Generalized synchronization versus phase synchronization

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The relation between generalized synchronization and phase synchronization is investigated. It was claimed that generalized synchronization always leads to phase synchronization, and phase synchronization is a weaker form than generalized synchronization. We propose examples that generalized synchronization can be weaker than phase synchronization, depending on parameter misfits. Moreover, generalized synchronization does not always lead to phase synchronization.

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Synchronization phenomena in coupled or driven chaotic systems have been extensively studied in the context of laser dynamics [1], electronic circuits [2], chemical and biological systems [3], and secure communications [4]. The entrainment of coupled or driven limit cycles has long been a well-studied topic, while synchronization of chaotic oscillators was an open area due to the presence of the intrinsic nonlinearity [5]. Pecora and Carroll [6] showed that two interacting identical chaotic oscillators can achieve synchronization, i.e., they evolve with the same orbit, $\mathbf{x}_1(t)=\mathbf{x}_2(t)$, even though they possess the well-known exponential instability of neighboring orbits. This study arouses extensive interest in synchronized entrainment of chaotic oscillators, and different degrees of synchronizations were found, for example, complete synchronization (CS) [6,7], generalized synchronization (GS) [8], phase synchronization (PS) [9], lag synchronization (LS) [10], and even measure synchronization in Hamiltonian systems [11]. CS appears only when interacting systems are identical. Otherwise, if the parameters of coupled oscillators slightly mismatch, the states can be very close, but remain different. GS was introduced for drive-response systems, and it is defined as the emergence of some functional relation between the states of response and drive, i.e., $\mathbf{x}_2(t)=\mathbf{F}[\mathbf{x}_1(t)]$. PS means the entrainment of phases of chaotic oscillators, whereas their amplitudes remain chaotic and generally noncorrelated.

Though a number of researches have been made on these synchronizations, studies of their relations and the scenarios of transitions among them have not been well addressed. CS is usually considered as the strongest form among these synchronizations. A weaker form should be GS, which calls for the stability of the functional manifold $\mathbf{x}_2(t)=\mathbf{F}[\mathbf{x}_1(t)]$. PS does not add restrictions on the oscillation amplitudes, and only the locking of phases is crucial. It is usually accepted that PS is weaker than GS. Parlitz *et al.* first studied this problem [12]. They claimed that in general GS leads always to PS, and GS is stronger, i.e., PS may occur in cases where the coupled systems show no GS. In many cases this is correct. However, as we will address in this paper, GS is not necessarily stronger than PS for two interacting chaotic oscillators with well-defined phases. In some cases, PS comes after GS with increasing coupling strength, depending on parameter mismatches.

The model we adopt in the context is a drive-response system, where both the drive and response systems are Rossler oscillators with mismatched parameters. In dimensionless form, the equation of motion of this system can be written as

$$\begin{aligned}
\dot{x}_d&=-\omega_d y_d-z_d,\\
\dot{y}_d&=\omega_d x_d+0.15y_d,\\
\dot{z}_d&=0.2+z_d(x_d-10.0),\\
\dot{x}_r&=-\omega_r y_r-z_r+\varepsilon(x_d-x_r),\\
\dot{y}_r&=\omega_r x_r+0.15y_r,\\
\dot{z}_r&=0.2+z_r(x_r-10.0).
\end{aligned}\tag{1}$$

Here the subscripts d and r denote the coordinates for the drive and response systems, respectively, and ε the coupling strength. ω_d and ω_r are usually nonidentical. For Rossler oscillators with parameters given above, a temporal phase can be well defined on the x-y plane [13] as

$$\theta(t)=\tan^{-1}[y(t)/x(t)].\tag{2}$$

Then the average winding number can be defined as the temporal average of the phase velocity

$$\Omega=\lim_{T\to\infty}\frac{1}{T}\int_0^T\dot{\theta}(t)dt.\tag{3}$$

The 1:1 phase locking (PS) of the drive and response systems can be defined as the temporal localization of the phases $\lim_{t\to\infty}|\theta_r(t)-\theta_d(t)|<\text{const}$ or $\Omega_r=\Omega_d$.

