Influence of noise on statistical properties of nonhyperbolic attractors

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We analyze effects of bounded white and colored noise on nonhyperbolic chaotic attractors in two-dimensional invertible maps. It is shown that first the nonhyperbolic nature is kept even in the presence of strong noise, but secondly already due to weak noise some properties of nonhyperbolic chaos can become similar to those of hyperbolic and almost hyperbolic chaos. We also estimate the stationary probability measure of noisy nonhyperbolic attractors. For this purpose two different methods for calculating the probability density are applied and the obtained results are compared in detail.

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I. INTRODUCTION

Small random perturbations are inevitably present in all dynamical systems. The degree and the character of their influence mainly depend on the peculiarities of a system under study, in particular, on its nonlinear properties and characteristics of its stability and robustness [1–11,13–20]. If the system's dynamics is complicated enough, then the influence of fluctuations can lead to various and sometimes unexpected effects. Noise can increase the degree of disorder in a system, including a transition to chaotic dynamics [2–4], or, on the contrary, can cause the system's behavior to be more ordered [5–7,12,13]. It was established numerically that under the influence of noise attractors of dissipative dynamical systems can undergo different bifurcations, which can significantly change their structure [8–10].

Stochastic nonlinear problems are of fundamental and practical importance. They can be treated as a natural extension of the nonlinear dynamics problems. The presence of noise in a system requires a transition to a statistical description. There are two main approaches of studying stochastic systems [16–18,37]. *The first one* is based on solving stochastic equations (SE's) and is also called Langevin's method. Each particular solution of SE's, even with the same initial state, produces a new realization of a random process. By this means, one is able to obtain a statistical ensemble of a great number of realizations and to find its intrinsic statistical characteristics. In practice, time averaging along a single, long enough realization is often used assuming that the process is ergodic. *The second approach* consists in solving evolution equations for the probability measure, called the Chapman-Kolmogorov equation, kinetic equation, or the Fokker-Planck equation. However the process in a system should be at least Markovian that imposes certain requirements on the noise sources. For the process to be Markovian, random kicks must be statistically independent. In this case the Chapman-Kolmogorov equation holds. If the noise possesses Gaussian properties, the process is diffusive, and, therefore, for the probability density we can write the Fokker-Planck equation. Under appropriate definitions of noise sources, the method of SE's and the method of evolution equations must yield the equivalent statistical description of the system [16–19].

Of special interest is the problem of statistical characteristics of dynamical chaos and of the role of fluctuations in chaotic systems [15,19,21–25]. For systems with chaotic dynamics of hyperbolic type, the transition to a statistical description is possible already in a purely deterministic case, e.g., without noise [23–25]. This means that the stationary solution of evolution equation for the probability density allows the presence of the limit $D\to 0$, where D is the noise intensity, and there is the possibility of deducing an expression for the probability measure in a purely deterministic case. As shown in [23,25], small fluctuations ($D\ll 1$) in hyperbolic systems cause small changes of the structure of the probability measure. So-called quasihyperbolic (almost hyperbolic) attractors, such as the Lozi attractor and the Lorenz attractor [26,27], behave in a similar manner. Almost hyperbolic attractors enclose nonrobust singular trajectories, for example, separatrix loops in the Lorenz attractor. Nevertheless, these attractors, as well as hyperbolic ones, do not contain stable periodic orbits. The characteristics of quasihyperbolic attractors, measured in numerical experiments, are robust relative to small perturbations of the evolution operator. Particularly, there is a rigorous proof for the existence of the probability measure of the Lorenz attractor without noise [24]. The system's own dynamics turns out to be much stronger than that imposed from outside by external noise [23].

However, the majority of chaotic dynamical systems demonstrate nonhyperbolic chaos [10,26,27]. In the phase space these dissipative dynamical systems correspond *nonhyperbolic chaotic attractors*. They enclose the limit subsets of stable and unstable chaotic[1] and periodic trajectories. With this, basins of attraction of a set of attractors are, as a rule, fractal. They can be vanishingly narrow and are often difficult to detect in numerical experiments. In such systems the effect of noise can play a significant role. In [11] it was shown that the mean distance between a noisy orbit and the noiseless nonhyperbolic attractor appears to be significantly larger than that in the hyperbolic case and depends on the information dimension of the attractor. It is well known that in systems with nonhyperbolic attractors noise can induce

[1]Chaotic subsets are assumed to be stable according to Poisson.

various phase transitions [28–30]. When Gaussian noise sources are added to a system, basins of attraction of all system's attractors can merge. As a result, a unified stationary probability density appears being independent of the initial state [31]. However, the statistical description of nonhyperbolic chaos encounters principal difficulties. In general case, there is no stationary probability measure on nonhyperbolic chaotic attractors without noise, being independent of the initial distribution. In this case the continuous limit transition $D\to 0$ does not exist for the probability density of noisy nonhyperbolic systems [31]. Moreover, the probability characteristics of nonhyperbolic chaos are very sensitive to even slightest changes of the system parameters [32,10,33].

In this paper we analyze effects of bounded noise on nonhyperbolic chaotic systems. We numerically study typical two-dimensional invertible maps. It is known that invertible maps can be considered as model systems qualitatively describing Poincaré maps in the section of certain three-dimensional continuous-time systems. The purpose of our study is to elucidate (as far as possible in numerical experiments) the following issues.

(1) Does a principal difference between hyperbolic and nonhyperbolic chaos hold if a system is obscured by noise?

(2) How do bounded white and colored noise perturbations influence the probability characteristics of nonhyperbolic chaos, particularly, the properties of the invariant probability measure of an attractor?

