Dimensionality effects in restricted bosonic and fermionic systems

Dragoş-Victor Anghel
Department of Physics, University of Jyväskylä, P.O. Box 35, 40351 Jyväskylä, Finland
and NIPNE –''Horia Hulubei,'' P. O. Box MG-6, R.O.-76900 Bucureşti - Măgurele, Romania
(Received 1 February 2000)

The phenomenon of Bose-like condensation, the continuous change of the dimensionality of the particle distribution as a consequence of freezing out of one or more degrees of freedom in the limit of low particle density, is investigated theoretically in the case of closed systems of massive bosons and fermions, described by general single-particle Hamiltonians. This phenomenon is similar for both types of particles and, for some energy spectra, exhibits features specific to multiple-step Bose-Einstein condensation, for instance, the appearance of maxima in the specific heat. In the case of fermions, as the particle density increases, another phenomenon is also observed. For certain types of single particle Hamiltonians, the specific heat is approaching asymptotically a divergent behavior at zero temperature, as the Fermi energy ϵ_F is converging towards any value from an infinite discrete set of energies $\{\epsilon_i\}_{i\geqslant 1}$. If $\epsilon_F=\epsilon_i$, for any i, the specific heat is divergent at $T=0$ just in infinite systems, whereas for any finite system the specific heat approaches zero at low enough temperatures. The results are particularized for particles trapped inside parallelepipedic boxes and harmonic potentials.

PACS number(s): 05.30.Ch, 64.90.+b, 05.30.Fk, 05.30.Jp

I. INTRODUCTION

Because of the advances in nanotechnology it has become possible to use very small structures in a broad range of applications. The importance of these applications and the fact that the physical properties of such structures could be very different from those of bulk materials, make the theoretical and experimental investigations very useful in this area.

The experimental findings in Ref. [1] motivated us to calculate the thermal properties of ultrathin dielectric membranes or wires by splitting the phonon spectra into discrete and continuous parts [2,3]. This framework implies crossovers between different phonon gas distributions, reflected, for example, in the exponent of the temperature dependence of the specific heat or heat conductivity. For example in a membrane, as the temperature drops, the population of the phonon modes parallel to the surfaces [which we shall call the two-dimensional (2D) ground state (g.s.)] becomes dominant, and the three-dimensional phonon gas distribution changes into a two-dimensional one [2]. The macroscopic population of the 2D g.s. (or one-dimensional ground state in the case of a wire [3]) and the qualitative differences between phonon gas distributions with various dimensions enabled us to make the analogy with the multiple-step Bose-Einstein condensation (BEC) [4,5] and to call this phenomenon Bose-like condensation (BLC). Yet, the number of phonons changes with temperature and features like maxima of the specific heat (c_V) observed in the case of BEC cannot be seen in the case of a phonon gas undergoing BLC.

The first purpose of this paper is to extend the previous work reported in Refs. [2,3] and to describe BLC in systems of massive bosons and fermions. This will be done in Sec. II. The mathematical technique used here is a straightforward extension of the one introduced by Pathria and Greenspoon in Ref. [6]. Nevertheless, the analytical approximations used there are not appropriate for our case. Therefore, after obtaining general expressions, we make numerical calculations to give concrete examples of BLC and to observe the behavior of the specific heat during the transition. The phenomenon occurs at low particle densities (this will be made more clear in Sec. II) and is specific to both bosons and fermions. At low temperatures, the number of massive particles in a closed system can be considered to be constant. The conservation of the particle number will allow us to observe resemblances with the BEC, such as, in some cases, maxima of the specific heat ($c_{V\max}$) at the condensation temperature. Anyway, the signature of BLC, as seen in the temperature dependence of c_V, is more complex and depends on the energy spectrum.

A consequence of the third law of thermodynamics is that the specific heat of any thermodynamical system should vanish at zero temperature. Li *et al.* showed in Ref. [7] that the heat capacity of a Fermi gas, confined in an external potential of quite general form, and for any space dimension, has the asymptotic behavior $c_V\propto T$ at low temperatures (where T is the temperature of the system). This is for the case of a continuous energy spectrum. In contrast to this we show in Sec. III that the specific heat of a Fermi gas with a single-particle Hamiltonian of the form $H=H_c+H_d$, with H_c having a (quasi)continuous spectrum $\epsilon_c\in[0,\infty)$ and H_d having the discrete eigenvalues ϵ_i, $i=0,1,\ldots$, may approach, depending on the density of the energy levels of H_c, divergent behavior at temperature $T=0$ K as the Fermi energy ϵ_F converges to ϵ_i, for any $i\geqslant 1$. In such a case, if the spectrum of H_c is continuous, then the specific heat diverges at $T=0$ and $\epsilon_F=\epsilon_i$, for any $i\geqslant 1$. However, in any finite system the energy spectrum is discrete, so the specific heat approaches zero if we go at low enough temperatures and the third law of thermodynamics is not violated.

Ultrathin (semi)conducting membranes and wires, nowadays widely used in mesoscopic applications, atoms in very anisotropic harmonic traps, wires or constrictions defined in 2D electronic gasses are just a few examples of systems where the phenomena presented here could be observed.

Also, they could provide an understanding of the behavior of very thin liquid He films.