The drive-response system (1) exhibits rich dynamical behaviors, depending on different parameters. For the case $\omega_r=\omega_d$, i.e., the two subsystems are identical, complete synchronization can be achieved as the coupling strength exceeds a critical value. When $\omega_r\neq\omega_d$, GS can usually be observed at a certain coupling strength. A necessary condition for CS and GS is the negativeness of the maximum conditional Lyapunov exponent (MCLE) [14]. The conditional Lyapunov exponent spectrum $\{\lambda_c^1\geqslant\lambda_c^2\geqslant\ldots\}$ can be numerically computed along the CS or GS manifold. For weak couplings, the maximum exponent $\lambda_c^1>0$, implying

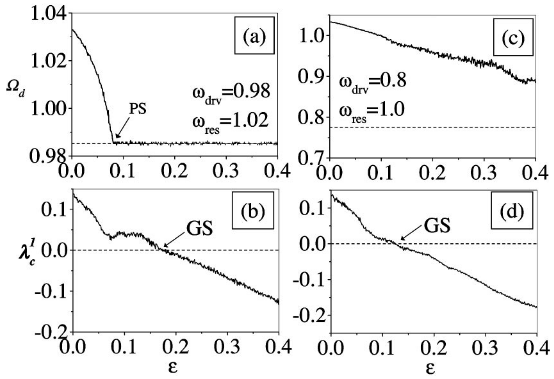

FIG. 1. (a) and (c) The average winding numbers Ω_d (dashed) and Ω_r (solid) against the coupling ε for different parameter mismatches. (a) $\omega_d=0.98$, $\omega_r=1.02$. (b) $\omega_d=0.8$, $\omega_r=1.0$. (b) and (d) The MCLE λ_c^1 varies with the coupling strength ε, corresponding to (a) and (c), respectively.

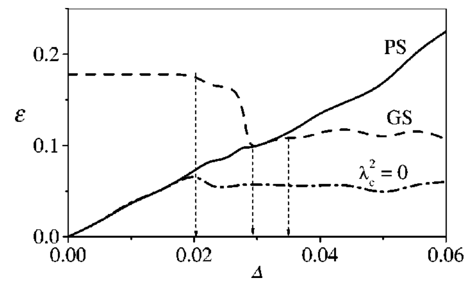

FIG. 2. The phase diagram on the Δ-ε plane with Δ being a measure of parameter misfit. Three lines: the critical line for PS ($\Omega_r=\Omega_d$) (solid line), the critical line for GS ($\lambda_c^1=0$) (dashed line), and the line for the boundary of $\lambda_c^2=0$ (dot-dashed line).

CS or GS between the drive and response, is not built. As one increases ε, λ_c^1 becomes negative at a certain strength, then the evolution of the response element follows the manifold of the drive, i.e., $\mathbf{x}_1(t)=\mathbf{x}_2(t)$ for identical cases and $\mathbf{x}_2(t)=\mathbf{F}[\mathbf{x}_1(t)]$ for nonidentical cases. On the other hand, for cases $\omega_r\neq\omega_d$, PS can be found if a well-defined phase can be introduced according to Eq. (2). Hence a problem arises: for the case $\omega_r\neq\omega_d$, PS and GS are all present, which one comes first as one increases the coupling? To check this problem, we computed Ω_d and Ω_r against ε for different parameter mismatches. In Fig. 1(a), $\omega_d=0.98$, $\omega_r=1.02$. It can be found that at $\varepsilon=\varepsilon_c^{PS}\approx0.08$, $\Omega_r=\Omega_d$, i.e., the response system is phase entrained by the drive. Let us check whether GS is also achieved at this point by computing the MCLE λ_c^1 against ε, as shown in Fig. 1(b). GS is implemented when $\varepsilon>\varepsilon_c^{GS}\approx0.18>\varepsilon_c^{PS}$. This result agrees with the conclusion of Parlitz *et al.* [12] that GS is stronger than PS. However, this does not exclude the possibility that GS comes before PS as one increases the interaction strength. For large parameter misfits, a large ε should be applied for the system to reach PS. If GS is achieved for a larger coupling, then PS still comes before GS with increasing couplings. On the contrary, if ε_c^{GS} keeps unchanged or decreases with increasing parameter misfits, sooner or later PS will emerge after GS. To check this possibility, in Fig. 1(c) we give the average frequency Ω_r varying against ε, and in Fig. 1(d) the corresponding MCLE is also plotted. Here $\omega_r=0.8$ and $\omega_d=1.0$. One finds quite a different picture from Figs. 1(a) and 1(b). At $\varepsilon\approx0.13$, λ_c^1 becomes negative, i.e., GS is reached at $\varepsilon\approx0.13$. On the other hand, it is found that $\Omega_r\neq\Omega_d$ even at $\varepsilon=0.4$. In this case GS is achieved before PS. This supports our argument and is contradict to the conclusions of Parlitz *et al.* [12].