To evaluate the probability density on a noisy nonhyperbolic attractor, two different methods are applied. The first one operates with a single, long enough trajectory of stochastic equations of a system under study. The second approach consists in solving the evolution equation for the probability density. In this connection the problem which we also address is in what degree the results for the probability density obtained using both methods are equivalent? And how significant can the error be connected with correlations in a sequence of random perturbations modeled by a special computer program (a source of pseudorandom numbers)?

In Sec. II of this paper we describe the models and the methods which we use. Section III is concerned with the study of some properties of nonhyperbolic attractors in the presence of noise. In Sec. IV we present our results for the probability density on nonhyperbolic attractors of two-dimensional invertible maps in the presence of bounded noise. In this section we compare the results of two numerical algorithms for estimating the probability density. We also analyze the influence of correlations in a sequence of random perturbations on the steady state probability density. In Sec. V we present noise-induced transitions in a two-dimensional map demonstrating nonhyperbolic chaos. Our conclusions are given in Sec. VI.

II. MODELS AND METHODS

In the present work we analyze statistical characteristics of evolutionary processes which are described by nonlinear discrete invertible maps $\mathbf{F}$ with additive noise:

$$\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n)+\mathbf{D}(\mathbf{x}_n)\xi_n, \quad \mathbf{x}_n, \quad \xi_n\in \mathrm{Re}^2. \tag{1}$$

In the stochastic equation (SE) (1), $\mathbf{x}_n$ is a two-dimensional vector of the system state, $\mathbf{F}$ is a nonlinear deterministic function defining a map, ξ_n is bounded noise, and $\mathbf{D}$ denotes the matrix of noise intensities. The noise sources ξ_n^1 and ξ_n^2 are chosen to be independent and represent white noise uniformly distributed in the interval $[-0.5,0.5]$. The matrix $\mathbf{D}$ is assumed to be constant and has the following elements: $D_{11}=D_{22}=D$, $D_{12}=D_{21}=0$. Thus, independent additive noise sources of the same intensity are added to both equations of the system (1). It should be noted that in the case of uniformly distributed noise D determines both the variance of the noise and the maximal strength of random kicks perturbing the system.

Here, we study the Henon map [34]

$$x_{n+1}=a-x_n^2+y_n, \quad y_{n+1}=bx_n \tag{2}$$

for $a=1.06$, $b=0.3$, and the cubic map

$$x_{n+1}=[(a-1)x_n-ax_n^3]\exp\left(\frac{-x_n^2}{b}\right)+y_n, \quad y_{n+1}=cx_n \tag{3}$$

for $a=2.95$, $b=0.5$, and $c=0.3$. For these parameter values, we have numerically verified that both maps have nonhyperbolic chaotic attractors. To check whether the chaotic attractor of the system is nonhyperbolic, we use a numerical procedure for calculating the angles ϕ between the stable and the unstable direction along a chaotic trajectory. This algorithm enables to establish a homoclinic tangency between the manifolds [35]. A chaotic attractor is nonhyperbolic if there are arbitrary close to zero angle ϕ values [33].

We explore the influence of noise on some characteristics of nonhyperbolic chaos using the Henon map which was proved to be a typical nonhyperbolic system [36]. In particular, for Eqs. (2) we analyze the distribution of angles ϕ between the manifolds in the presence of noise and the largest Lyapunov exponent under variation of the system control parameter and the noise intensity.

The main point in our study is to estimate the probability distribution on nonhyperbolic attractors of two-dimensional invertible maps in the presence of bounded noise. We first consider the Henon map and apply two different approaches briefly described in the Introduction. According to the first approach we compute a chaotic sequence $\mathbf{x}_n$ by iterating stochastic equations (1) $n=10^9$ times from a set of different initial conditions $\mathbf{x}(0)$ and for given noise intensities D. Then we estimate the probability measure $P_{\mathbf{x}}^*(i,j)$ as normalized residence times of the trajectory in a square element (i,j) on the phase plane. Thus, we obtain the probability measure by means of the SE's method with the implicit assumption that the process is ergodic.

The second method is based on solving the evolution equation for the probability density. Taking into account that random perturbations are statistically independent in consecutive time moments, we can consider the process $\mathbf{x}_n=\mathbf{x}(n)$ to be Markovian. In this case the probability density of $\mathbf{x}_n$ obeys the Chapman–Kolmogorov equation [37]:

$$p_{\mathbf{x}}(\mathbf{x}_{n+1})=\int_W \frac{p_\xi(\mathbf{D}^{-1}(\mathbf{x}_n)[\mathbf{x}_{n+1}-f(\mathbf{x}_n)])}{|\det \mathbf{D}(\mathbf{x}_n)|}p_{\mathbf{x}}(\mathbf{x}_n)d\mathbf{x}_n, \tag{4}$$

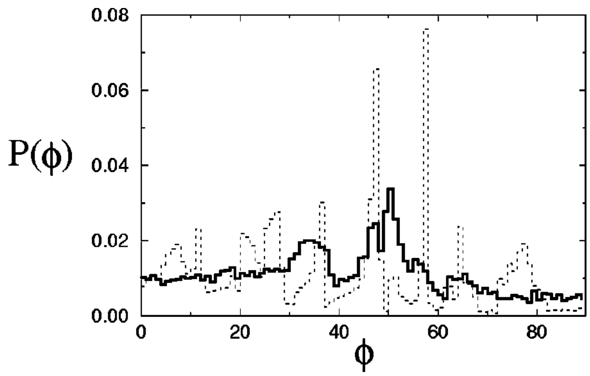

FIG. 1. The angle probability distribution $P(\phi)$ for the nonhyperbolic attractor of the Henon map at $a=1.06$, $b=0.3$ without noise (*dashed line*) and in the presence of noise with intensity $D=0.28$ (*solid line*). The angle ϕ is plotted in degrees.

where p_ξ is the probability density of noise, $\mathbf{D}^{-1}$ is the matrix inverse to the matrix of noise intensities. The integral is taken over all possible $\mathbf{x}_n$ values. Setting initial probability density $p_{\mathbf{x}}(\mathbf{x}_0)$ and implementing successively the transformation (4), one can see how the probability density of the process $\mathbf{x}_n$ evolves in time. Our calculations have shown that to estimate the stationary probability density by solving Eq. (4), the number of iterations $100<n<140$ is quite sufficient. But in practice this procedure requires significant computer time.

