Steady states of a Boltzmann equation for driven granular media

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We study a three-dimensional model for driven granular media in which particles interact inelastically while they follow Brownian dynamics in between collisions. A steady Boltzmann-type kinetic equation associated with a pseudo-Maxwellian model is analyzed. Homogeneous steady states are found by a small inelasticity expansion. These states are given by a Maxwellian distribution corrected by the second Sonine polynomial up to third order in the expansion. The resulting correction is a quartic polynomial in velocity space. This result agrees qualitatively with the molecular dynamics simulation in C. Bizon, M. D. Shattuck, J. B. Swift, and H. L. Swinney, Phys. Rev. E **60**, 4340 (1999).

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I. INTRODUCTION

In recent years significant interest has been focused on the study of kinetic models for rapid granular flows. Experimental and numerical data from molecular dynamics (MD) simulations indicate that particle distribution functions are far from Gaussian distributions when particles collide inelastically. Our work is motivated by a recent one [1], where molecular dynamics simulations of a homogeneous granular flow, driven by a heat bath, show a clear deviation from Gaussian states. As a consequence, we look for steady distributions of granular flows driven by random accelerations. Our aim is to find an approximate steady solution for a simplified homogeneous inelastic Boltzmann-Enskog model. Our expansion parameter is the energy dissipation rate.

Following the initial work of Ref. [2], to simplify we assume the Boltzmann-Enskog inelastic collision operator introduced as an analog to the case of Maxwellian molecules in the classical elastic Boltzmann equation. This pseudo-Maxwellian approximation assumes a collision frequency independent of the relative velocities, but proportional to the square root of the kinetic temperature through a constant S. This constant S is fixed in such a way that the energy loss coincides with the one from the hard-spheres collision operator Q_{HS}. In particular this model reproduces the steady temperature according to a recent three-dimensional (3D) molecular dynamics simulation. Such a reduction allows the explicit computation of stationary isotropic homogeneous solutions for small energy dissipation perturbations. Then we find that the corrections to these perturbations can only be Gaussian distributions multiplied by a factor given by the second Sonine polynomial. The second Sonine polynomial is related to the second isotropic eigenvalue of the linearized classical Boltzmann elastic operator.

In particular, the stationary solution, to second order accuracy in energy dissipation rate, is given by a Maxwellian distribution multiplied by a factor proportional to a quartic polynomial in velocity space. Such a solution qualitatively agrees with the one computed in Ref. [1], Fig. 3. Let us remark that our computation of the solution is rigorous once the expansion in the dissipation rate is performed.

Being more precise, we perform a statistical mechanics analysis of the dynamics of perfect spheres of diameter $\sigma>0$ colliding inelastically in a thermal bath of infinite temperature. Because the inelastic collision particles are constantly losing energy, the inclusion of an energy input mechanism allows us to achieve a steady state. Then, a uniformly heated system is obtained by assuming Brownian motion of the particles between collisions. The corresponding equation of motion can be written as the Langevin equation

$$x''=\Gamma(t),$$

where $\Gamma(t)$ is a white noise stochastic force with independent, identically Gaussian distributed processes of variance F, that is, $\langle\Gamma(t),\Gamma(t')\rangle=2F\delta(t-t')$.

Concerning the collision mechanism, if (x,v) and $(x-\sigma n,w)$ are the states of two particles before a collision, where $n\in S^2$ is the unit vector along the center of both spheres, the post-collisional velocities are found by assuming that the total momentum is preserved, but part of the normal relative velocity is lost, that is,

$$n\cdot(v'-w')=-e[(v-w)\cdot n],$$

where $0<e\leqslant 1$ is called the restitution coefficient.

We can then easily construct the post-collisional velocities as

$$v'=\frac{1}{2}(v+w)+\frac{V'}{2}. \tag{1.1}$$

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$$w' = \frac{1}{2}(v+w) - \frac{V'}{2}, \qquad (1.2)$$

where $V' = V - (1+e)(V\cdot n)n$, $V = v - w$, and $V' = v' - w'$. Let us denote by v^* and w^* the precollisional velocities corresponding to v and w.

Therefore, following the standard procedures of kinetic theory [3–6], we can find a Boltzmann-Enskog equation for inelastic hard spheres in a thermal bath. This equation reads

$$\frac{\partial f}{\partial t} = Q_{\mathrm{HS}}(f,f) + L_{\mathrm{FP}}f, \qquad (1.3)$$

where Q_{HS} is the collision operator for inelastic hard spheres [4,5] and L_{FP} is the Fokker-Planck operator. This operator takes into account the white noise interaction between collisions and using Ito's stochastic calculus is given by

$$L_{\mathrm{FP}}f = F\Delta_v f. \qquad (1.4)$$

The corresponding homogeneous Boltzmann equation for inelastic particles under the pseudo-Maxwellian approximation in a heat bath is given by

$$\frac{\partial f}{\partial t} = B(\rho,t)Q_\varepsilon(f,f) + L_{\mathrm{FP}}f, \qquad (1.5)$$

where

$$Q_\varepsilon(f,f) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{S^2}[f(t,v^*)f(t,w^*)\,J - f(t,v)f(t,w)]dn\,dw \qquad (1.6)$$

with

$$J = \frac{1}{e^2}\frac{|v-w|}{|v^*-w^*|}$$

and

$$B(\rho,t) = \pi S\sigma^2 G(\rho)\sqrt{\theta(t)} = B(\rho)\sqrt{\theta(t)} \qquad (1.7)$$

and the label ε refers to the temperature dissipation rate $\varepsilon = (1-e^2)/4$. Here, v^*, w^* are the precollisional velocities associated with the postcollisional velocities

$$v' = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|n, \qquad (1.8)$$

$$w' = \frac{1}{2}(v+w) - \frac{1-e}{4}(v-w) - \frac{1+e}{4}|v-w|n. \qquad (1.9)$$

Equation (16) corrects the strong form of the pseudo-Maxwellian collision integral given in [2].