It should be intuitive to give a phase diagram for the above picture. In Fig. 2, a diagram on the Δ-ε plane is shown, where $\omega_{r,d}=1.0\pm\Delta$, with Δ being a measure of parameter misfit. Three lines are given in the diagram. The first line is the critical coupling for PS ($\Omega_r=\Omega_d$) varying against the parameter misfit Δ, which gives the boundary of the PS and desynchronization regimes [labeled by a solid line]. We call this line the *PS line*. The second line corresponds to the critical coupling for GS ($\lambda_c^1=0$), which gives the GS regime above it (the dashed line in Fig. 2, called the *GS line*). To analyze the mechanism for previous results, we also plot a dash-dot line that gives the maximum coupling $\varepsilon_{\max}$ for $\lambda_c^2=0$. We name it the *topological line*. When $\varepsilon<\varepsilon_{\max}$, $\lambda_c^2=0$, and $\lambda_c^2<0$ as ε exceeds $\varepsilon_{\max}$. By observing this diagram, three bifurcation points are found. The *first* bifurcation occurs at $\Delta_{c1}\approx0.02$. When $\Delta\leqslant\Delta_{c1}$, the PS line and the topological line coincide, where PS implies the second conditional Lyapunov exponent becomes negative. In this regime, the emergence of PS leads directly to a *topological change* of the phase space. When $\Delta>\Delta_{c1}$, these two lines separate. The second conditional exponent becomes negative at a weaker coupling than that for PS. The coincidence of the PS and topological lines is thus broken. This broken coincidence in consequence leads to a topological change of the GS (note that the GS line is horizontal for $\Delta\leqslant\Delta_{c1}$, and decreases when $\Delta>\Delta_{c1}$). At $\Delta_{c2}\approx0.028$, the *second* bifurcation is observed, where the PS and GS lines merge together. For a parameter misfit Δ lying in the regime [0.028,0.035], PS and GS are achieved at the same critical coupling. This regime is in agreement with the observation of Parlitz *et al.* [12] that GS leads to PS. The *third* bifurcation is shown at $\Delta_{c3}\approx0.035$, above which the PS line exceeds and lies above the GS line. For parameter misfit $\Delta\geqslant\Delta_{c3}$, GS is first achieved, and PS comes with a stronger coupling. In this regime, PS is a stronger form of synchronization. Therefore from this phase diagram, we show a cascade of transitions from PS-GS to PS-GS merge and GS-PS. More important is that this diagram exhibits the possibility that PS may be stronger than GS. The key mechanism is due to the broken coincidence of the PS and topological lines.

It is instructive to determine the functional form of GS. In Fig. 3 we show the relation between $x_r(t)$ and $x_d(t)$. In Fig. 3(a) for $\Delta=0.01$ and $\varepsilon=0.3$, the projection of the attractor on the plane is shown. In Fig. 3(b) this relation is exhibited by a delayed-coordinate plot $x_r(t-\tau)$ vs $x_d(t)$, where $\tau=0.12$. It can be found that $x_r(t-\tau)\approx x_d(t)$, i.e., a lag form is built for small Δ. Hence for small parameter misfits, the GS is identified as LS. In Figs. 3(c) and 3(d), we plot a case of large parameter misfit $\Delta=0.06$ for $\varepsilon=0.3$, where GS and PS are both attained. A complicated relation between $x_r(t)$ and $x_d(t)$ is built. A delay-coordinate plot shows that they cannot build a good LS. To get a better picture, we compute the lag function (called the similarity function in [10]) between the response $x_r(t)$ and the drive $x_d(t)$ taken with a time shift τ

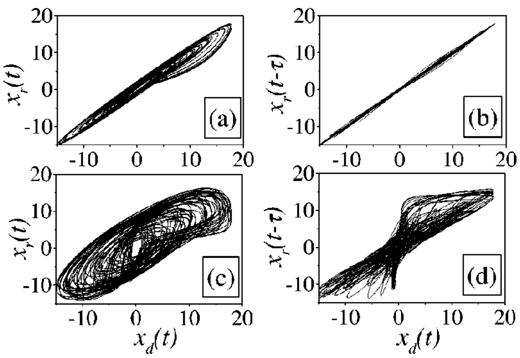

FIG. 3. (a) and (c) The relation between $x_r(t)$ and $x_d(t)$ for $\Delta=0.01$ [(a)] and $\Delta=0.06$. $\varepsilon=0.3$. (b) and (d) The delayed-coordinate plot $x_r(t-\tau)$ vs $x_d(t)$ corresponding to (a) and (c), where $\tau=0.12$ in (b) and $\tau=0.84$ in (d).