III. PROPERTIES OF NONHYPERBOLIC ATTRACTORS OF TWO-DIMENSIONAL INVERTIBLE MAPS IN THE PRESENCE OF NOISE

We start with exploring the influence of noise on the distribution of angles ϕ between the stable and unstable directions of a trajectory on a chaotic attractor in the Henon map (2). We fix $a=1.06$ and $b=0.3$. First we compute the probability distribution of angles ϕ for the Henon attractor in the noise-free case. The results shown in Fig. 1 by the dashed line clearly indicate the $P(\phi)>0$ for close to zero angle ϕ values. This implies that the chaotic attractor of Eqs. (2) is nonhyperbolic. Such an angle probability distribution was proved to be typical for chaotic attractors in the Henon map. Now, we add noise of intensity $D=0.28$ to both equations of Eq. (2). It is evident that by definition we cannot speak about stable and unstable manifolds in SE's. However, as follows from numerical experiments, we can identify the averaged directions of stability and instability and calculate the angle ϕ between them. The graph pictured in Fig. 1 by the solid line represents the angle probability distribution in the presence of additive noise in Eqs. (2). Comparing both graphs, the additive noise in the Henon map does not qualitatively change the probability distribution of angles ϕ between the manifolds. Most important, the probability of close to zero angle values is still strongly larger than zero. Thus, even strong noise perturbations do not alter the nature of the attractor, that is, the nonhyperbolic attractor is not transformed into a hyperbolic one.

As we have already mentioned, one of the peculiarities of nonhyperbolic chaos is its nonrobustness under small perturbations of the system. For this reason, even with arbitrary

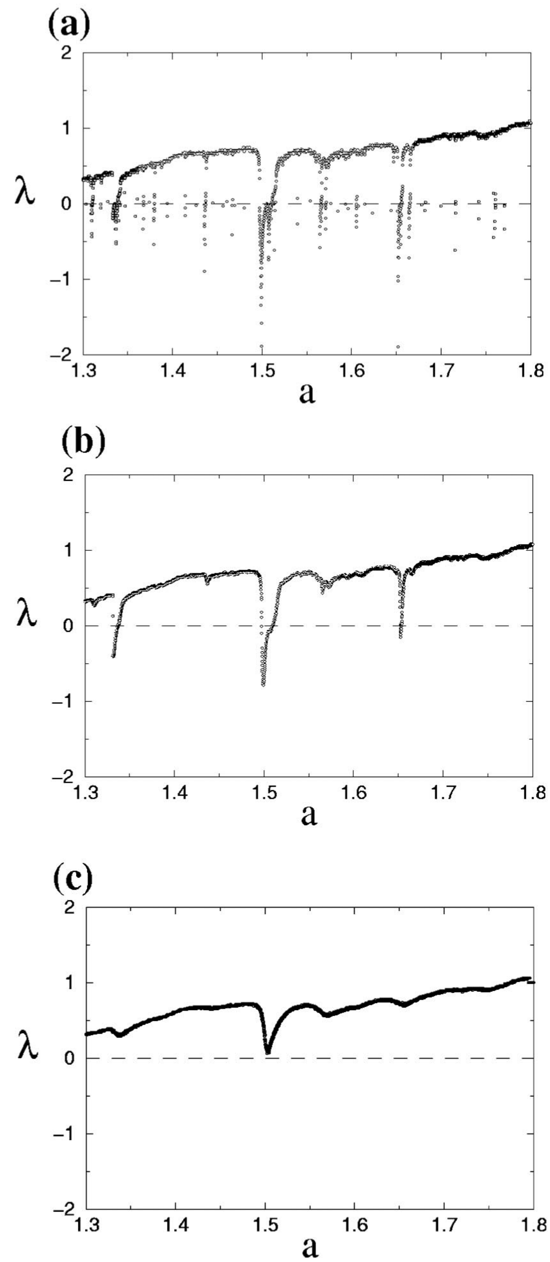

FIG. 2. Largest Lyapunov exponent of Eqs. (2) as a function of the control parameter a for $b=0.3$ without noise (a) and in the presence of additive noise with $D=0.002$ (b) and $D=0.01$ (c).

small control parameter variations, a set of bifurcations can be observed resulting in switchings between different chaotic and regular regimes. The dependence of a certain quantitative characteristic of the regime under observation on the control parameter is found to be a strongly rugged nonsmooth function. To illustrate this we now estimate the largest Lyapunov exponent (LE) for the map (2). In Fig. 2(a) the largest LE λ of the noise-free map is plotted as a function of the control parameter a. It can be seen that when the parameter a is varied over a rather small range, the behavior is chaotic for most values of a, but we find also regions with negative Lyapunov exponent values which correspond to stable periodic windows. However, adding weak noise ($D=0.002$) to Eqs. (2) can smooth the dependence $\lambda(a)$ by eliminating most of the narrow periodic windows. As seen