II. BOSE-LIKE CONDENSATION

The BEC in cuboidal boxes with small dimensions drew a lot of attention many years ago, in the beginning in connection with very thin films of liquid He [5,6,8–11]. It is now well known that, as the dimensions of the box are reduced, at constant density, the cusplike maximum of c_V is rounded off and the condensation temperature (in this situation taken as the temperature corresponding to the maximum) increases with respect of the bulk value. The maximum of the specific heat is usually smaller in restricted geometries than in the bulk, for all the boundary conditions imposed on the walls of the container, with the exception (the only one known by the present author) of Dirichlet boundary conditions [10,11,6]. The theoretical investigation of BEC in harmonic traps (see Ref. [12], and references therein) was motivated recently by its realization in ultracold trapped atomic gases [13]. In this situation, the specific heat of an infinite system presents a discontinuity at the condensation temperature. In finite systems, the discontinuity is again rounded off, as shown by analytical and numerical calculations, for example in Refs. [14–17]. As the number of particles is decreased the condensation temperature decreases [14–17]. Furthermore, the multiple-step BEC was introduced in Refs. [4,5] for the cases of very anisotropic boxes or confining potentials. In this case a finite Bose gas is condensing gradually to the ground state, exhibiting in between 2D and/or 1D macroscopic populations.

In a very anisotropic Bose system, as particle density decreases, the multiple-step BEC (MSBEC) temperature becomes lower than the temperature at which some of the degrees of freedom of our system freeze out. During both processes (MSBEC and freezing) the 3D particle distribution transforms gradually into a lower dimensional distribution. On the other hand, the two processes change into each other at the variation of the particle density or of the dimensions of the system. Moreover, the reduction of the dimensionality of the particle distribution due to the freezing out of some of the degrees of freedom can happen also for fermions at low densities. The analogies and differences between the two processes mentioned above, justify (arguably, of course) the use of the simpler expression of Bose-like condensation for the freezing out of some degrees of freedom, in the limit of low particle density.

The temperature at which BLC occurs (as in the case of BEC in finite systems, this temperature cannot be uniquely defined) depends on the energy spectrum and reaches a finite positive value in the low particle density limit. This type of condensation is identical for both bosons and fermions (see Fig. 1). To show this, let us consider a closed system of massive bosons and fermions described by a single-particle Hamiltonian of the form $H=H_c+H_d$, with the eigenvalues $\epsilon=\epsilon_c+\epsilon_i$, as explained in the Introduction. The mean occupation numbers of single particle energy levels ϵ, are $\langle n_\epsilon^{(\pm)}\rangle=[\exp(\alpha+\epsilon/k_BT)\pm 1]^{-1}$, where $(-)$ is the superscript for bosons, and $(+)$ for fermions, $\alpha=-\mu/k_BT$, and μ is the chemical potential. We introduce the functions

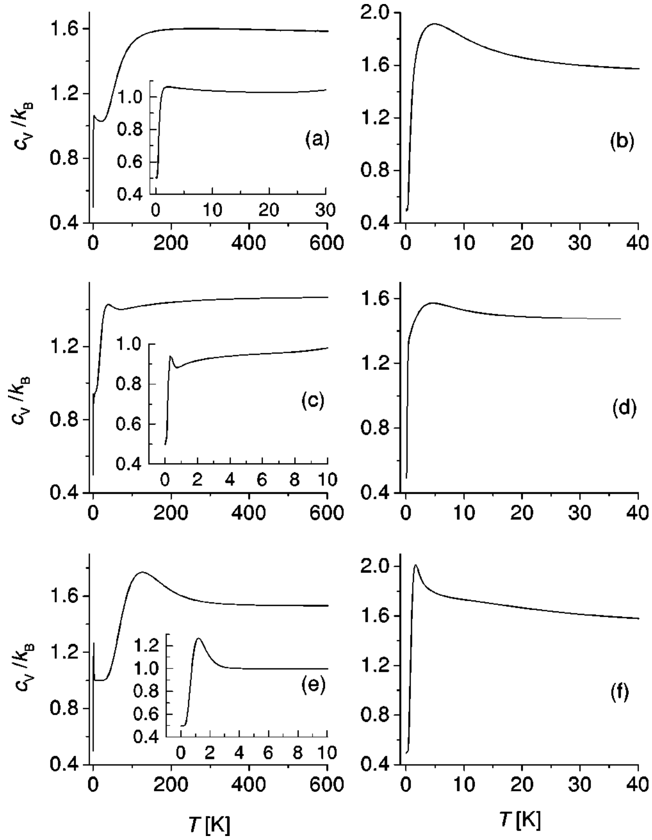

FIG. 1. Specific heat in units of k_B vs temperature for ideal Bose and Fermi gases trapped inside cuboidal boxes with Dirichlet [(a) and (b)], Neumann [(c) and (d)], and periodic [(e) and (f)] boundary conditions on the walls. (a), (c), and (e) correspond to $l_1\to\infty$, $l_2=10^{-9}$ m, and $l_3=10^{-10}$ m, while (b), (d), and (f) correspond to $l_1\to\infty$ and $l_2=l_3=10^{-9}$ m. The particle density is 10^{25} m^{-3} in each case. In each situation the results for bosons (solid line) and fermions (dashed line) are both plotted, but they cannot be distinguished. In the insets of (a), (c), and (e) we show low temperature details of the larger graphs (the axes are the same).