$$s(\tau)=\sqrt{\frac{\langle[x_r(t+\tau)-x_d(t)]^2\rangle}{\sqrt{\langle x_r^2(t)\rangle\langle x_d^2(t)\rangle}}}, \tag{4}$$

where $\langle\cdot\rangle$ is taken as a long time average. When $x_r(t+\tau)\approx x_d(t)$, i.e., there is a time shift between the two signals, $s(\tau)\approx 0$, implying a good LS. When two signals are noncorrelated, i.e., no LS is achieved, $s(\tau)\approx\sqrt{2}$. In Fig. 4, $s(\tau)$ is plotted against τ for $\varepsilon=0.30$ and different Δ's. It can be found that when Δ is small, $s(\tau)$ has a very small minimum at some τ_0, indicating a good LS. For large Δ, $s(\tau)$ also has a minimum, but this minimum is very large. Thus one cannot observe a good LS [see the line for $\Delta=0.06$ in Fig. 4].

The transition to PS is accompanied by the temporal localization of the phase difference $\Delta\theta(t)=\theta_r(t)-\theta_d(t)$. In Fig. 5(a) we give the evolution of $\Delta\theta(t)$ for $\Delta=0.05$ and $\varepsilon=0.1$, 0.125, and 0.15. It is found that for ε far from ε_c^{PS}, $\Delta\theta(t)$ evolves smoothly from $2n\pi$ to $2(n+1)\pi$. While for ε near to the critical point for PS, $\Delta\theta(t)$ exhibits a stick-slip feature. Furthermore, the stick-slip motion is irregular, i.e., the slip time T (the time interval between two adjacent 2π slips) is random. In Fig. 5(b) the statistics of T, $P(T)$, is shown for $\Delta=0.05$ and $\varepsilon=0.05$, 0.075, 0.10, and 0.15. It is very interesting that all these distributions possess multiple peaks, composed of a central peak and a number of small peaks. The interval among these peaks equals the average period for a 2π rotation of the drive system, $2\pi/\Omega_d$. This gives an interesting picture that the rotation of the response is controlled by the drive in a quantized manner. A comparison of subfigures in Fig. 5(b) indicates that as ε approaches

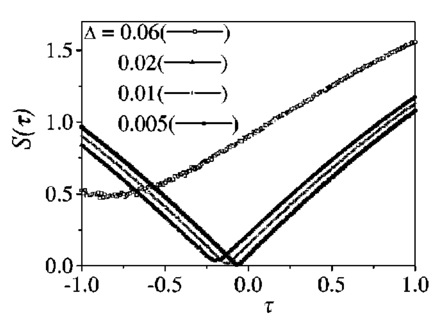

FIG. 4. The lag function $s(\tau)$ is plotted for $\varepsilon=0.30$ and $\Delta=0.06$, 0.02, 0.01, and 0.005.

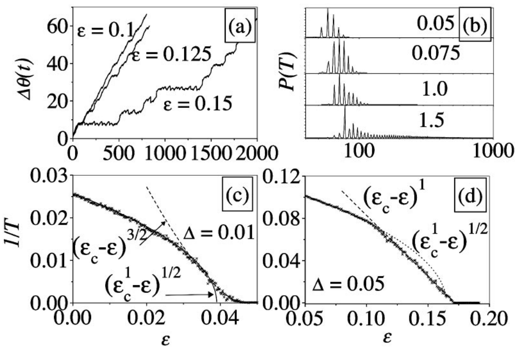

FIG. 5. (a) The evolution of $\Delta\theta(t)=\theta_r(t)-\theta_d(t)$ for $\Delta=0.05$ and $\varepsilon=0.1$, 0.125, and 0.15. (b) The statistics of the slip time T for $\Delta=0.05$ and $\varepsilon=0.05$, 0.075, 0.10, and 0.15. Multiple peaks can be observed, where the interval among these peaks equals $2\pi/\Omega_d$. Logarithm scale for the T axis is adopted to show these peaks. (c) and (d) The inverse mean slip time $\langle T\rangle^{-1}$ varying against the ε for $\Delta=0.01$ and 0.05. Dashed and dotted lines are fitting functions.