Also, ρ, u, and $\theta(t)$ are the density, mean velocity, and temperature of the distribution f. Since both $Q_\varepsilon(f,f)$ and $L_{\mathrm{FP}}f$ preserve density and mean velocity, these quantities are just constants. The function $G(\rho)$ takes into account dense gas effects [5,1].

The operator $Q_\varepsilon(f,f)$ acts on functions $\psi \in C_0^\infty(\mathbb{R}^3)$ as

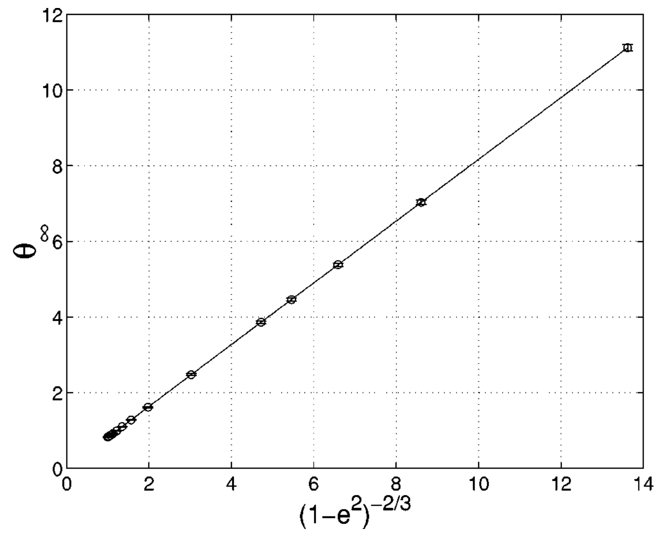

FIG. 1. Validation of formula (2.11) by a 3D MD simulation. Kinetic temperature versus $(1-e^2)^{-2/3}$.

$$\langle Q_\varepsilon(f,f),\psi\rangle = \frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} f(t,w)f(t,v)[\psi(v') - \psi(v)]dv\,dw\,dn,$$

where v' is computed by

$$v' = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|n. \qquad (1.10)$$

The existence of such steady states in heated granular media was proved in the one-dimensional case for a different collision operator in Ref. [7] and discussed for a discrete number of particles in Ref. [8]; see also Ref. [9,10]. As a first result we find the equation of state for the steady state which is given by

$$\theta_\infty(\rho) = \left(\frac{2F}{B(\rho)\rho\varepsilon}\right)^{2/3}.$$

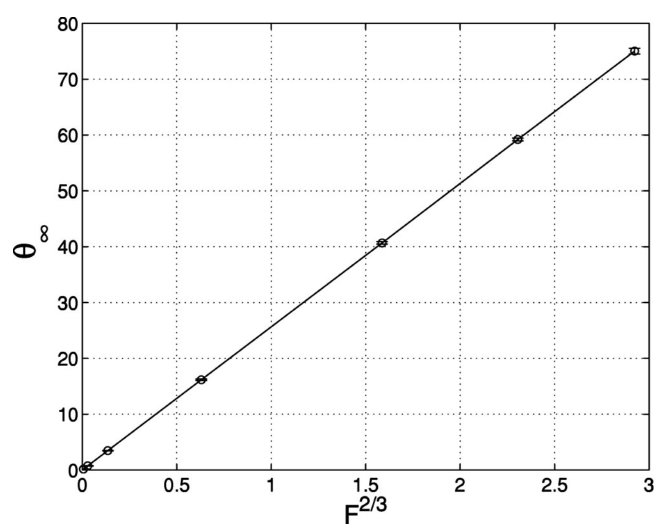

FIG. 2. Validation of formula (2.11) by a 3D MD simulation. Kinetic temperature versus $F^{2/3}$.

Precisely, this prediction coincides with the results shown in Figs. 1 and 2, corresponding to a recent three-dimensional molecular dynamics simulation. Figures 1 and 2 show the steady state kinetic temperature as a function of the heat bath temperature F and as a function of the energy dissipation rate $\varepsilon=(1-e^2)/4$, which clearly coincide with the above formula for θ_∞. These simulations are performed in a finite box with a fixed number of particles. In addition we point out that this dependence of θ_∞ on ρ, ε, and F coincides with the one-dimensional molecular dynamics results shown in Ref. [8] (Figs. 3, 4, and 5).

Though our work has been strongly motivated by the MD simulation of a 2D hard-spheres model in a finite box [1], we performed it in 3D were the collision operator has a natural invariance in spherical coordinates that yields the representation (1.6). Similar data corresponding to 3D MD simulations are not available, but those simulations are currently under way.

II. TEMPERATURE DISSIPATION

We first find the equation for the evolution of the second moment of the distribution function. Let us consider ρ, u, and $\theta(t)$ the density, mean velocity, and temperature of $f(t,v)$. Thus, f must satisfy

$$\int_{\mathbb{R}^3} f\,dv=\rho,\quad \int_{\mathbb{R}^3} v f\,dv=\rho u,\quad \int_{\mathbb{R}^3}|v-u|^2 f\,dv=3\rho\theta. \tag{2.1}$$

Let us remark that the computation of the second moment without the Fokker-Planck operator was done for isotropic solutions in Ref. [2] using Fourier transform techniques. We include here this computation in a different way which is valid for general distributions.