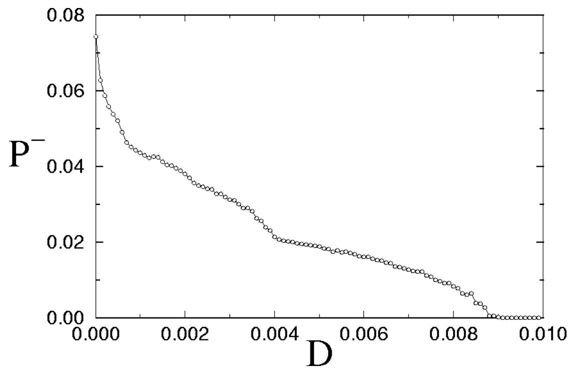

FIG. 3. For the Henon map ($b=0.3$), the measure P^- of regular regimes in the parameter a range $[1.3,1.8]$ is depicted as a function of the noise intensity D.

from Figs. 2(b) and 2(c), with increasing noise intensity the dependence of the largest LE on the parameter a is becoming more and more smooth and similar to that for quasihyperbolic chaos, which is known to be always a smooth positive definite function of a system parameter. We also estimate the measure P^- of regular regimes (periodic windows) in the given parameter a range, which is shown in Fig. 3 as a function of the noise intensity D. To quantify P^- we sum the lengths of all intervals corresponding to regular regimes and then calculate the ratio of this sum to the total length of the considered range of parameter a variation.

It is evident that other averaged over an attractor characteristics behave in a similar manner. Therefore, although noise perturbations do not influence the angle distribution between stable and unstable directions of a chaotic trajectory, i.e., they do not change the nonhyperbolic nature of the attractor, the noise can significantly affect some of its characteristics. As a result, the nonhyperbolic attractor becomes more similar in its properties to a hyperbolic one.

IV. EXPLORATION OF THE PROBABILITY DENSITY ON NONHYPERBOLIC ATTRACTORS IN THE PRESENCE OF NOISE

We begin our numerical experiments by considering again the Henon map (2) with $a=1.06$ and $b=0.3$. The nonhyperbolic chaotic attractor generated by Eqs. (2) for the given parameter values consists of four parts, also called bands. In this case a representative point visits regularly during four iterations each part of the chaotic attractor. The presence of additive noise leads to connected bifurcations, i.e., the attractor bands merge. For example, for the noise intensity $D=0.05$, there already exists a two-band attractor. We now construct the probability distribution $P_{xy}(i,j)$ on this chaotic attractor by applying the two methods described above. The results are shown in Fig. 4. $P_{xy}(i,j)$ denotes the probability of finding a representative point in a small covering box of edge length ε centered at the point (x_i,y_j). In the computation, the value of ε does not exceed 10^{-2}, and thus the equality $P_{xy}(i,j)=\varepsilon^2 p_{xy}(x_i,y_j)$ is valid, where p_{xy} is the probability density. In addition, the probability $P_{xy}(i,j)$ being of the order 10^{-12} and less is put to be equal to zero. The steady state probability distribution $P_{xy}(i,j)$, where i and j denote a partition element from the ε-grid, is periodically nonstationary with period 2. When calculating

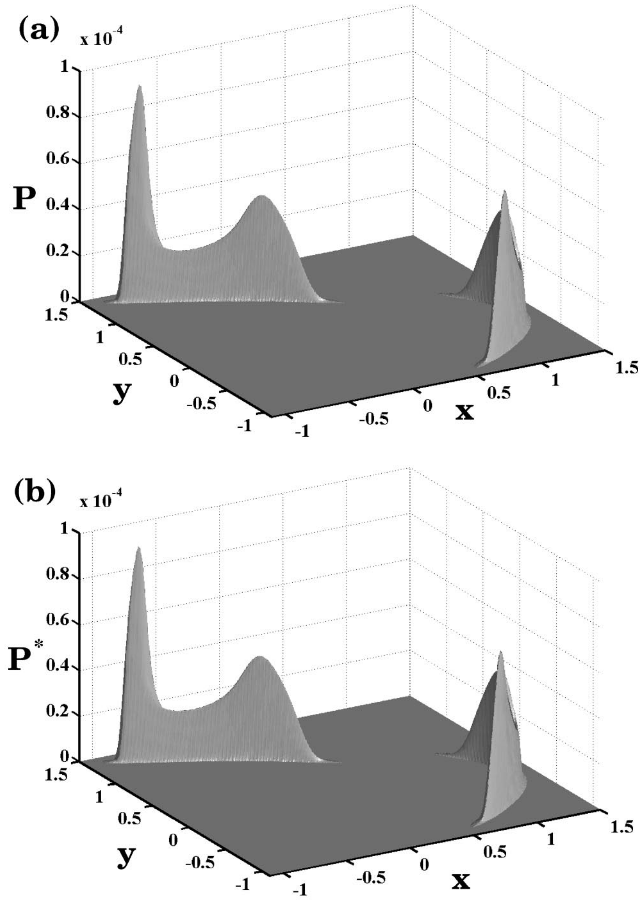

FIG. 4. For $a=1.06$, $b=0.3$ in Eqs. (2), the probability distribution on the chaotic attractor at $D=0.05$ and $\epsilon=2\times10^{-3}$. (a) The steady probability distribution calculated from Eq. (4); (b) the time-averaged probability distribution obtained from the corresponding stochastic equations.

the probability distribution $P^*_{xy}(i,j)$ from a single long enough noisy time series, we obtain the distribution which consists of two parts, each having a joint probability equal to 1/2. The shape of each of these parts completely conforms with that of the distribution $P_{xy}(i,j)$ being considered in the relevant time moments. It is worth noting the following. Let the attractor of a map arising in the section of a certain continuous-time system consist of a few bands. In this case the attractor can be characterized by a periodically nonstationary probability distribution. Nevertheless, the probability distribution on the attractor of the original continuous-time system will be stationary.