$$Z_n^{(\pm)}=\sum_\epsilon \left(\frac{\epsilon}{k_BT}\right)^n \langle n_\epsilon^{(\pm)}\rangle, \tag{1}$$

$$G_n^{(\pm)}=\sum_\epsilon \left(\frac{\epsilon}{k_BT}\right)^n [\langle n_\epsilon^{(\pm)}\rangle \mp \langle n_\epsilon^{(\pm)}\rangle^2]=-\frac{\partial Z_n^{(\pm)}}{\partial\alpha}, \tag{2}$$

in a similar way as Pathria and Greenspoon did for bosons in Ref. [6]. Then, for example, the number of particles, the internal energy, and the heat capacity can be written as $N^{(\pm)}=Z_0^{(\pm)}$, $U^{(\pm)}=k_BTZ_1^{(\pm)}$, and $C_V^{(\pm)}=k_B(G_2^{(\pm)}-G_1^{(\pm)2}/G_0^{(\pm)})$, respectively (in all this paper we shall consider spinless particles). To avoid divergent terms that occur in the functions introduced when T approaches zero, in the case when the ground-state energy ϵ_0 is positive, we redefine α as $\alpha-\epsilon_0/kT$ and ϵ as $\epsilon-\epsilon_0$. Making these replacements we do not change the thermodynamics of the canonical ensemble [18,19]. If the density of the energy levels of the (quasi)continuous spectrum, as a function of energy, is $\sigma(\epsilon_c)$, then we can write

$$Z_0^{(\pm)}=\sum_{i=0}^{\infty}\int_0^{\infty}\frac{\sigma(\epsilon)}{\exp(\alpha+\beta\epsilon_i+\beta\epsilon)\pm 1}d\epsilon, \tag{3}$$

where $\beta=1/k_BT$. If in the temperature range of interest for the study of BLC ($\epsilon_1/k_BT\approx 1$) $\alpha\gg 1$, then we can write Z_0 in terms of two functions, corresponding to the continuous and to the discrete spectra, respectively,

$$Z_0^{(\pm)}=e^{-\alpha}Z_c^{(\pm)}Z_d^{(\pm)},$$

where $Z_c^{(\pm)}=\int_0^\infty\sigma(\epsilon)e^{-\beta\epsilon}d\epsilon$ and $Z_d^{(\pm)}=\Sigma_{i=0}^\infty e^{-\beta\epsilon_i}$. In this approximation it is no difference between bosons and fermions and, according to Eq. (2), $G_n^{(\pm)}=Z_n^{(\pm)}$. Using the relation

$$\left(\frac{\partial^n Z_n^{(\pm)}}{\partial\alpha^n}\right)_\beta=\beta^n\left(\frac{\partial^n Z_0^{(\pm)}}{\partial\beta^n}\right)_\alpha, \tag{4}$$

that holds for bosons [6], as well as for fermions, we can write the specific heat $c_V^{(\pm)}=C_V^{(\pm)}/N^{(\pm)}$, in units of k_B, as

$$\frac{c_V}{k_B}=\beta^2\frac{\partial^2}{\partial\beta^2}\ln Z_0(\beta,\alpha)=\beta^2\frac{\partial^2}{\partial\beta^2}\ln Z_c(\beta)+\beta^2\frac{\partial^2}{\partial\beta^2}\ln Z_d(\beta) \tag{5}$$

[where we have droped the superscript $(\pm)$ as insignificant in this case]. Acording to Eq. (5), the specific heat is nothing else then the sum of the heat capacities of two systems, each of them containing a single particle under canonical conditions, and it is described by the Hamiltonian H_c and H_d, respectively.

Explicit expressions for $Z_n^{(\pm)}$ and $G_n^{(\pm)}$ can be obtained if we assume that the density of states of the continuous spectrum has the form $\sigma(\epsilon_c)=C\epsilon_c^s(C$ and s are constants, such that $C>0$ and $s>-1)$, as it happens in most of the cases [21]. Using the Eqs. (1), (2), and (4), we can write

$$Z_n^{(\pm)}=\frac{C}{\beta^{s+1}}\sum_{j=0}^{n}C_n^j\Gamma(s+1+n-j)\times\sum_{i=0}^{\infty}n_i(\beta\epsilon_i)^j g_{s+1+n-j}^{(\pm)}(\alpha+\beta\epsilon_i)$$

and

$$G_n^{(\pm)}=\frac{C}{\beta^{s+1}}\sum_{j=0}^{n}C_n^j\Gamma(s+1+n-j)\times\sum_{i=0}^{\infty}n_i(\beta\epsilon_i)^j g_{s+n-j}^{(\pm)}(\alpha+\beta\epsilon_i), \tag{6}$$