Computing the integral of $Q_\varepsilon(f,f)$ multiplied by $|v-u|^2$ we deduce

$$\begin{aligned}3\rho\theta'(t)&=\frac{d}{dt}\int_{\mathbb{R}^3}|v-u|^2 f(v)\,dv\\&=\frac{B(\rho,t)}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} f(v)f(w)\\&\quad\times(|v'-u|^2-|v-u|^2)\,dv\,dw\,dn+6F\rho.\end{aligned} \tag{2.2}$$

We change variables in the collision integral finding

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} f(v)f(w)(|v'-u|^2-|v-u|^2)\,dv\,dw\,dn\\&\quad=\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} f(v+u)f(w+u)(|v'|^2-|v|^2)\,dv\,dw\,dn.\end{aligned} \tag{2.3}$$

Now, we need to compute $|v'|^2$. In order to simplify this computation we use the unitary linear change of variables given by the velocity of the center-of-mass–relative-velocity system

$$(\bar{v},V)=\left(\frac{v+w}{2},v-w\right),$$

so that

$$v'=\bar{v}+\frac{1-e}{4}V+\frac{1+e}{4}|V|\cdot n \tag{2.4}$$

then

$$\begin{aligned}|v'|^2=|\bar{v}|^2+\frac{1+e^2}{8}|V|^2+\frac{1-e}{2}\bar{v}\cdot V\\+\frac{1+e}{2}|V|\bar{v}\cdot n+\frac{1-e^2}{8}|V|V\cdot n.\end{aligned}$$

Since the integral on the unit vector n goes through in Eq. (2.3), the integrals containing $|V|\bar{v}\cdot n$ and $|V|V\cdot n$ will vanish. On the other hand, due to $\bar{v}\cdot V=1/2(|v|^2-|w|^2)$ and Eq. (2.1) the integral containing $\bar{v}\cdot V$ vanishes too. In addition, rewriting $\bar{v}$ and V in the original variables yields

$$|\bar{v}|^2+\frac{1+e^2}{8}|V|^2=\frac{3+e^2}{8}(|v|^2+|w|^2)+\frac{1-e^2}{4}v\cdot w; \tag{2.5}$$

therefore the collision integral of Eq. (2.2) can be computed as

$$\begin{aligned}&B(\rho)\,\theta^{1/2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v+u)f(w+u)\\&\quad\times\left[\frac{3+e^2}{8}(|v|^2+|w|^2)-|v|^2\right]dw\,dv\end{aligned} \tag{2.6}$$

since the term with $v\cdot w$ vanishes too due to Eq. (2.1). Again, using properties (2.1), (2.6) results in

$$\begin{aligned}&=B(\rho)\,\theta^{1/2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v+u)f(w+u)\\&\quad\times\left[\frac{(-5+e^2)}{8}|v|^2+\frac{3+e^2}{8}|w|^2\right]dv\,dw\\&=-3B(\rho)\rho^2\frac{1-e^2}{4}\,\theta^{3/2}\\&=-3B(\rho)\rho^2\varepsilon\,\theta^{3/2}.\end{aligned} \tag{2.7}$$

Therefore combining Eqs. (2.2) and (2.7) yields the equation

$$\rho\theta'(t)=-B(\rho)\rho^2\varepsilon\,\theta^{3/2}+2F\rho \tag{2.8}$$

or equivalently the temperature dissipation equation

$$\theta'=-B(\rho)\rho\varepsilon\,\theta^{3/2}+2F. \tag{2.9}$$

Now, we fix the value of the constant S. The dissipation term in temperature arising from the collision operator Q_ε is given by $\gamma_\varepsilon=\pi S\sigma^2 G(\rho)\rho\varepsilon=B(\rho)\rho\varepsilon$ while the dissipation term arising from the hard-spheres original operator Q_{HS} is given by [5,1] $\gamma_{\rm HS}=8\sqrt{\pi}\sigma^2 G(\rho)\rho\varepsilon$. We set S by $\gamma_\varepsilon=\gamma_{\rm HS}$ and then $S=8/\sqrt{\pi}$. Thus, finally the temperature equation can be written as

$$\theta'=-\gamma_\varepsilon\theta^{3/2}+2F. \tag{2.10}$$

Let us remark that this equation is also valid for the hard-spheres case.

The equilibrium point corresponds to a possible steady state $f^s_{\rho,u}$ whose temperature $\theta_\infty(\rho)$ is explicitly given by

$$\theta_\infty(\rho)=\left(\frac{2F}{B(\rho)\rho\varepsilon}\right)^{2/3}=\left(\frac{2}{\gamma_\varepsilon}F\right)^{2/3}. \tag{2.11}$$

This temperature is asymptotically stable for Eq. (2.10). Moreover, it is easy to see that a steady state $f^s_{\rho,u}$ can be obtained from a normalized steady state $f^s_{1,0}=f^s$ by the self-similar relation

$$f^s_{\rho,u}(v)=\rho A(\rho)^{-3/2}f^s[A(\rho)^{-1/2}(v-u)], \tag{2.12}$$

where

$$A(\rho)=\rho^{-2/3}\left(\frac{G(1)}{G(\rho)}\right)^{2/3}=\frac{\theta_\infty(\rho)}{\theta_\infty}=\frac{\sqrt{\theta_\infty}G(1)}{\rho\sqrt{\theta_\infty(\rho)}G(\rho)}$$

and $\theta_\infty=\theta_\infty(1)$. In this sense we can say that the steady states are self-similar solutions.