ε_c^{PS}, $P(T)$ is expanded in a quantized manner, i.e., new peaks appear with the interval $2\pi/\Omega_d$. Furthermore, as shown in Fig. 5(b), the highest peaks shift to the right as the coupling is increased. For couplings very near ε_c^{PS}, $P(T)$ has a large number of peaks. This type of slip-time distribution was not observed before, and a deep study is necessary, which is beyond our scope in the present paper. For both small and large Δ's, we find the similar feature of $P(T)$. In Figs. 5(c) and 5(d), we give the inverse mean slip time $\langle T\rangle^{-1}$ varying against the ε for $\Delta=0.01$ and 0.05. For $\varepsilon\ll\varepsilon_c^{PS}$, we find for both small and large parameter misfits, $\langle T\rangle\propto(\varepsilon_0-\varepsilon)^{-1/2}$, as shown in Fig. 5(c), where ε_0 is a fitting value. This scaling law is in agreement with that for coupled periodic oscillators [15]. The scaling law of $\langle T\rangle$ near ε_c^{PS} for small and large Δ's, however, are extremely different. In Fig. 5(c), for $\Delta=0.01$, $\langle T\rangle\propto(\varepsilon_c^{PS}-\varepsilon)^{-3/2}$, while in Fig. 5(d) for $\Delta=0.05$, $\langle T\rangle\propto(\varepsilon_c^{PS}-\varepsilon)^{-1}$. This difference reflects the competition of PS and GS. For small Δ, PS comes before GS, thus the scaling around ε_c^{PS} exhibits a chaotic feature. Note that the exponent $-3/2$ is typical for turbulence. In the case of large Δ, GS has already reached before PS, the chaoticity is then suppressed.

Finally, we should emphasize that the fact that PS could be stronger than GS can be found in a variety of systems. For example, we studied the response of a Lorenz oscillator to the drive of another Lorenz oscillator with mismatched parameters,

$$\begin{aligned}
\dot{x}_d&=\sigma(y_d-x_d),\\
\dot{y}_d&=r_dx_d-y_d-x_dz_d,\\
\dot{z}_d&=x_dy_d-bz_d,\\
\dot{x}_r&=\sigma(y_r-x_r)+\varepsilon(x_d-x_r),\\
\dot{y}_r&=r_rx_r-y_r-x_rz_r,\\
\dot{z}_r&=x_ry_r-bz_d,
\end{aligned} \tag{5}$$

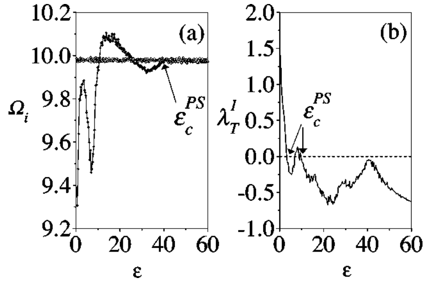

FIG. 6. The relation $\Omega_{r,d} \sim \varepsilon$ [(a)] and $\lambda_c^1 \sim \varepsilon$ [(b)] for the Lorenz drive-response system, where $\sigma = 10$, $b = 8/3$, and $r_d = 39$, $r_r = 35$.

where $\sigma = 10$, $b = 8/3$, and $r_d = 39$, $r_r = 35$. A definition of the phase for a Lorenz oscillator has been given as follows:

$$\theta_{r,d}(t) = \tan^{-1}\left\{ \frac{[\sqrt{x_{r,d}^2 + y_{r,d}^2} - \sqrt{2b(r_{r,d}-1)}]}{[z_{r,d} - (r_{r,d}-1)]} \right\}. \qquad (6)$$

Then similar average frequencies $\Omega_{r,d}$ can be defined according to Eq. (3), and one may study the phase entrainment of the response system to the drive. In Figs. 6 we give the relations $\Omega_{r,d} \sim \varepsilon$ [6(a)] and $\lambda_c^1 \sim \varepsilon$ [6(b)]. PS is achieved at $\varepsilon \approx 39$, as seen in Fig. 6(a). GS occurs first at $\varepsilon \approx 3$ and loses its stability at $\varepsilon \approx 6.5$, and then recurs at $\varepsilon \approx 10$. Obviously $\varepsilon_c^{GS} < \varepsilon_c^{PS}$. This supports our result that GS is not necessarily stronger than PS.

To conclude, in this paper we studied the relation between PS and GS. GS could be stronger than PS, and they can also occur at the same critical coupling. It is equally possible that PS can be stronger than GS. We show these features by using a phase diagram. For small parameter misfits, GS is identified as LS. The stick-slip statistics is studied before the onset of PS. The quantized feature of slip time is shown. For small and large parameter misfits, the average slip time vs the coupling obeys different scaling relations near the PS threshold.

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[13] A. Pikovsky, M.G. Rosenblum, G.V. Osipov, and J. Kurths, Physica D **104**, 219 (1997).
[14] The negativeness of the MCLE does not necessarily mean CS or GS. A possibility is the period-doubled synchronization, found by Rulkov *et al.* (see [8]), where the response is a period-doubled version of the drive, which is not a one-to-one relation. For the systems proposed in the present paper, the negative NCLE does correspond to the implement of GS.
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