With increasing noise intensity another connected bifurcation takes place resulting in the merging of the remaining two attractor bands. Thus, the noise eliminates the periodic non-stationarity of the process, and a merged one-band attractor occurs in the system (2). Setting the noise intensity $D=0.28$, we solve numerically stochastic equations (1) for the map (2) and the corresponding evolution equation for the probability density. Figure 5(a) shows the stationary probability distribution $P_{xy}(i,j)$ calculated on the basis of Eq. (4). The corresponding probability distribution $P^*_{xy}(i,j)$, constructed from the SE's, is presented in Fig. 5(b). As follows from the obtained results, in time $n\geqslant100$ the time-invariant probability distribution is established on the

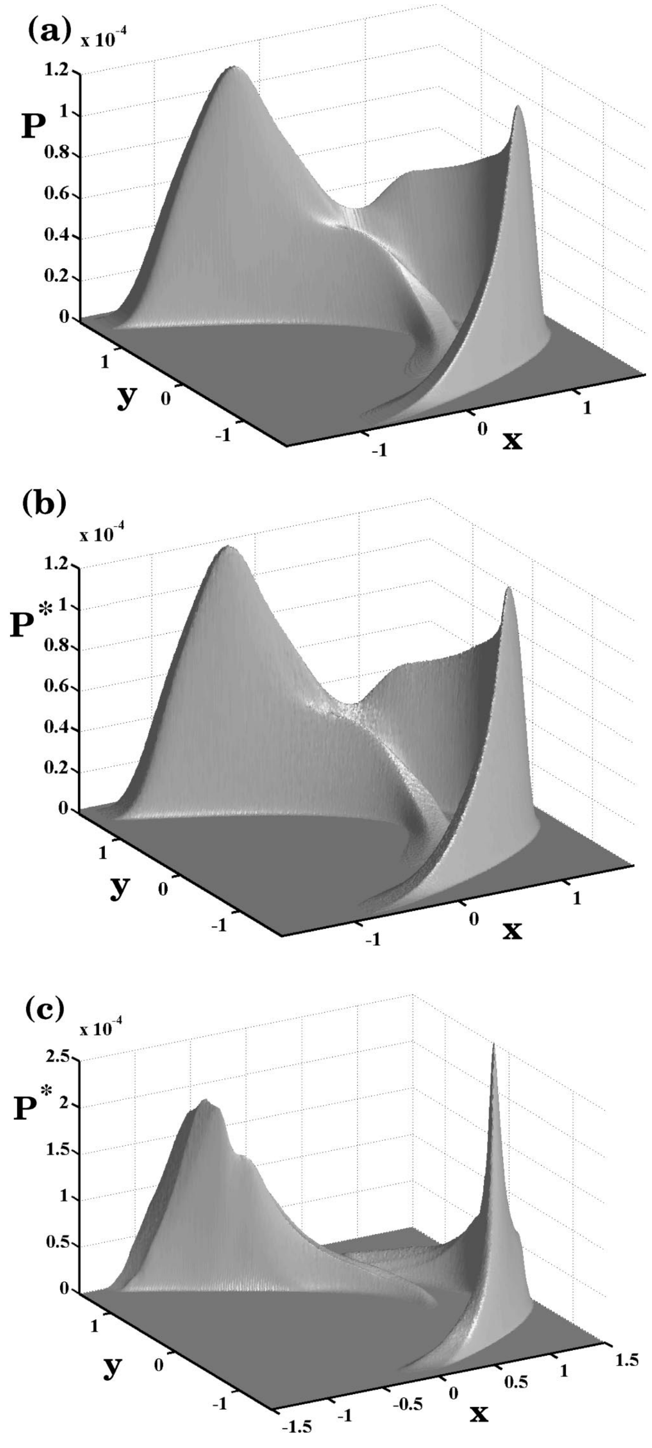

FIG. 5. Statistical characteristics of the Henon map (2) for $a=1.06$, $b=0.3$, $D=0.28$, and $\epsilon=10^{-2}$. (a) The stationary probability distribution calculated from Eq. (4); (b) the time-averaged distribution obtained from Eqs. (2) with white noise sources; (c) the probability distribution obtained by solving Eqs. (2) with independent color bounded noise sources whose autocorrelation function is shown in Fig. 6 by curve 3.

merged chaotic attractor of the Henon map. Our numerical experiments have shown that the probability distribution does not depend on the choice of initial distribution. Comparing Figs. 5(a) and 5(b), it can be seen that the probability distribution, obtained from a single time series of length $n=10^9$ of the SE's, is the same as that one computed from the evolution equation. Our calculations have shown that for $n>120$ both distributions coincide within a round-off error.

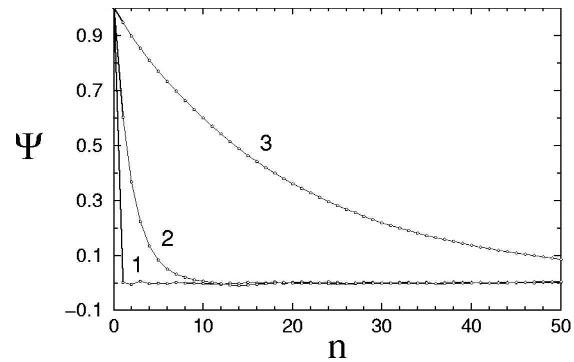

FIG. 6. Autocorrelation functions of different bounded noise sources used in numerical calculations. The corresponding correlation time is $\tau_c^1=0$ (white noise), $\tau_c^2=2$, and $\tau_c^3=20$.

