

Interface scaling in the contact process

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A correspondence between lattice models with absorbing states and models of pinned interfaces in random media can be established by defining local height variables $h(\mathbf{x}, t)$ as integrals of the activity at point \mathbf{x} up to time t . Within this context we study the interface representation of a prototypical model with absorbing states, the contact process, in dimensions 1–3. Simulations confirm the scaling relation $\beta_w = 1 - \theta$ between the interface-width growth exponent β_w and the exponent θ governing the decay of the order parameter. A scaling property of the height distribution, which serves as the basis for this relation, is also verified. The height-height correlation function shows clear signs of anomalous scaling, in accord with López' analysis [Phys. Rev. Lett. **83**, 4594 (1999)], but no evidence of multiscaling.

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I. INTRODUCTION

Scaling and criticality in nonequilibrium systems continue to be of great interest in statistical physics. Among the various classes of systems that have been subject to intensive study are models of growing interfaces [1–5], and absorbing-state phase transitions, typified by directed percolation (DP) and the contact process [6–9]. The latter class of models generally involve variables $\sigma_i(t) = 1(0)$ indicating the presence (absence) of activity at site i at time t . Introducing a set of height variables $h_i(t) = \int_0^t \sigma_i(t') dt'$ establishes a connection between the particle model and a corresponding interface model. The absorbing state of the particle model (global absence of activity) corresponds to pinning of the interface. At the critical point, the scaling properties of the interface can be related to those of the original particle model. It is therefore of considerable interest to investigate interface dynamics at an absorbing-state critical point.

Absorbing-state phase transitions have also been linked to self-organized criticality (SOC) in sandpile models [10–12], as have driven interface models [13–18]. [The latter connection is established by defining $h_i(t)$ as the number of topplings at site i up to time t .] It turns out that the interface of the Bak-Tang-Wiesenfeld model [19] is described by an Edwards-Wilkinson equation with columnar noise [17,18]. In view of the connections between absorbing-state phase transitions, SOC, and surface growth, it is worthwhile to study the dynamics of the interface representation of a simple model in the DP class, the contact process (CP). Precise results on the scaling properties of the CP interface should prove useful when trying to assign interface models (or height representations of other models, such as sandpiles) to universality classes.

In this paper we examine the dynamics of the contact process in dimensions 1–3, studying the interface width and the height-height correlation function, as well as the height probability distribution. In Sec. II we define the model and

then present a brief discussion of the associated continuum equation, and of a scaling theory. Numerical results are presented in Sec. III. Section IV contains a brief summary.

II. MODEL

The contact process is a simple particle system (lattice Markov process) exhibiting a phase transition to an absorbing (frozen) state at a critical value of the creation rate [20]. This model belongs to the universality class of directed percolation [6] and Reggeon field theory [21], and is pertinent to models of epidemics [8], catalysis [22], and damage spreading [23], among many others. In the CP each site of the hypercubic lattice \mathcal{Z}^d is either vacant or occupied by a particle. Particles are created at vacant sites at rate $\lambda n/2d$, where n is the number of occupied nearest neighbors, and are annihilated at unit rate, independent of the surrounding configuration. The order parameter is the particle density ρ ; the vacuum state, $\rho = 0$, is absorbing. As λ is increased beyond λ_c , there is a continuous phase transition from the vacuum to an active state; for $\lambda > \lambda_c$, $\rho \sim (\lambda - \lambda_c)^\beta$ in the stationary state. In one dimension, $\lambda_c \approx 3.297\,848$.

There are a number of ways (equivalent as regards scaling behavior) of implementing the CP in a simulation algorithm; this work follows the widely used practice of maintaining a list of all occupied sites. In this study the initial condition is always that of all sites occupied. Subsequent events involve selecting (at random) an occupied site \mathbf{x} from the N_p sites on the list, selecting a process [creation with probability $p = \lambda/(1 + \lambda)$, annihilation with probability $1 - p$], and, in the case of creation, selecting one of the $2d$ nearest neighbors \mathbf{y} of \mathbf{x} . (The creation attempt succeeds if \mathbf{y} is vacant.) The time increment Δt associated with an event is $1/N_p$, where N_p is the number of occupied sites immediately prior to the event. A trial ends when all the particles have vanished, or at the first event with time $\geq t_m$, a predetermined maximum time.