where n_i is the degeneracy of the level with energy ϵ_i and $C_k^j=n!/j!(n-j)!$. The functions $g_l^{(\pm)}(\alpha)$ are the lth order polylogarithmic functions (see, for example, Ref. [20], and references therein for more details) of argument $e^{-\alpha}$ (bosons) or $-e^{-\alpha}$ (fermions). In the case of ideal particles inside a rectangular box of dimensions $l_x\gg l_y,l_z$, we can write $\epsilon_c=\hbar^2k_x^2/2m$ and $\epsilon_{\{i,j\}}=\hbar^2(k_{yi}^2+k_{zj}^2)/2m$, where k_x, k_y, and k_z are the wave vectors along the x, y, and z axes, respectively. The mass of one particle is m and the discrete values of k_{yi} and k_{zj} depend on the boundary conditions. In this case $s=-1/2$. If $l_x,l_y\gg l_z$, then $s=0$ and $\epsilon_c=\hbar^2(k_x^2+k_y^2)/2m$, while $\epsilon_i=\hbar^2k_{zi}^2/2m$. Let us now concentrate on the BLC of particles inside such rectangular boxes. In Fig. 1 we can see the results of the exact numerical calculation of c_V [using the formulas from Eq. (6) for $Z_n^{(\pm)}$ and $G_n^{(\pm)}$] as a function of temperature, for two different kinds of geometries and for Dirichlet [Figs. 1(a), 1(b)], Neumann [Figs. 1(c), 1(d)], and periodic [Figs. 1(e), 1(f)] boundary conditions. In geometry I [see Figs. 1(a), 1(c), and 1(e)] $l_2=10^{-9}$ m, $l_3=10^{-10}$ m, and $l_1\gg l_2$, while in geometry II [see Figs. 1(b), 1(d), and 1(f)] $l_2=l_3=10^{-9}$ m and $l_1\gg l_2$. To make concrete calculations we choose $\lambda^2\equiv 2\pi\hbar^2/mk_BT=10^{-18}T^{-1}$ which corresponds to a mass of about 3 atomic mass units for all the particles in the systems investigated. In the figure, the results for bosons and fermions are indistinguishable, as expected for low particle densities. The choice of the dimensions in geometry I allows us to observe the BLC from 3D to 2D and, at lower temperature, from 2D to 1D. We observe the formation of a maximum (at, let us say, temperature $T_{\max}$) in each of these two cases and for all boundary conditions. The height of this maximum and, in general, the shape of the function $c_V(T)$ around $T_{\max}$ depend on the spectrum of H_d. For example, for Neumann boundary conditions, we observe the formation of a minimum at a temperature a bit higher than $T_{\max}$. In geometry II we observe the BLC from 3D to 1D. In this case, the maxima are more pronounced and the minima observed in geometry I for Neumann boundary conditions disappear.

In Fig. 2 we plot $T_{\max}/T_c$ and $c_{V\max}/k_B$ vs l_3/l, for Dirichlet, Neumann, and periodic boundary conditions, in the cases when $l_1,l_2\gg l_3$ [Figs. 2(a), 2(c)] and $l_1\gg l_2=l_3$ [Figs.

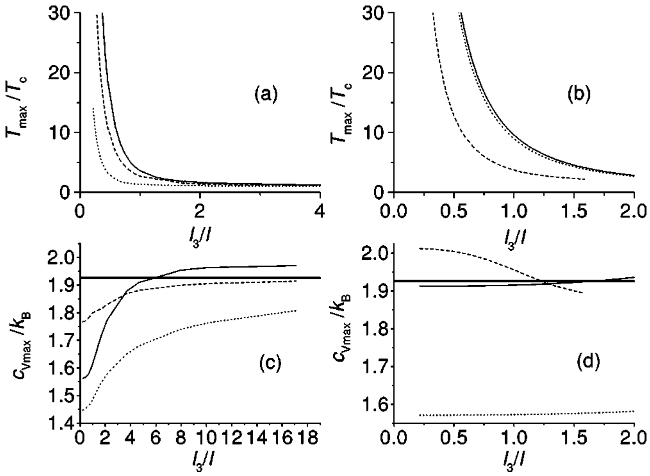

FIG. 2. The temperature of $c_{V\max}$ of *Bose gases*, scaled by the bulk critical temperature T_c (see the text), as a function of l_3/l (where $l=\rho^{-1/3}$, and ρ is the density), is shown for (a) the membrane geometry ($l_1,l_2\gg l_3$) and (b) the wire geometry ($l_1\gg l_2=l_3$). The value of $c_{V\max}$, in units of k_B, vs l_3/l, is plotted for (c) membrane geometry and (d) wire geometry [the same as in (a) and (b)]. Solid, dashed and dotted lines are used for Dirichlet, periodic and Neumann boundary conditions, respectively. The thick horizontal lines in (c) and (d) correspond to the 3D bulk value of c_V at the BEC temperature. In the numerical calculations we varied ρ, keeping $l_3=10^{-9}$ m.

2(b), 2(d)]. T_c is the bulk BEC temperature, given by the equation $\rho(2\pi\hbar^2/mk_BT_c)^{3/2}=\zeta(3/2)$, ρ is the particle density, ζ is the Riemann zeta function, and $l=\rho^{-1/3}$ is the mean interparticle distance. Since $T_{\max}$ converges to a finite value and $T_c\to 0$ when $\rho\to 0$, the ratio $T_{\max}/T_c$ diverges in this limit. As ρ increases, T_c increases and BLC is gradually replaced by BEC. As a consequence, $\lim_{l_3/l\to\infty}T_{\max}/T_c=1$. Figure 2(c) can make the connection between these numerical calculations and the analytical approximations reported in Ref. [6]. We observe that $c_{V\max}$ is higher in Fig. 2(d) then in Fig. 2(c), at the same value of l_3/l, for any boundary conditions. Nevertheless, the maximum value of $c_{V\max}$, which is about $2.02k_B$, is obtained for periodic boundary conditions in the limit $\rho\to 0$, while at higher densities this decreases under its bulk value, as expected from previous calculations [6].