As already mentioned, Eq. (4) holds when noise kicks ξ_n perturbing the system under study in the time moments n and $(n+1)$ are statistically independent. The probability distributions obtained by means of the two methods described can be in good agreement only in the case when in numerical simulation of stochastic systems random perturbations represent practically white noise. Noise sources $\xi_{1,2}(t)$, which we use to estimate the probability distributions shown in Figs. 4(b) and 5(b), satisfy this requirement. Their normalized autocorrelation function is presented in Fig. 6 (curve 1). If the noise sources are less ''good,'' then the probability distribution obtained from the solution of SE's may be considerably different from the solution of Eq. (4). Figure 5(c) illustrates the probability distribution obtained by processing a realization of Eqs. (2) with colored noise sources. The autocorrelation function for $\xi_1(t)$ and $\xi_2(t)$ is plotted in Fig. 6 (curve 3). As before noise sources $\xi_{1,2}$ are chosen to be uniformly distributed in the interval $[-0.5,0.5]$ and with intensity $D=0.28$. The difference between the distributions shown in Figs. 5(a) and 5(c) is noticeable by estimation.

It is of interest to compare the convergence rate of the two methods used for calculating the probability distribution and to estimate the error related to correlations in a source of pseudorandom numbers, which is utilized in numerical experiments. The divergence of these two numerical procedures can be estimated by considering the magnitude $\langle d\rangle$, which reads

$$\langle d\rangle=\frac{\langle l^2\rangle}{\sigma_{xy}^2}, \tag{5}$$

where $\langle l^2\rangle$ is the mean square of the difference[2] $[P_{xy}(i,j,n)-P_{xy}^*(i,j)]^2$, and σ_{xy}^2 is the variance of the steady state distribution $P_{xy}(i,j)$. In our computations, we use three different kinds of random perturbations, namely, $\xi_{1,2}^1(t)$, $\xi_{1,2}^2(t)$, and $\xi_{1,2}^3(t)$. Their normalized autocorrelation functions are shown in Fig. 6. The first noise source (1) corresponds to white noise, and the two other (2 and 3) represent colored noise with correlation time $\tau_c=2$ and τ_c

[2]Here we compute the arithmetic mean over all partition elements.

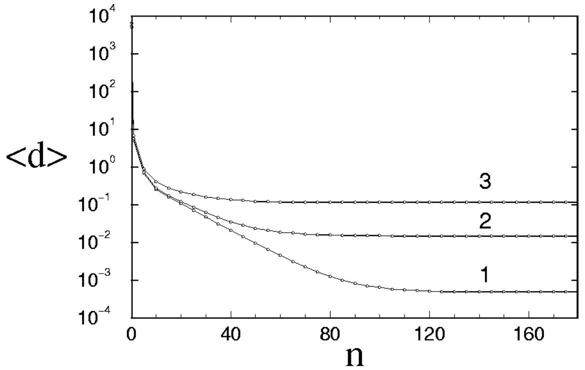

FIG. 7. Estimate of the divergence between $P_{xy}(i,j,n)$ and $P^*_{xy}(i,j)$ when utilizing different bounded noise sources with the same noise intensity $D=0.28$. Curves 1, 2, and 3 correspond to the noise sources with autocorrelation functions shown in Fig. 6 by curves 1, 2, and 3.

$=20$, respectively. The correlation time is defined as the time in the course of which the autocorrelation function decreases in e times. With this, random noise sources added to the first and the second equation of map (2) are assumed to be identical, uncorrelated and uniformly distributed in $[-0.5,0.5]$. Their intensity is fixed at $D=0.28$. The calculation results for $\langle d\rangle$ are shown in Fig. 7. For the first kind of noise (white noise), $\langle d\rangle$ becomes constant for $n=120$ and constitutes less than 0.05% of the variance (curve 1). This fact testifies a good agreement between the results obtained using the two different techniques for estimating the probability distribution. Hence, one may conclude that the SE's method and the solution of evolution equation yield equivalent results for the typical nonhyperbolic system (2), at least starting from a certain noise intensity D. Curves 2 and 3 drawn in Fig. 7 reflect the divergence of the two methods when introducing colored noise sources 2 and 3 to SE's, respectively. For the noise of the second kind (2), $\langle d\rangle$ settles at the level corresponding to 1.5% of the variance σ^2_{xy}, and for the noise sources 3 at the level 10%. The latter value of $\langle d\rangle$ indicates a significant divergence of the two methods applied. For this value, there are strong correlations between sequential states of random sources used in numerical experiments.

We have also analyzed nonhyperbolic chaos in the cubic map (3), in which a noise-induced crisis of separatrix tangency is realized. For this system we fix $a=2.95$, $b=0.5$, $c=0.3$. These parameter values correspond to the existence of two symmetric chaotic attractors in the map (3). Noise sources $\xi_{1,2}(t)$ are taken to be white, uncorrelated and uniformly distributed in $[-0.5,0.5]$. Our calculations have shown that for insufficient crisis noise intensity $D=0.15$, each of the two attractors possesses its own stationary probability distribution independent of a given initial distribution. For the noise intensity $D=0.4$, there occurs a crisis of separatrix tangency. As a result, a merged chaotic attractor is realized in Eqs. (3). For this case we have also calculated probability distributions using both approaches. Figure 8(a) shows the stationary probability distribution obtained by solving integral equation (4) for map (3). The time-averaged distribution calculated from the solution of SE's is presented in Fig. 8(b). It is seen that the merged chaotic attractor of Eqs. (3) is characterized by the invariant probability distribution.