The important scaling laws pertinent to the critical contact process on a lattice of L^d starting with all sites occupied are (1) the mean survival time $\tau \propto L^{d/\nu_\perp}$, and (2) the average particle density decaying as a power law, $\rho(t) \propto t^{-\theta}$ for $1 < t < \tau$. (In practice, the power law is already found for t

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≈ 2 .) In one dimension, $\nu_{\parallel}/\nu_{\perp} \approx 1.5808$ and $\theta \approx 0.1595$ [24].

Occupied sites represent activity, which spreads from site to site. (The absence of any activity corresponds to the absorbing state.) In a growing surface or driven interface, activity corresponds to the motion of the interface. We therefore define the *height* $h_i(t)$ at site i as the amount of time (up to time t) that site i has been occupied. In our numerical studies we use a real-valued height (recall that time is not restricted to integer values in our implementation). By keeping a record of the last time t_i at which the state of site i changed, we are able to evaluate $h_i(t)$ at any moment in the simulation. (While the results reported here are for real h , we find the same scaling properties for integer h .) The surface $h_i(t)$ may be thought of as a driven interface. Since the critical contact process must eventually enter the absorbing state, the interface, in this analogy, will eventually be pinned.

The large-scale properties of the CP and related models such as DP can be described via a field theory (so-called Reggeon field theory) framed in terms of a coarse-grained density $\rho(\mathbf{x}, t) \geq 0$ [21]. Retaining only relevant terms, the stochastic partial differential equation for $\rho(\mathbf{x}, t)$ reads

$$\frac{\partial \rho}{\partial t} = \nabla^2 \rho - a\rho - b\rho^2 + \eta(\mathbf{x}, t). \quad (1)$$

Here $\eta(\mathbf{x}, t)$ is a Gaussian noise with zero mean and autocorrelation

$$\overline{\eta(\mathbf{x}, t) \eta(\mathbf{x}', t')} = \Gamma \rho(\mathbf{x}, t) \delta^d(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (2)$$

In our continuum description, the height is given by

$$h(\mathbf{x}, t) = \int_0^t dt' \rho(\mathbf{x}, t'). \quad (3)$$

Integrating Eq. (1) from time zero to time t , we obtain

$$\frac{\partial h}{\partial t} = \nabla^2 h - ah - b \int_0^t \left(\frac{\partial h}{\partial t'} \right)^2 dt' + \zeta(\mathbf{x}, t), \quad (4)$$

with the noise autocorrelation

$$\overline{\zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t')} = \Gamma \delta^d(\mathbf{x} - \mathbf{x}') h(\mathbf{x}, t_{<}), \quad (5)$$

where $t_{<} \equiv \min(t, t')$. Thus the equation governing the height includes a nonlinear memory term and a noise with nonvanishing correlations between different times. This equation does not seem to shed much light on the scaling properties of the interface. Rather, the relation between h and the density in the contact process, Eq. (3), yields some properties that are not immediately obvious from Eq. (4), for example, that $\partial h / \partial t \geq 0$, and that the nonlinear term is relevant for $d < d_c = 4$. In any event, Eq. (4) does serve to point up some differences between the CP interface and conventional surface-growth models such as the Edwards-Wilkinson [25] or Kardar-Parisi-Zhang equations [26]. First, the linear ‘‘drive’’ term $[-ah(\mathbf{x}, t)]$ with $a < 0$ in the active regime is proportional to the local height and so cannot be transformed away. Second, in the active state, the moduli of the last three terms on the right-hand side (RHS) of Eq. (4) grow without limit, suggesting that $\nabla^2 h$ does as well, so that the width of the active phase never saturates. We note that it is not our inten-

tion in this paper to simulate Eq. (4) directly; rather, we shall study the dynamics of $h(x, t)$ via simulations of the CP.