Let us also finally point out that the dependency on ρ of $\theta_\infty(\rho)$ (2.11) and the relation (2.12) for $G=1$ coincides with the one-dimensional granular media models developed in Refs. [7,10]. Let us remark again that the formula (2.11) appears in Ref. [8] for a one-dimensional problem.

III. AN APPROXIMATION FORMULA FOR THE STEADY STATE

Assuming the existence of a steady state solution $f^s_{\rho,u}$ we shall perform a small inelasticity expansion and a linearization of the operator to compute a small inelasticity approximation of this steady state. Let us take unit density and zero mean velocity and focus on an expansion of f^s since using Eq. (2.12) we produce an expansion for $f^s_{\rho,u}$. Let us remember that f^s has temperature

$$\theta_\infty=\left(\frac{2F}{B\varepsilon}\right)^{2/3} \tag{3.1}$$

with $B=8\sqrt{\pi}\sigma^2 G(1)$ and assumed to satisfy

$$B\sqrt{\theta_\infty}Q_\varepsilon(f^s,f^s)+L_{\rm FP}f^s=0. \tag{3.2}$$

The expansion parameter is given by $\varepsilon=(1-e^2)/4$. Therefore, we can approximate the restitution coefficient by $e=\sqrt{1-4\varepsilon}\approx 1-2\varepsilon-2\varepsilon^2$. The reason for this choice is twofold: on one hand, ε is the parameter involved in the dissipation term for the temperature equation, on the other hand, ε appears as a small eigenvalue of the linearized operator Q_ε about Dirac's delta distribution which has a lot of important consequences in the asymptotic behavior of the distribution for small inelasticity in the unheated case $\theta_b=0$, see Ref. [2].

We perform the expansion over the action of the operator Q_ε since it is easier to evaluate it in this form. The postcollisional velocity v' is given by

$$\begin{aligned}v'&=\frac{1}{2}(v+w)+\frac{1-e}{4}(v-w)+\frac{1+e}{4}|v-w|n\\&=\frac{1}{2}(v+w)+\frac{1}{2}|v-w|n+\frac{1}{2}(\varepsilon+\varepsilon^2)(v-w)\\&\quad-\frac{1}{2}(\varepsilon+\varepsilon^2)|v-w|n\\&=v_0+(\varepsilon+\varepsilon^2)v_1.\end{aligned} \tag{3.3}$$

Here $v_0=\frac{1}{2}(v+w)+\frac{1}{2}|v-w|n$ is the postcollisional velocity corresponding to an elastic collision and $v_1=\frac{1}{2}(V-|V|n)$ the dissipated part of the postcollisional velocity, which depends only on the relative velocity and the collision angle.

We look for the steady state as an ε^2 perturbation of a Maxwellian distribution with temperature θ_∞; therefore

$$f^s(v)=M_{\theta_\infty}(1+\varepsilon^2 g)=M_{\theta_\infty}+\tilde{g}$$

with $\int\tilde{g}\,dv=0$. Linearizing $Q_\varepsilon(f,f)$ about M_{θ_∞} on Eq. (3.2) yields an equation for g

$$B\sqrt{\theta_\infty}Q_\varepsilon(M_{\theta_\infty},M_{\theta_\infty})+2B\sqrt{\theta_\infty}Q_\varepsilon(M_{\theta_\infty},\tilde{g})+L_{\rm FP}(M_{\theta_\infty})+L_{\rm FP}(\tilde{g})=0. \tag{3.4}$$

On the other hand, since Q_ε depends on the restitution coefficient $e\approx 1-2\varepsilon-2\varepsilon^2$ we expand Q_ε expressed in a weak form by using a Taylor series in ε for $\psi(v')$, that is, for $\psi\in C_0^\infty(\mathbb{R})$ and $\overline{v}=(v+w)/2$,

$$\begin{aligned}\psi(v')&=\psi[v_0+(\varepsilon+\varepsilon^2)v_1]\\&=\psi(v_0)+(\varepsilon+\varepsilon^2)[v_1\cdot(\nabla_{\overline{v}})\psi](v_0)\\&\quad+\frac{\varepsilon^2}{2}v_1^{\otimes 2}\cdot(H_{\overline{v}}\psi)(v_0)+\mathcal{O}(\varepsilon^3),\end{aligned} \tag{3.5}$$

where $H_{\overline{v}}\psi(v_0)$ denotes the Hessian of ψ with respect to $\overline{v}$ at v_0. Then inserting Eq. (3.5) in the weak formulation of the collision integral and integrating by parts we have

$$\langle Q_\varepsilon(f,f),\psi\rangle=\langle Q_0(f,f),\psi\rangle+(\varepsilon+\varepsilon^2)\langle R(f,f),\psi\rangle+\frac{\varepsilon^2}{2}\langle S(f,f),\psi\rangle, \tag{3.6}$$

where Q_0 is the classical elastic Boltzmann collision operator for Maxwellian molecules

$$\langle R(f,f),\psi\rangle = -\frac{1}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}(V-|V|n)\cdot\nabla_{\bar{v}}[f(v)f(w)]\psi(v_0)dv\,dw\,dn \quad (3.7)$$

and

$$\langle S(f,f),\psi\rangle = \frac{1}{16\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}(V-|V|n)^{\otimes 2}\cdot H_{\bar{v}}\times[f(v)f(w)]\psi(v_0)dv\,dw\,dn. \quad (3.8)$$