The study of the BLC of ideal particles in harmonic traps is easier since in this situation Z_d has a very simple analytical expression. If we denote the characteristic frequencies of the harmonic trap by ω_x, ω_y, and ω_z, with $\omega_x\ll\omega_y,\omega_z$, then $Z_c=k_BT/\hbar\omega_x$, and $Z_d=\{[(1-\exp(\hbar\omega_x/k_BT)][(1-\exp\hbar\omega_y/k_BT)]\}^{-1}$. In this case $dc_V/dT\geqslant 0$ for any temperature, so BLC is not accompanied by the formation of a maximum. The dimensionality of the system (say, nD) is reflected in the value of c_V, which is nk_B, and the fraction of the particle number in the 1D g.s., has the expression $N_{1\mathrm{D}}/N=(1-e^{-(\hbar\omega_y/k_BT)})(1-e^{-(\hbar\omega_z/k_BT)})$.

III. DIVERGENT BEHAVIOR OF C_V IN FERMIONIC SYSTEMS

In this section we shall concentrate on Fermi systems close to $T=0$ K. We consider again that the Hamiltonian of the system can be approximated by single-particle operators of the form $H=H_c+H_d$, as explained in the Introduction. At the increase of the particle density or of the density of the eigenvalues ϵ_i of the operator H_d, we would expect to approach the limit in which both H_c and H_d have continuous spectra (3D bulk limit). In such a limit we should recover the results from Ref. [7], namely, $c_V\propto T$ at low temperatures. As it will be shown next, this is not the case in general. The continuous limit is not attained in a smooth way. Instead, in some situations, the specific heat would become divergent at zero temperature, for certain values of the Fermi energy.

At temperatures close to 0 K the chemical potential of a Fermi system approaches the Fermi energy ϵ_F. For $\alpha\ll-1$, the polylogarithmic functions of negative argument can be written in the form [7]

$$g_n^{(+)}(\alpha)=\frac{|\alpha|^n}{\Gamma(n+1)}\left[1+O\left(\frac{1}{\alpha^2}\right)\right]. \tag{7}$$

The cases for $n=0$ and 1 are included in Eq. (7), but can be refined further to write $g_n^{(+)}(\alpha)=|\alpha|^n[1+O(e^{\alpha})]$. In the other extreme case, when $\alpha\gg 1$, all the polylogarithmic functions have a behavior of the form $g_n^{(\pm)}(\alpha)=e^{-\alpha}[1+O(e^{-\alpha})]$. Using these asymptotic expressions we can return to the study of the specific heat close to zero temperature, for a density of energy levels of H_c similar to the one introduced in the previous section, namely $\sigma(\epsilon_c)=C\epsilon_c^s$. The ground state of H_d is nondegenerate since we discuss a finite system. We shall use the notation $\alpha_0\equiv-\beta\epsilon_F$.

Since we know that $\mu\to\epsilon_F$ as $T\to 0$, let us now calculate $\lim_{T\to 0}(|\alpha_0|-|\alpha|)$ when $\epsilon_F=\epsilon_i$, $i>0$ (in all the other cases will turn out that the limit is zero). Using $N=Z_0^{(+)}(\alpha)$, Eqs. (6), and the definition of the Fermi energy, we write two different expressions for the total number of particles in the system:

$$\begin{aligned}N=&\frac{C}{(s+1)\beta^{s+1}}\{|\alpha|^{s+1}+\cdots+(|\alpha|-\beta\epsilon_{i-1})^{s+1}\}\\&\times\left[1+O\left(\frac{1}{\alpha^2}\right)\right]+n_iC\frac{\Gamma(s+1)}{\beta^{s+1}}g_{s+1}^{(+)}(\alpha+\beta\epsilon_i)\\&+C\frac{\Gamma(s+1)}{\beta^{s+1}}\sum_{j=i+1}^{\infty}n_je^{|\alpha|-\beta\epsilon_j}\end{aligned} \tag{8}$$

$$=\frac{C}{(s+1)\beta^{s+1}}\{|\alpha_0|^{s+1}+\cdots+(|\alpha_0|-\beta\epsilon_{i-1})^{s+1}\}. \tag{9}$$

If we denote $\xi\equiv\alpha+\beta\epsilon_i$, then from Eqs. (8) and (9), neglecting the exponentials and assuming that $\lim_{T\to 0}(\xi/|\alpha_0|)=\lim_{T\to 0}(\xi/|\alpha|)=0$, we obtain, in the case $\alpha_0,\alpha\ll-1$, an equation for ξ:

$$n_i\frac{g_{s+1}^{(+)}(\xi)}{\xi}=\frac{|\alpha_0|^s}{\Gamma(s+1)}\chi_s, \tag{10}$$

where $\chi_s\equiv 1+\cdots+n_{i-1}(1-x_{i-1})^s$ and $x_j\equiv\epsilon_j/\epsilon_F$. We now notice that we have three distinct situations: (a) $s>0$, in which case $\xi\to 0$ as $T\to 0$, (b) $s=0$, and ξ converges to a finite positive value, and (c) $s\in(-1,0)$, when $\xi\to\infty$ as $T\to 0$.