We turn now to a simple scaling analysis of the interface. At its basis lies a scaling hypothesis for the probability density $p(h; t)$ of the height h (at any lattice site) at time t : the time dependence of this density enters only through the *mean height* $\bar{h}(t)$. The conjectured scaling property is

$$p(h; t) = \frac{1}{\bar{h}(t)} \mathcal{P}(h/\bar{h}(t)), \quad (6)$$

where the scaling function $\mathcal{P} \geq 0$ with $\int \mathcal{P}(u) du = 1$, and the prefactor guarantees normalization. We have no proof for this scaling hypothesis. While it is known that an analogous scaling relation holds for the probability $p(\rho; t)$ of the *local particle density* in the critical CP [9], it is not obvious that the scaling property for ρ implies the same for h . [The problem is that $h(x, t)$ is the sum of many random variables, i.e., $\rho(x, t')$ for $t' < t$, which are, however, strongly correlated.]

It follows from Eq. (6) that the moments of h all scale with the mean height $\bar{h}^n = \bar{u}^n [\bar{h}(t)]^n$ with \bar{u}^n the n th moment \mathcal{P} . In particular, the mean-square width

$$W^2(t, L) = \text{var}(h) \sim [\bar{h}(t)]^2, \quad (7)$$

if Eq. (6) holds. On the other hand, we have that for times $t < \tau$ in the critical contact process,

$$\bar{h}(t) = \int_0^t dt' \rho(t') \sim t^{1-\theta}, \quad (8)$$

which immediately implies that $W^2 \sim t^{2\beta_W}$ with $\beta_W = 1 - \theta$.

In surface growth studies the crossover time is expected to scale as $t_{\times} \sim L^2$; for a process in the DP universality class, we may write $z = \nu_{\parallel}/\nu_{\perp}$ (clearly t_{\times} and τ should scale with the same exponent). Then the roughness exponent α , defined via $W^2(t, L) = W_{\text{sat}}^2(L) \sim L^{2\alpha}$ for $t \geq t_{\times}$, is given by the scaling relation $\alpha = \beta_W z = (1 - \theta) \nu_{\parallel}/\nu_{\perp}$. It is perhaps worth stressing that the expressions relating α and β_W to DP exponents depend on the validity of the scaling hypothesis Eq. (6), which should be tested. Inserting the known DP exponent values, scaling theory yields $\alpha \approx 1.3287, 0.97, 0.51$, and zero for dimensions 1, 2, 3, and 4, respectively. In other words, the interface associated with the critical contact process is super-rough in one dimension, and asymptotically flat in $d \geq d_c = 4$, where $\theta = 1$.

The family of height-difference correlation functions

$$G_q(r, t) \equiv \overline{|h(x, t) - h(x+r, t)|^q} \quad (9)$$

is also much studied in surface growth. Starting with a flat interface at $t=0$, we expect power-law growth $G_q \sim r^{q\alpha_q}$ for $r < \xi(t) \sim t^{1/z}$, the time-dependent correlation length. If α_q depends on q the interface is said to be multiaffine. For $r > \xi$, G_q will saturate; in particular, G_2 should approach $W^2(t, L) \sim t^{2\beta_W} \sim \xi^{2\alpha}$ for $r \approx \xi$ and $t \ll t_{\times}$. Since ξ is the only length scale relevant to correlations at short times (i.e., for $t < t_{\times}$, so that the system size L does not come into play), it is reasonable to expect the scaling form

$$G_2(r, t) = \xi^{2\alpha} \mathcal{G}(r/\xi), \quad (10)$$

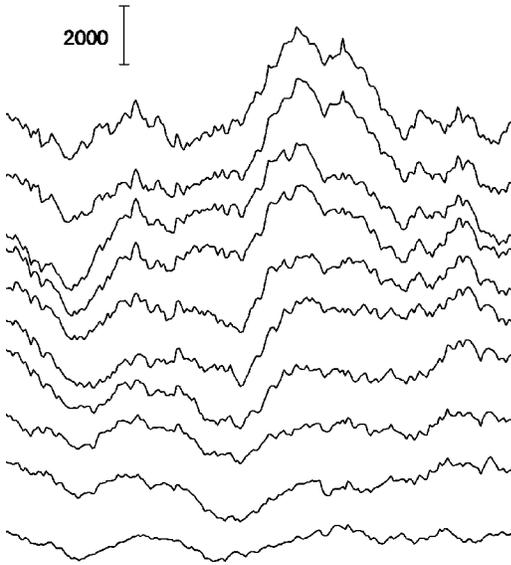


FIG. 1. Interface of the critical contact process in a system of 200 sites, shown at intervals of 5000 time units.