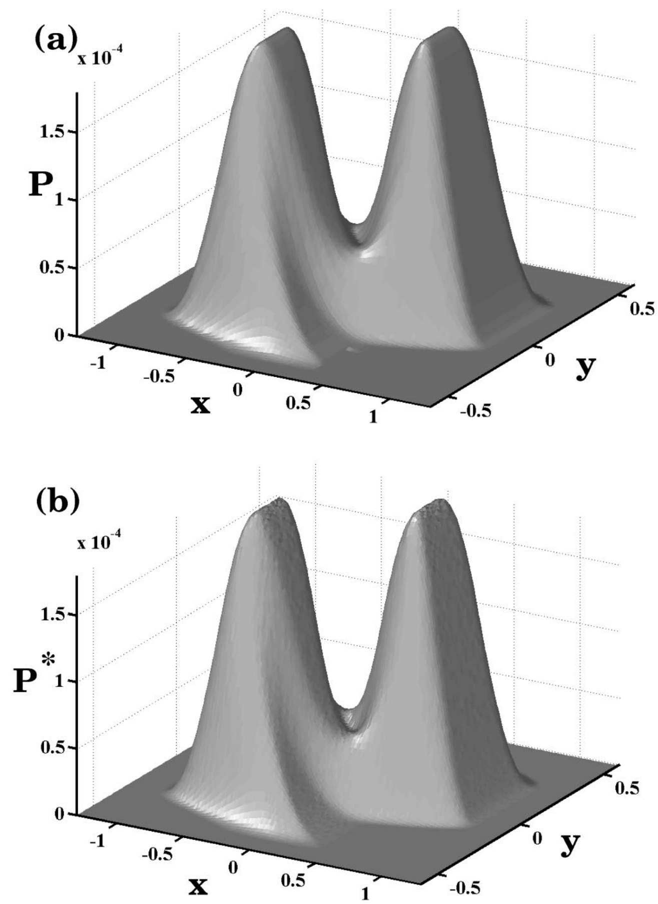

FIG. 8. Probability distributions calculated for the cubic map (3) at $a=2.95$, $b=0.5$, $c=0.3$, $D=0.4$, and $\epsilon=10^{-2}$ by solving Eq. (4) (a), and from stochastic equations (b).

Our computations clearly indicate that both methods for estimating the probability measure also yield the same results for the case of another kind of noise-induced phase transitions.

Therefore, to estimate the steady state probability distribution on a nonhyperbolic noisy attractor, one can employ the ordinary calculation procedure, i.e. to use a single sufficiently long trajectory from stochastic equations. Periodic nonstationarity of the process may manifest itself only in getting simultaneously the probability distribution for all parts of a many-band attractor with a general normalization to unity.

V. NOISE-INDUCED TRANSITIONS IN NONHYPERBOLIC SYSTEMS

The results described above indicate the possibility of applying the SE's method (1) to estimate numerically a steady state probability measure on chaotic attractors of nonhyperbolic systems. This conclusion is of great practical importance by virtue of several reasons. First, as compared to the method based on Eq. (4), the SE's method requires essentially less computer time. Secondly, the method (4) becomes extremely difficult to use in cases when multistability (the presence of a large number of attracting subsets in the phase space) is well expressed, and an arbitrary low noise level