Since all the Maxwellian distributions are in the kernel of the classical operator Q_0 we have $Q_0(M_{\theta_\infty},M_{\theta_\infty})=0$. Therefore, $\tilde{g}$ must satisfy the equation

$$\begin{aligned}0 = B\sqrt{\theta_\infty}\Big\langle &(\varepsilon+\varepsilon^2)[R(M_{\theta_\infty},M_{\theta_\infty})+2R(M_{\theta_\infty},\tilde{g})]\\&+\frac{\varepsilon^2}{2}[S(M_{\theta_\infty},M_{\theta_\infty})+2S(M_{\theta_\infty},\tilde{g})],\psi\Big\rangle\\&+2B\sqrt{\theta_\infty}\langle Q_0(M_{\theta_\infty},\tilde{g}),\psi\rangle+\langle L_{\mathrm{FP}}M_{\theta_\infty},\psi\rangle\\&+\langle L_{\mathrm{FP}}\tilde{g},\psi\rangle+\mathcal{O}(\varepsilon^3).\end{aligned} \quad (3.9)$$

It is straightforward to find

$$L_{\mathrm{FP}}M_{\theta_\infty}(v) = \frac{F}{\theta_\infty}\left(-3+\frac{|v|^2}{\theta_\infty}\right)M_{\theta_\infty}(v) = -\frac{\varepsilon}{2}B\sqrt{\theta_\infty}M_{\theta_\infty}\left[3-\frac{|v|^2}{\theta_\infty}\right]. \quad (3.10)$$

Next, replacing Eq. (3.10) into Eq. (3.9) and taking into account that $\tilde{g}=\varepsilon^2 M_{\theta_\infty}g$ the following equation is valid for g:

$$\begin{aligned}0 = &\varepsilon^{-1}B\sqrt{\theta_\infty}\left[\langle R(M_{\theta_\infty},M_{\theta_\infty}),\psi\rangle-\left\langle\frac{1}{2}M_{\theta_\infty}\left(3-\frac{|v|^2}{\theta_\infty}\right),\psi\right\rangle\right]\\&+B\sqrt{\theta_\infty}\left\langle\left(R+\frac{1}{2}S\right)(M_{\theta_\infty},M_{\theta_\infty}),\psi\right\rangle\\&+2B\sqrt{\theta_\infty}\langle Q_0(M_{\theta_\infty},gM_{\theta_\infty}),\psi\rangle\\&+\langle L_{\mathrm{FP}}(M_{\theta_\infty}g),\psi\rangle+\mathcal{O}(\varepsilon).\end{aligned} \quad (3.11)$$

The terms R and S evaluated on $(M_{\theta_\infty},M_{\theta_\infty})$ can be computed exactly. The computation is given in the Appendix and yields

$$\langle R(M_{\theta_\infty},M_{\theta_\infty}),\psi\rangle = \left\langle\frac{1}{2}M_{\theta_\infty}(v)\left[3-\frac{|v|^2}{\theta_\infty}\right],\psi\right\rangle \quad (3.12)$$

and

$$S(M_{\theta_\infty},M_{\theta_\infty}) = M_{\theta_\infty}(v)\left[\frac{|v|^4}{3\theta_\infty^2}-\frac{7}{3}\frac{|v|^2}{\theta_\infty}+2\right]. \quad (3.13)$$

This shows that the term of order ε^{-1} vanishes in Eq. (3.11). Next, we can replace Eqs. (3.12) and (3.13) in the weak formulation of the equation for g, given by Eq. (3.11), obtaining the following equation for g:

$$\begin{aligned}&-2B\sqrt{\theta_\infty}Q_0(M_{\theta_\infty},gM_{\theta_\infty})-L_{\mathrm{FP}}(gM_{\theta_\infty})\\&=B\sqrt{\theta_\infty}M_{\theta_\infty}\left[\frac{1}{2}\left(3-\frac{|v|^2}{\theta_\infty}\right)+\frac{|v|^4}{6\theta_\infty^2}-\frac{7}{6}\frac{|v|^2}{\theta_\infty}+1\right]\\&=\frac{1}{6}B\sqrt{\theta_\infty}M_{\theta_\infty}\left[\frac{|v|^4}{\theta_\infty^2}-\frac{10|v|^2}{\theta_\infty}+15\right]=-h(v).\end{aligned} \quad (3.14)$$

Next, let us consider the linear operator acting on g

$$\mathcal{L}(gM_{\theta_\infty}) = 2B\sqrt{\theta_\infty}Q_0(M_{\theta_\infty},gM_{\theta_\infty})+L_{\mathrm{FP}}(gM_{\theta_\infty}). \quad (3.15)$$

We need to solve the problem

$$\mathcal{L}(gM_{\theta_\infty}) = h(v). \quad (3.16)$$

We first discuss the eigenvalue problem associated with the operator $\mathcal{L}(gM_{\theta_\infty})$, that is,

$$\mathcal{L}(gM_{\theta_\infty}) = -\lambda gM_{\theta_\infty}. \quad (3.17)$$

We restrict ourselves to the case of isotropic eigenfunctions. We recall from Refs. [3,11] that the isotropic eigenfunctions of the operators $Q_0(M_{\theta_\infty},gM_{\theta_\infty})$ are given by Sonine polynomials multiplied by $M_{\theta_\infty}(v)$. We recall that the second order Sonine polynomial [11] is

$$S_{1/2}^{(2)}(\xi) = \frac{3}{24}(15-20\xi+4\xi^2) \quad (3.18)$$

for any $\xi\in\mathbb{R}$, and if we define the function

$$P(v) = S_{1/2}^{(2)}\left(\frac{|v|^2}{2\theta_\infty}\right) = \frac{3}{24}\left(15-\frac{10|v|^2}{\theta_\infty}+\frac{|v|^4}{\theta_\infty^2}\right) \quad (3.19)$$

then, the function $h(v)$ is given by

$$h(v) = -\frac{4}{3}B\sqrt{\theta_\infty}M_{\theta_\infty}P. \quad (3.20)$$

We also recall [12,13] that

$$2Q_0\left[M_{\theta_\infty},S_{1/2}^{(2)}\left(\frac{|v|^2}{2\theta_\infty}\right)M_{\theta_\infty}\right] = -\lambda_{20}S_{1/2}^{(2)}\left(\frac{|v|^2}{2\theta_\infty}\right)M_{\theta_\infty}(v) \quad (3.21)$$

with $\lambda_{20}=1/3$.