Let us analyze now the asymptotic behavior of ξ in the case (c). For $\xi\gg 1$ we can write

$$\frac{e^{-\xi}}{\xi}=\frac{|\alpha_0|^s}{n_i\Gamma(s+1)}\chi_s, \tag{11}$$

so $\xi=(-s)\ln|\alpha_0|-\ln\xi-\ln[\chi_s/n_i\Gamma(s+1)]$. Therefore, at $\alpha_0\ll-1$, $\xi\approx|s|\ln|\alpha_0|-\ln[\ln|\alpha_0|]+\cdots$. We can see now that the assumption $\lim_{T\to 0}(\xi/|\alpha_0|)=\lim_{T\to 0}(\xi/|\alpha|)=0$ was justified. Also, following the same kind of reasoning, one can prove that when $\epsilon_F\neq\epsilon_i$, for any i, then $\lim_{T\to 0}(|\alpha_0|-|\alpha|)=0$ for any s.

Using the Eqs. (10) and (11) we can calculate the specific heat close to 0 K. For that we have to evaluate the functions $G_2^{(+)}$, $G_1^{(+)}$, $G_0^{(+)}$, and $Z_0^{(+)}$. We analyze again the case when $\epsilon_F=\epsilon_i$, $i>0$. After some algebra and dropping out the factors that become exponentially small in the limit $T\to 0$, we can write

$$G_2^{(+)}=\frac{C|\alpha|^{s+2}}{\beta^{s+1}}\left\{\chi_s\left[1+O\left(\frac{1}{\alpha^2}\right)\right]+n_i\left[\Gamma(s+3)\frac{g_{s+2}^{(+)}(\xi)}{|\alpha|^{s+2}}\right.\right.$$
$$+2\Gamma(s+2)y_i\frac{g_{s+1}^{(+)}(\xi)}{|\alpha|^{s+1}}$$
$$\left.\left.+\Gamma(s+1)y_i^2\frac{g_s^{(+)}(\xi)}{|\alpha|^s}\right]-\frac{s\xi}{|\alpha|}\Upsilon_s\right\}, \tag{12}$$

$$G_1^{(+)}=\frac{C|\alpha|^{s+1}}{\beta^{s+1}}\left\{\chi_s\left[1+O\left(\frac{1}{\alpha^2}\right)\right]+n_i\left[\Gamma(s+2)\frac{g_{s+1}^{(+)}(\xi)}{|\alpha|^{s+1}}\right.\right.$$
$$\left.\left.+\Gamma(s+1)y_i\frac{g_s^{(+)}(\xi)}{|\alpha|^s}\right]-\frac{s\xi}{|\alpha|}\Upsilon_s\right\}, \tag{13}$$

$$G_0^{(+)}=\frac{C|\alpha|^{s}}{\beta^{s+1}}\left\{\chi_s\left[1+O\left(\frac{1}{\alpha^2}\right)\right]\right.$$
$$\left.+n_i\Gamma(s+1)\frac{g_s^{(+)}(\xi)}{|\alpha|^s}-\frac{s\xi}{|\alpha|}\Upsilon_s\right\}, \tag{14}$$

$$Z_0^{(+)}=\frac{C|\alpha|^{s+1}}{(n+1)\beta^{s+1}}\left\{\chi_{s+1}\left[1+O\left(\frac{1}{\alpha^2}\right)\right]\right.$$
$$\left.+n_i\Gamma(s+2)\frac{g_{s+1}^{(+)}(\xi)}{|\alpha|^{s+1}}-\frac{(s+1)}{|\alpha|}\Upsilon_{s+1}\right\}, \tag{15}$$

where $y_j=\beta\epsilon_j/|\alpha|$ and $\Upsilon_s\equiv\Sigma_{k=1}^{i-1}n_k(1-x)^{s-1}x_k$. To see the asymptotic behavior, we calculate c_V separately for the cases (a), (b), and (c). Using Eqs. (10),(12)–(15) and working consistently in the orders of $|\alpha|$, we obtain the following asymptotic results:

Case (a)

$$\frac{c_V}{k_B}=\frac{(s+1)|\alpha|}{\chi_{s+1}+O(|\alpha|^{-(s+1)})}\left\{\frac{n_i^2\Gamma^2(s+1)}{\chi_s}\frac{g_s^{(+)2}(\xi)}{|\alpha|^{2s}}\right.$$
$$\left.+\frac{n_i^3\Gamma^3(s+1)}{\chi_s^2}\frac{g_s^{(+)3}(\xi)}{|\alpha|^{3s}}+O\left(\frac{1}{|\alpha|^m}\right)\right\}, \tag{16}$$

where $m=\min\{s+1,4s,2\}$.

