. These parameters are used in all the following figures. (a) The largest Lyapunov exponent Re σ_m vs r for different L. (b) Re σ_m vs L for different r. Note, as $L\to\infty$, Re σ_m crosses zero at $r=r_c\approx 1.735$.

into Eq. (13) and considering $|a|$, $|\phi|\ll 1$, we reach a set of linear equations

$$\begin{aligned} a_t &= a_{xx}+(r-2kc_1)a_x-2(1-k^2)a-c_1\phi_{xx} \\ &\quad -2k\phi_x-\varepsilon\delta(x)a, \\ \phi_t &= \phi_{xx}+(r-2kc_1)\phi_x+c_1a_{xx}+2ka_x \\ &\quad -2c_2(1-k^2)a-\varepsilon\delta(x)\phi. \end{aligned} \tag{15}$$

Assuming an eigenfunction $\binom{a}{\phi}$, corresponding to an eigenvalue σ, we have

$$\begin{pmatrix} a \\ \phi \end{pmatrix} = \sum_{i=1}^{4} \begin{pmatrix} a_i \\ \phi_i \end{pmatrix} e^{\sigma t} e^{p_i x}, \tag{16}$$

with all quantities σ and $a_i, \phi_i, p_i, i=1,2,3,4$, being unknown complex constants. Inserting Eq. (16) to Eq. (15) we obtain the coefficient relations

$$\phi_i = \frac{p_i^2+(r-2kc_1)p_i-2(1-k^2)-\sigma}{c_1p_i^2+2kp_i}a_i, \quad i=1,2,3,4, \tag{17}$$

and the eigenvalue relations

$$\begin{aligned} \sigma &= \frac{1}{2}(F_{11}^{(i)}+F_{22}^{(i)})+\frac{1}{2}\sqrt{(F_{11}^{(i)}+F_{22}^{(i)})^2-4(F_{11}^{(i)}F_{22}^{(i)}-F_{12}^{(i)}F_{21}^{(i)})}, \\ F_{11}^{(i)} &= p_i^2+(r-2kc_1)p_i-2(1-k^2), \quad F_{12}^{(i)}=-c_1p_i^2-2kp_i, \\ F_{21}^{(i)} &= c_1p_i^2+2kp_i-2c_2(1-k^2), \quad F_{22}^{(i)}=p_i^2+(r-2kc_1)p_i, \\ & i=1,2,3,4. \end{aligned} \tag{18}$$

In Eq. (16) there are 13 unknown complex constants and we need 12 constraints to fix them (one constant, say a_1, is arbitrary). Apart from the eight equations of Eqs. (17) and (18), we have four more equations for the boundary conditions at $x=0$ and L,

$$\begin{aligned} & a(0,t)=a(L,t), \quad \phi(0,t)=\phi(L,t), \\ & a_x(0,t)-a_x(L,t)-c_1\phi_x(0,t)+c_1\phi_x(L,t)=\varepsilon a(0,t), \\ & \phi_x(0,t)-\phi_x(L,t)+c_1a_x(0,t)-c_1a_x(L,t)=\varepsilon\phi(0,t). \end{aligned} \tag{19}$$

Then the eigenvalue σ and eigenvector can be solved from Eqs. (17)–(19). As $\varepsilon\to\infty$, the system is pinned to the target state at the injecting point, Eq. (13) is then reduced to a boundary-injecting model $A(0,t)=A(L,t)=\hat{A}(0,t)$, and the constraints (19) is simplified to