Next, we show that the function $M_{\theta_\infty}P=M_{\theta_\infty}S_{1/2}^{(2)}(|v|^2/2\theta_\infty)$ is an eigenvector with zero eigenvalue for the operator $L_{\mathrm{FP}}=F\Delta_v$ up to an order of $\varepsilon^{2/3}$. We first compute

$$L_{\rm FP}(M_{\theta_\infty})=FM_{\theta_\infty}\left[P\left(\frac{|v|^2}{\theta_\infty}-3\right)\frac{1}{\theta_\infty}-\frac{v}{\theta_\infty}\cdot\nabla_v P+\Delta_v P\right]$$
$$=\frac{1}{\theta_\infty}FM_{\theta_\infty}\frac{3}{24}\left(-105+85\frac{|v|^2}{\theta_\infty}-17\frac{|v|^4}{\theta_\infty^2}+\frac{|v|^6}{\theta_\infty^3}\right). \tag{3.22}$$

Now, by Eq. (3.1) $\theta_\infty^{-1}=\mathcal{O}(\varepsilon^{2/3})$ and thus, $L_{\rm FP}(M_{\theta_\infty}P)=\mathcal{O}(\varepsilon^{2/3})$.

To finish we come back to Eq. (3.16), take into account Eq. (3.20) and the previous results for the eigenvalues of the operators to find out that

$$g(v)=4S_{1/2}^{(s)}\left(\frac{|v|^2}{2\theta_\infty}\right)M_{\theta_\infty}(v)$$

is a solution of Eq. (3.16) up to order ε. We conclude by writing the expansion of the stationary solution

$$f^s(v)=M_{\theta_\infty}(v)\left[1+4\varepsilon^2S_{1/2}^{(2)}\left(\frac{|v|^2}{2\theta_\infty}\right)\right]+\mathcal{O}(\varepsilon^3).$$

Now, we use the self-similar relation (2.12) to obtain the main result of this paper.

Theorem III.1. The steady state $f^s_{\rho,u}$ for Eq. (1.5), up to order ε^3, is given by

$$f^s_{\rho,u}(v)=\rho M_{\theta_\infty(\rho)}(v-u)\left[1+4\varepsilon^2S_{1/2}^{(2)}\left(\frac{|v-u|^2}{2\theta_\infty(\rho)}\right)\right]+\mathcal{O}(\varepsilon^3)$$

with

$$\theta_\infty(\rho)=\left(\frac{2F}{B(\rho)\rho\varepsilon}\right)^{2/3}=\left(\frac{2}{\gamma_\varepsilon}F\right)^{2/3}$$

and $S_{1/2}^{(2)}$ given by formula (3.18).

IV. HEAT BATH WITH FRICTION

We may also include friction on the particles between collisions assuming that the particles are in some sort of surrounding heat bath with a fixed finite temperature θ_b. In this case, the paths of the particles are governed by the Langevin equation

$$x''+\frac{1}{\tau}x'=\Gamma(t),$$

where $\Gamma(t)$ is a white noise stochastic force with independent, identically Gaussian distributed processes of variance F, that is, $\langle\Gamma(t),\Gamma(t')\rangle=2F\delta(t-t')$ with $F=\theta_b/\tau$ and $\tau>0$ the relaxation time corresponding to the damping force. The Fokker-Planck operator now reads

$$L^2_{\rm FP}f=\frac{1}{\tau}{\rm div}_v(vf+\theta_b\nabla_v f).$$

In this case, we can perform the same computations as before and the temperature dissipation equation becomes

$$\theta'+\frac{2}{\tau}\theta=-\gamma_\varepsilon\theta^{3/2}+\frac{2}{\tau}\theta_b. \tag{4.1}$$

Also, the mean velocity is no longer preserved and is dissipated according to $u(t)=e^{-t/\tau}u(0)$.

The equilibrium point corresponds to a possible steady state f^s_ρ with zero mean velocity where $\theta_\infty(\rho)$ is given by the unique positive solution of the equation

$$\theta_b=\theta+\frac{\tau}{2}\gamma_\varepsilon\theta^{3/2}. \tag{4.2}$$

This temperature is globally asymptotically stable for the evolution of Eq. (4.1). In this case we do not have a simple relation between f^s_ρ and the normalized f^s.

The same expansion procedure can be applied to this operator. The main differences are that we do not have a relation as Eq. (2.12) to reduce the computation to f^s and the computation of $L^2_{\rm FP}(M_{\theta_\infty(\rho)}P)$ with P given by

$$P(v)=S_{1/2}^{(2)}\left(\frac{|v|^2}{2\theta_\infty(\rho)}\right)$$
$$=\frac{3}{24}\left(15-\frac{10|v|^2}{\theta_\infty(\rho)}+\frac{|v|^4}{\theta_\infty(\rho)^2}\right).$$

The first one is easily solved by directly expanding f^s_ρ. Now, we compute

$$L^2_{\rm FP}(M_{\theta_\infty(\rho)}P)=\frac{M_{\theta_\infty(\rho)}}{\tau}\left[P\left(3-\frac{|v|^2}{2}\right)\left(1-\frac{\theta_b}{\theta_\infty(\rho)}\right)+v\left(1-\frac{2\theta_b}{\theta_\infty(\rho)}\right)\nabla_vP+\theta_b\Delta_vP\right]. \tag{4.3}$$