Case (b)

$$\frac{c_V}{k_B}=\frac{n_i}{|\alpha|\chi_1 O(|\alpha|^{-1})}$$
$$\times\left\{\frac{\chi_0 g_0^{(+)}(\xi)}{\chi_0+n_i g_0^{(+)}(\xi)}\xi^2+\frac{2\chi_0 g_1^{(+)}(\xi)}{\chi_0+n_i g_0^{(+)}(\xi)}\xi+2g_2^{(+)}(\xi)\right.$$
$$\left.-\frac{n_i g_1^{(+)2}(\xi)}{\chi_0+n_i g_0^{(+)}(\xi)}\frac{\pi^2}{3}\chi_0+O(e^{\alpha})\right\}. \tag{17}$$

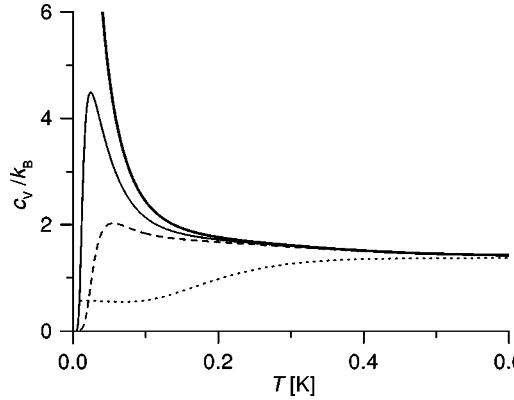

FIG. 3. The specific heat (in units of k_B) of a Fermi gas trapped inside a cuboidal box ($l_1\to\infty$, $l_2=l_3=10^{-9}$ m) with Neumann boundary conditions on the walls. The four curves correspond to the following densities: 1.5×10^{26} m^{-3} (dotted line), 9.2×10^{26} m^{-3} (dashed line), 9.6×10^{26} m^{-3} (solid line), and 1×10^{27} m^{-3} (thick solid line). The last case corresponds to $\epsilon_F=\epsilon_1$.

Case (c)

$$\frac{c_V}{k_B}=\frac{s+1}{\chi_{s+1}+O|\alpha|^{-(s+1)})}\frac{|\alpha|}{\xi}\left\{\chi_s+\frac{\chi_s}{\xi}+O\left(\frac{1}{|\alpha|}\right)\right\}. \tag{18}$$

So, for $\epsilon_F=\epsilon_i$, $i>0$, from Eqs. (16)–(18) we distinguish the following situations.

(a1) $s>1/2$, then $c_V/k_B\propto(\epsilon_F/k_BT)^{1-2s}$, so $c_V\to0$ as $T\to0$ (note that if $s>1$ some of the orders of α interchange, but the function c_V converges fast to zero as T approaches 0 K).

(a2) $s=1/2$, then $\lim_{T\to0}(c_V/k_B)=(3-2\sqrt{2})\times(3\pi/8)\zeta^2(1/2)n_i^2/\chi_{3/2}\chi_{1/2}$.

(a3) $s\in(0,1/2)$, then $c_V/k_B\propto(\epsilon_F/k_BT)^{1-2s}$, so $c_V\to\infty$ as $T\to0$.

(b) $s=0$, then $c_V/k_B\propto k_BT/\epsilon_F$, so $c_V\to0$ as $T\to0$.

(c) $s\in(-1,0)$, then $c_V/k_B\propto(\epsilon_F/k_BT)/\ln(\epsilon_F/k_BT)$, so $c_V\to\infty$ as $T\to0$.

Therefore, in the cases (a3) and (c), c_V presents a divergent behavior at $T=0$ K, while in case (a2) approaches a finite limit. These situations seem to be in contradiction with the third law of thermodynamics. To clarify this we mention that the divergency appears just if the spectrum of H_c is continuous. In any finite system this is not the case, so at low enough temperatures c_V decreases towards zero.

Without getting into details we state that when $\epsilon_F\neq\epsilon_i$, $\forall i\geqslant0$, similar calculations lead us to the results $\lim_{T\to0}(|\alpha_0|-|\alpha|)=0$ and $\lim_{T\to0}c_V=0$ for any s. Moreover, in the low-temperature limit we reobtain the known result [7] $c_V\propto T$. On the other hand, the continuity of μ as a function of ϵ_F implies the continuity of α and c_V as functions of ϵ_F, for any $T>0$ K. In other words, the divergent behavior in the cases (a3) and (c) can be approached asymptotically for any $T>0$ K, as $\epsilon_F\to\epsilon_i$ (for any i), by the functions $c_V(T)$. This leads to the formation of a maximum at finite temperature, with the properties $c_{V\max}\to\infty$ and $T_{\max}\to0$, as $\epsilon_F\to\epsilon_i$, for any $i\geqslant1$.

Let us make now the connections with familiar systems, namely, with the ones discussed in Sec. II. In the case of a cuboidal box with dimensions $l_x\gg l_y,l_z$, $s=-1/2$, so we are in the case (c). In Fig. 3 we plot the exact numerical calculation of such a fermionic system, with dimensions $l_1 \to\infty$, $l_2 = l_3 = 10^{-9}$ m. The mass of the particles is chosen as in Sec. II, such that $\lambda^2 = 10^{-18} T^{-1}$. We observe the formation of the maximum as the Fermi energy approaches the first excited energy level of H_d, and the divergent behavior at $\epsilon_F = \epsilon_1$. If the fermions are inside a cuboidal box with dimensions $l_x, l_y \gg l_z$ or a harmonic potential with the characteristic frequencies $\omega_x \ll \omega_y, \omega_z$, then $s=0$ and we are in case (b), therefore we do not observe the formation of a similar maximum. This was checked by exact numerical calculations and was found to be correct.

IV. CONCLUSIONS

In Sec. II of this paper it is presented in general the phenomenon of Bose-like condensation in the case of massive bosons and fermions. This denomination was introduced in Ref. [2], where, according to my knowledge, a crossover between different dimensionalities of the phonon gas distribution in ultrathin dielectric membranes was reported for the first time. This phenomenon appears to be identical for both types of massive particles and resembles the multiple-step Bose-Einstein condensation [5,4]. Nevertheless, the two phenomena are different in nature. The results are exemplified for the familiar cases of ideal particles trapped inside cuboidal boxes and harmonic potentials.