$$a(0,t)=a(L,t)=\phi(0,t)=\phi(L,t)=0. \tag{20}$$

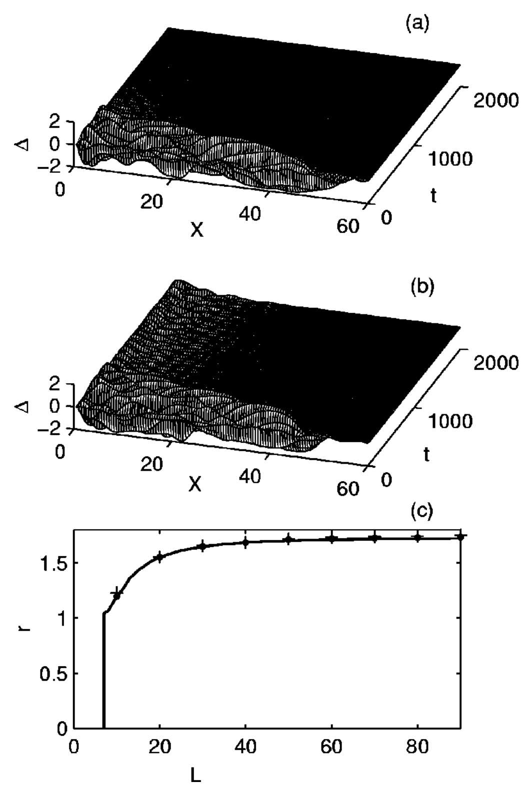

FIG. 3. (a) The space-time evolution of Eq. (13) for $\varepsilon\to\infty$ at $L=80$, $r=1.72$. $\Delta A(x,t)=A(x,t)-\hat{A}(x,t)$ with $\hat{A}(x,t)$ given in Eq. (12) for $k=0$. As t goes on the turbulence is successfully suppressed in the whole system. (b) The same as (a) with r reduced to $r=1.65$. A turbulent tail exists in the left side, which can never be eliminated for infinite time. (c) The controllability distribution in L-r plane. Turbulence can be well controlled in the region above the curves. Solid line, theoretical prediction of Eqs. (17), (18), and (20); black disks, numerical simulation of Eq. (13) by taking initial condition in the vicinity of the target; crosses, simulation of Eq. (13) from arbitrary initial condition.

To end our theoretical analysis, let us make a brief discussion on the form of eigenfunction (16). Without control the eigenfunction is always assumed to have a single mode

$$\begin{pmatrix} a \\ \phi \end{pmatrix} = \begin{pmatrix} a_1 \\ \phi_1 \end{pmatrix} e^{\sigma t} e^{p_1 x},$$

with p_1 being an imaginary value $p_1 = (2\pi n/L)i$ [4–6], given by the periodic boundary condition. With control this assumption should be replaced by the four-wave-mixing eigenfunction (16) to meet the δ-function control at the boundary $x=0$. And the wave numbers p_i become complex. The real parts represent the spatial amplifications (or contractions) of perturbation, which are crucial for the convective instability. In the limit $\varepsilon \to 0$ we will find $a_1, \phi_1 \neq 0$, $p_1 = (2\pi n/L)i$, and $a_i, \phi_i \to 0$, $i=2,3,4$, which recover the well known results without control. Thus, at the onset of instability, four waves (they together serve as an unstable mode) are amplified simultaneously, this is essentially different from the single wave instability without control. We find that this four-wave-mixing mode satisfactorily predicts the system behavior in the controlled systems at the onset of instability.

The above computation is exact. However, no explicit solution like Eqs. (8) and (9) can be available. We should perform numerical computation, based on the analytical results of Eqs. (17)–(19). For simplicity, we consider $\varepsilon \to \infty$ and the target state $k=0$, and then solve Eqs. (17), (18), and (20) for obtaining eigenvalues and eigenvectors. In Figs. 2(a) and 2(b) we calculate the largest LE of the system, i.e., the largest real part of the eigenvalues of the system, for different r and L. Two important features can be observed in the figures. First, increasing the gradient force r is favorable for reducing the largest LE of the system, and for any given L we can always stabilize the target state (i.e., reduce Re σ to negative) by increasing r. Second, there is a critical r, r_c, for $r > r_c$, such that we can control the system with infinitely long size by injecting a single point. In our case, $r_c \approx 1.735$ [in Fig. 2(a) the crossing points of the Re σ curves on the r axis accumulate to r_c as $L \to \infty$].

In order to confirm the analytical prediction in Figs. 2(a) and 2(b), we numerically simulate Eq. (13) with $\varepsilon \to \infty$ by taking the homogeneous oscillation, $k=0$ in Eq. (12), as our target $\hat{A}(x,t)$. In Figs. 3(a) and 3(b) two parameter combinations, corresponding to small positive and negative largest LE's, respectively, are used. The numerical simulations fully agree with the stability analysis. In Fig. 3(c) we study the controllability in the r-L plane. In the region above the curves the target homogeneous oscillation can be stabilized and the defect turbulence of the uncontrolled system can be successfully suppressed. It is striking, that the linear stability analysis (solid line) can perfectly agree with the global behavior of the system (crosses) in the whole parameter plane. In Fig. 3(c) the curves obviously saturate to a finite r value, $r_c \approx 1.735$, above which controlling turbulence in infinite CGLE system by injecting a single space point becomes possible, this confirms the prediction of Fig. 1.

In summary, we have analytically studied the influence of gradient force on the control efficiency of spatiotemporal chaos. The possibility of controlling spatiotemporal chaos of infinitely long system by injecting a single control signal at the upper string point is revealed, based on the eigenvalue analysis. This analytical computation is applied to control CGLE turbulence, and a four-wave-mixing mode is obtained to predict the system dynamics at the onset of the instability of the controlled system.

The investigation in this work is mainly based on the local control in the infinite dissipation limit [see Eqs. (4) and (20)]. This limit is appled only for the convenience of analytical computation. An analysis for general finite feedback control [Eqs. (3) and (19)], which are more realistic in practical cases, will appear in a forthcoming paper publication.

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