Now, by Eq. (4.2) we have $\theta_\infty(\rho)=\theta_b+\mathcal{O}(\varepsilon)$ and therefore, we can estimate the right-hand side of Eq. (4.3) as

$$\frac{M_{\theta_\infty(\rho)}}{\tau}[\mathcal{O}(\varepsilon)-v\nabla_vP+\theta_b\Delta_vP]=-4\frac{M_{\theta_\infty(\rho)}}{\tau}P+\mathcal{O}(\varepsilon)$$

thus

$$L^2_{\rm FP}(M_{\theta_\infty(\rho)}P)=-\frac{4}{\tau}M_{\theta_\infty(\rho)}P+\mathcal{O}(\varepsilon).$$

As a consequence, the expansion for the steady state in this case is

$$f^s_\rho(v)=\rho M_{\theta_\infty(\rho)}(v)\left[1+\varepsilon^2\bar{A}S_{1/2}^{(2)}\left(\frac{|v|^2}{2\theta_\infty(\rho)}\right)\right]+\mathcal{O}(\varepsilon^3)$$

with

$$\bar{A}=4\left(\frac{\tau\rho B(\rho)\theta_\infty(\rho)+3}{\tau\rho B(\rho)\theta_\infty(\rho)}\right),$$

$\theta_\infty(\rho)$ being the unique solution of

$$\theta_b=\theta+\frac{\tau}{2}\gamma_\varepsilon\theta^{3/2}$$

and $S^{(2)}_{1/2}$ given by formula (3.18). Formally, we recover theorem III.1 by taking the limit $\tau\to\infty$, $\theta_b\to\infty$ in such a way that $F=\theta_b/\tau$ remains constant.

V. CONCLUSIONS

We have found that a reduced model to the inelastic hard-spheres Boltzmann-Enskog equation, has stationary isotropic homogeneous solutions for small energy dissipation perturbations. These solutions are explicitly given and they must be, to quadratic order, Maxwellian distributions multiplied by a factor given by the second Sonine polynomial evaluated at the temperature of the Maxwellian. The second Sonine polynomial is related to the second eigenvalue of the linearized classical Boltzmann elastic operator. Moreover, we have obtained rigorously the equation for the temperature of the steady state (2.11).

Though our work has been strongly motivated by the MD simulation of a 2D hard-spheres model in a finite box [1], we performed it in 3D. Thus, our results are not directly comparable to those in Ref. [1]. Nevertheless, it reproduces qualitatively the difference of the steady state distribution function with respect to a Maxwellian as shown in Ref. [1], Fig. 3. Moreover, Figs. 1 and 2 show that formula (2.11) agrees very well with the molecular dynamics simulation.

Let us finally remark that very recent papers [14–16], dealing with the homogeneous inelastic hard-spheres Boltzmann-Enskog model uniformly heated by a bath, show that there can be solutions given by Maxwellians multiplied by a factor depending on the second Sonine polynomial. In these papers different additional assumptions are made in order to compute this factor depending on the energy dissipation rate. One one hand, the present paper shows a more rigorous approach to find an approximation to the steady solution for the pseudo-Maxwellian model showing that it must be a Maxwellian distribution modified by a factor containing the second Sonine polynomial. However, these papers show that the coefficient corresponding to this polynomial correction is not necessarily positive as a function of the energy dissipation rate. This may indicate that a solution for the stationary inelastic pseudo-Maxwellian model might be better approximated by a double expansion in energy dissipation rate at all orders and higher order Sonine polynomials as well. Also, the validity of the pseudo-Maxwellian model should be examined. These tasks are presently under way.

APPENDIX

Here, we calculate Eqs. (3.7) and (3.8) in order to replace in Eq. (3.11). First from Eq. (3.7), using $(\bar{v},V)$ coordinates

$$\langle R(M_{\theta_\infty},M_{\theta_\infty}),\psi\rangle=-\frac{1}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}(V-|V|n)\cdot\nabla_{\bar{v}}\times\left[M_{\theta_\infty}(\sqrt{2}\bar{v})M_{\theta_\infty}\left(\frac{V}{\sqrt{2}}\right)\right]\times\psi\left(\bar{v}+\frac{|V|n}{2}\right)d\bar{v}\,dV\,dn. \quad \text{(A1)}$$

Since $\nabla_{\bar{v}}M_{\theta_\infty}(\sqrt{2}\bar{v})=-(2\bar{v}/\theta_\infty)M_{\theta_\infty}(\sqrt{2}\bar{v})$ we have

$$\frac{1}{4\pi\theta_\infty}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}M_{\theta_\infty}(\sqrt{2}\bar{v})M_{\theta_\infty}\left(\frac{V}{\sqrt{2}}\right)\times(V\cdot\bar{v}-|V|\bar{v}\cdot n)\psi\left(\bar{v}+\frac{|V|n}{2}\right)d\bar{v}\,dV\,dn. \quad \text{(A2)}$$

Now, considering spherical coordinates $V=|V|k$ with $k\in S^2$, swapping the roles of k and n, interchanging the integrals over dn by the one over dk and rewriting $V=|V|n$, (A2) becomes

$$\frac{1}{4\pi\theta_\infty}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}M_{\theta_\infty}(\sqrt{2}\bar{v})M_{\theta_\infty}\left(\frac{V}{\sqrt{2}}\right)\times(|V|\bar{v}\cdot k-\bar{v}\cdot V)\psi(v)\,d\bar{v}\,dV\,dk. \quad \text{(A3)}$$