The analysis made in Sec. III, lead us to the observation of interesting divergences of the specific heat of a Fermi system at zero temperature. The phenomenon is described in general, for a single-particle Hamiltonian of the form $H = H_c + H_d$, with H_c having a (quasi)continuous spectrum $\epsilon_c \in [0,\infty)$ with the energy levels density $\sigma(\epsilon_c) = C\epsilon_c^s$ $(s > -1)$ and H_d having the discrete eigenvalues ϵ_i, $i = 0,1,\ldots$. It was found that $c_v(T) \to \infty$ for any $s \in (-1,0) \cup (0,1/2)$ if $\epsilon_F = \epsilon_i$, for any $i \geq 1$. This divergent behavior is approached asymptotically for any $T > 0$, as $\epsilon_F \to \epsilon_i$, $\forall i \geq 1$, leading in this way to the formation of very high maxima (in the limit, infinitely high) of the fermionic specific heat close to zero temperature. This is an unexpected new phenomenon, since it seems to contradict the third law of thermodynamics. Anyway, this does not happen since in any finite system the energy spectrum is discrete and at low enough temperature the specific heat decreases towards zero. Nevertheless, this phenomenon might have interesting consequences on the entropy of the system in the vicinity of zero temperature. On the other hand it should be investigated if systems obeying fractional statistics [22] or interacting Bose systems (see, for example, Ref. [23] and references therein for similarities between these two types of systems) exhibit similar behavior.

ACKNOWLEDGMENTS

The author wants to thank Professor M. Manninen, Professor J. P. Pekola, and Professor E. B. Sonin for discussions. This work was supported by the Academy of Finland under the Finnish Center of Excellence Programme 2000-2005 (Project No. 44875, Nuclear and Condensed Matter Programme at JYFL).

[1] M. M. Leivo and J. P. Pekola, Appl. Phys. Lett. **72**, 1305 (1998).
[2] D. V. Anghel, J. P. Pekola, M. M. Leivo, J. K. Suoknuuti, and M. Manninen, Phys. Rev. Lett. **81**, 2958 (1998).
[3] D. V. Anghel and M. Manninen, Phys. Rev. B **59**, 9854 (1999).
[4] N. J. van Druten and W. Ketterle, Phys. Rev. Lett. **79**, 549 (1997).
[5] E. B. Sonin, Sov. Phys. JETP **29**, 520 (1969).
[6] R. K. Pathria, Phys. Rev. A **5**, 1451 (1972); S. Greenspoon and R. K. Pathria, *ibid.* **8**, 2657 (1973); **9**, 2105 (1974); **11**, 1080 (1975).
[7] M. Li, Z. Yan, J. Chen, L. Chen, and C. Chen, Phys. Rev. A **58**, 1445 (1998).
[8] D. A. Krueger, Phys. Rev. **172**, 211 (1968).
[9] M. F. M. Osborne, Phys. Rev. **76**, 396 (1949); J. M. Ziman, Philos. Mag. **44**, 548 (1953).
[10] D. F. Goble and L. E. H. Trainor, Can. J. Phys. **44**, 27 (1965); D. F. Goble and L. E. H. Trainor, Phys. Rev. **157**, 167 (1967).
[11] Z. R. Hasan and D. F. Goble, Phys. Rev. A **10**, 618 (1974).
[12] M. Holthaus, E. Kalinowski, and K. Kirsten, Ann. Phys. (N.Y.) **270**, 198 (1998).
[13] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollet, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *ibid.* **75**, 3969 (1995).
[14] S. Grossmann and M. Holthaus, Phys. Lett. A **208**, 188 (1995).
[15] K. Kirsten and D. J. Toms, Phys. Rev. A **54**, 4188 (1996).
[16] H. Haugerud, T. Haugset, and F. Ravndal, Phys. Lett. A **225**, 18 (1997).
[17] W. Ketterle and N. J. van Druten, Phys. Rev. A **54**, 656 (1996).
[18] The bosonic systems with nonzero ground state energy create one more problem to the grand-canonical ensemble, in addition to the well known one related to the fluctuations of the number of particles of a Bose-Einstein condensed system [19,12]. To achieve BEC in such a system $(\alpha \to -\epsilon_0/kT)$ one should request the ground state energy of the reservoir to be greater than or equal to the ground state energy of the system under investigation. This implies restrictions on the geometry or on the trapping potential of the reservoir.
[19] R. M. Ziff, G. E. Uhlenbeck, and M. Kac, Phys. Rep. **32**, 169 (1977).
[20] M. H. Lee, J. Math. Phys. **36**, 1217 (1995).
[21] V. Bagnato, D. E. Pritchard, and D. Kleppner, Phys. Rev. A **35**, 4354 (1987); V. Bagnato and D. Kleppner, *ibid.* **44**, 7439 (1991).
[22] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991); Yong-Shi Wu, *ibid.* **73**, 922 (1994).
[23] R. K. Bhaduri, S. M. Reimann, S. Viefers, A. Ghose Choudhury, and M. K. Srivastava, J. Phys. B (to be published); e-print cond-mat/9908210v2.