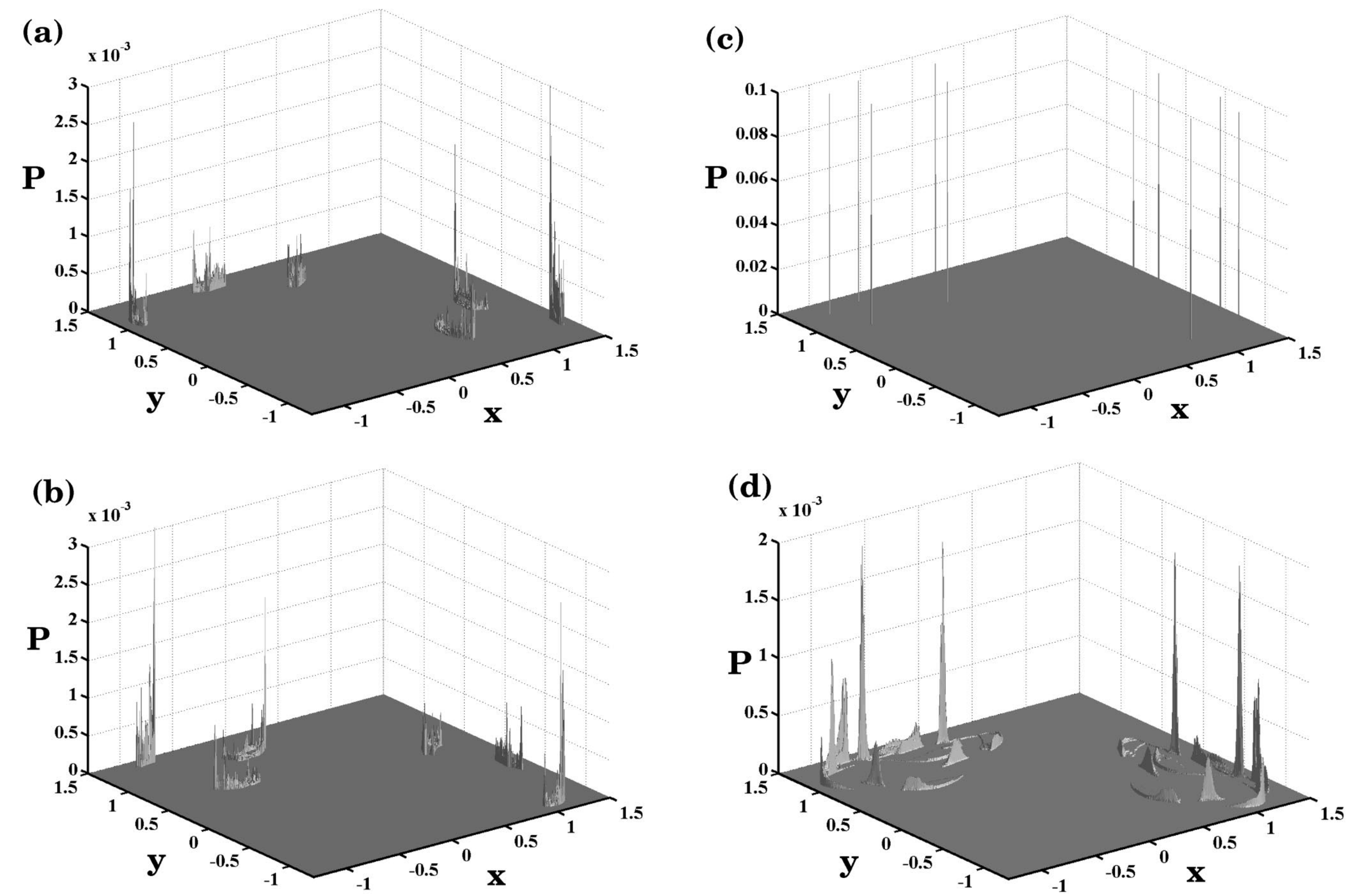

FIG. 9. Probability distributions of the coexisting attractors in map (6) for $\alpha = 0.78$, $\gamma = 0.2876$ in the absence of noise (a)–(c) and in the presence of additive noise with intensity $D = 6 \times 10^{-6}$ (d).

enables to change the probability measure. The decrease of the noise intensity when applying the algorithm (4) causes a significant increase of the calculation time. Ultimately, this method does not yield truly reliable results. The SE's method does not depend on the noise intensity that is undoubtedly important in numerical simulation.

For illustrative purposes we consider a non-invertible system in the form of two coupled logistic maps being a typical example of a discrete system with nonhyperbolic properties:

$$x_{n+1} = 1 - \alpha x_n^2 + \gamma(y_n - x_n) + D\xi_1(n),$$

$$y_{n+1} = 1 - \alpha y_n^2 + \gamma(x_n - y_n) + D\xi_2(n), \quad (6)$$

where $\xi_{1,2}(n)$ are independent bounded white noise sources uniformly distributed in the interval $[-0.5, 0.5]$. For the system parameter values $\alpha = 0.78$, $\gamma = 0.2876$, and $D = 0$ two symmetric chaotic and one regular attractors are found in numerical experiments. The influence of noise of very low intensity $D = 6 \times 10^{-6}$ already leads to the formation of a single attractor. This case can be treated as a noise-induced transition. We compute the probability measure by solving the stochastic equations (6). The calculation results are shown in Fig. 9. Note that the use of method (4) with reference to system (6) proved to be impossible due to a very small noise intensity.

VI. CONCLUSIONS

In the paper we have discussed the effect of bounded white and colored noise on statistical properties of nonhyperbolic chaotic attractors of two-dimensional invertible maps. It has been established that noise perturbations introduced in both equations of the system under study do not affect the probability distribution of angles ϕ between the stable and unstable directions along a chaotic trajectory. This fact implies that an important property of the chaotic dynamics does not change under the influence of noise, i.e., a nonhyperbolic attractor is not transformed into a hyperbolic one, and vice versa.

However, our numerical calculations have shown that in the presence of noise some properties of nonhyperbolic chaos can be quite similar to those of hyperbolic and quasihyperbolic chaos. This statement can be motivated, for example, by smoothing of the dependence of the largest Lyapunov exponent on a control parameter.

We have also demonstrated that for bounded noise of relatively high intensity acting on chaotic regimes of nonhyperbolic systems, it is possible to obtain the steady state stationary probability distribution being invariant to the choice of the initial distribution. Our numerical experiments have shown that the SE's approach is appropriate for estimating the steady state probability measure on noisy nonhyperbolic attractors. The presence of correlations in a sequence of states of the noise source utilized in computations may distort the obtained result. However, the error will be significant only in the case if the autocorrelation function of this noise source decreases slowly enough (the correlation time must be of order 20 characteristic time intervals of the system).

Our numerical results for two-dimensional maps (2) and (3) have provided evidence that the SE's method can be reliably used for calculating the steady state probability measure on chaotic attractors of nonhyperbolic three-dimensional differential systems.

From a theoretical viewpoint, in the presence of Gaussian noise of finite intensity the stationary probability measure of nonhyperbolic attractors always exists, is unique and does not depend on the initial conditions [19]. However, in numerical experiments the probability measure, as a rule, is estimated over a finite time interval being sometimes significantly less than possible times of transitions between different attracting subsets of a nonhyperbolic attractor. For this reason the following effects may take place both in the presence of Gaussian noise and in the case of bounded noise. Assume that a system under study has a nonhyperbolic attractor which in the absence of noise encloses m attracting subsets. Noise added to the system reduces them to $(m-n)>1$ subsets, each possessing its own probability distribution and its own basin of attraction. Then the selection of the $(m-n)$ probability distributions observed in numerical experiments strongly depends on the choice of initial conditions.

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