The integration with respect to k on S^2 can be performed and the integral of the term $\bar{v}\cdot k$ vanishes. Returning to (v,w) coordinates and taking into account $\bar{v}\cdot V=(|v|^2-|w|^2)/2$, Eq. (A3) becomes

$$-\frac{1}{2\theta_\infty}\int_{\mathbb{R}^3\times\mathbb{R}^3}M_{\theta_\infty}(v)M_{\theta_\infty}(w)(|v|^2-|w|^2)\psi(v)\,dv\,dw. \quad \text{(A4)}$$

Therefore the operator $R(M_{\theta_\infty},M_{\theta_\infty})$ can be expressed as

$$-\tfrac{1}{2}\int_{\mathbb{R}^3}\frac{1}{\theta_\infty}M_{\theta_\infty}(v)M_{\theta_\infty}(w)(|v|^2-|w|^2)\,dw=-\frac{1}{2\theta_\infty}M_{\theta_\infty}(v)(|v|^2-3\theta_\infty), \quad \text{(A5)}$$

which finally gives Eq. (3.12).

We proceed similarly in order to compute $S(M_{\theta_\infty},M_{\theta_\infty})$. Expressing the first integral on S in $(\bar{v},V)$ variables, computing

$$H_{\bar{v}}[M_{\theta_\infty}(\sqrt{2}\bar{v})]=M_{\theta_\infty}(\sqrt{2}\bar{v})\left(-\frac{2}{\theta_\infty}I+\frac{4}{\theta_\infty^2}\bar{v}^{\otimes 2}\right) \quad \text{(A6)}$$

and swapping the role of n and k as done for R we have

$$\frac{1}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2}M_{\theta_\infty}\left(\frac{V}{\sqrt{2}}\right)M_{\theta_\infty}(\sqrt{2}\bar{v})\left(-\frac{1}{\theta_\infty}I+\frac{2}{\theta_\infty^2}\bar{v}^{\otimes 2}\right)\times(|V|^2k^{\otimes 2}-2|V|V\otimes k+V^{\otimes 2})\psi(v)\,dk\,d\bar{v}\,dV.$$

Now, performing the integration with respect to $k\in S^2$, recalling $\int_{S^2}k^{\otimes 2}dk=(4\pi/3)I$, $\int_{S^2}V\otimes k\,dk=0$ and $\int_{S^2}dk=4\pi$ and reducing the matrices, Eq. (A6) becomes

$$\tfrac{1}{2}\int_{\mathbb{R}^6}M_{\theta_\infty}\left(\frac{V}{\sqrt{2}}\right)M_{\theta_\infty}(\sqrt{2}\bar{v})\left[-\frac{|V|^2}{\theta_\infty}+\frac{2}{3\theta_\infty^2}|\bar{v}|^2|V|^2-\frac{|V|^2}{\theta_\infty}+\frac{2}{\theta_\infty^2}\bar{v}^{\otimes 2}\cdot V^{\otimes 2}\right]\psi(v)\,d\bar{v}\,dV,$$

thus, rewriting thus integral in the original coordinates

$$\int_{\mathbb{R}^6} M_{\theta_\infty}(v) M_{\theta_\infty}(w) \times \Bigg[-\frac{|v-w|^2}{2\theta_\infty} + \frac{|v-w|^2|v+w|^2}{12\theta_\infty^2} - \frac{|v-w|^2}{2\theta_\infty} + \frac{1}{4\theta_\infty^2}(v+w)^{\otimes 2}(v-w)^{\otimes 2} \Bigg] \psi(v) dv\, dw$$

$$= \int_{\mathbb{R}^6} M_{\theta_\infty}(v) M_{\theta_\infty}(w) \times \Bigg[-\frac{|v|^2+|w|^2-2vw}{\theta_\infty} + \frac{(|v|^2+|w|^2)^2-4(vw)^2}{12\theta_\infty^2} + \frac{(|v|^2-|w|^2)^2}{4\theta_\infty^2} \Bigg] \psi(v) dv\, dw. \tag{A7}$$

Now, we have to integrate with respect to w: we make use of the integral

$$\int_{\mathbb{R}^3} M_{\theta_\infty}(w)(vw)^2 dw = \int_0^\infty M_{\theta_\infty}(w)|v|^2|w|^4 d|w| \int_{S^2} (n_1 n_2)^2 dn_2 = \frac{4\pi}{3} \int_{\mathbb{R}^3} M_{\theta_\infty}(w)|v|^2|w|^4 d|w| = \tfrac{1}{3} \int_{\mathbb{R}_3} M_{\theta_\infty}(w)|v|^2|w|^2 dw,$$

and of the fact that the integral in w of the term $v \cdot w$ vanishes. Therefore, the term $S(M_{\theta_\infty}, M_{\theta_\infty})$ becomes

$$M_{\theta_\infty}(v) \int_{\mathbb{R}^3} M_{\theta_\infty}(w) \times \left(-\frac{|v|^2+|w|^2}{\theta_\infty} + \frac{|v|^4+|w|^4}{3\theta_\infty^2} - \frac{4}{9\theta_\infty^2}|v|^2|w|^2 \right) dw. \tag{A8}$$

Taking moments of $M_{\theta_\infty}(w)$ one finally proves Eq. (3.13). In particular

$$\int R(M_{\theta_\infty}, M_{\theta_\infty}) dv = 0 = \int S(M_{\theta_\infty}, M_{\theta_\infty}) dv.$$

Thus, we have checked our computations in the sense that the first two order terms in the expansion of Q_ε preserve mass.

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