where the scaling function $\mathcal{G}(x) \sim x^{2\alpha_2}$ for small x and is constant for large x . The case $\alpha_2 = \alpha$ is referred to as “conventional” scaling while $\alpha_2 < \alpha$ is characterized as “anomalous” scaling [4,27].

III. SIMULATION RESULTS

We study the standard CP on hypercubic lattices with periodic boundary conditions. All interface properties are derived from the height $h_i(t)$ variables defined in Sec. II. In one dimension, we determined the square width $W^2(t, L)$ for rings of $L = 500, 1000, 2000,$ and 5000 sites, in samples of 2000, 1000, 1000, and 400 trials, respectively. The maximum time ranged from about 1.6×10^5 for $L = 500$ to 8.8×10^6 for $L = 5000$. All simulations were performed at the critical point, $\lambda_c \approx 3.297848$. In the contact process, interfacial properties can be studied over the full sample, or over a sample restricted to those trials that survive to time t . When

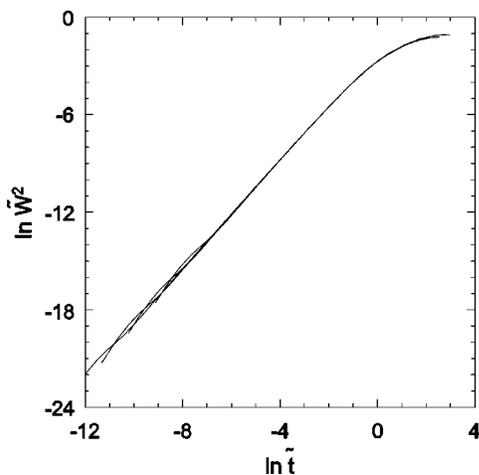


FIG. 2. Scaled mean-square width versus reduced time for system sizes $L = 500, 1000, 2000,$ and 5000 .

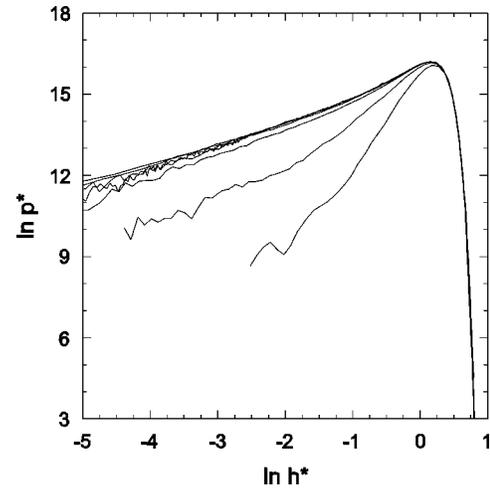


FIG. 3. Scaling plot of the height probability distribution (un-normalized) for $L = 1000$ at (from top to bottom on left-hand side) times 500, 1000, 2000, 5000, 10^4 , 2×10^4 , 5×10^4 , and 10^5 .

a trial enters the absorbing state, W^2 naturally remains fixed for all subsequent times, and since all trials do eventually reach the absorbing state in the critical contact process, the interface width saturates for large t . If we restrict the sample to surviving trials, however, there is no saturation. It is important to note that the same scaling laws apply in either case. (The power-law growth regime, for example, corresponds to times less than τ , for which all trials still survive.)

Figure 1 shows a series of snapshots of the interface in a single trial with $L = 200$, at intervals of 5000 time units. The progressive roughening of the interface, without evidence of saturation, is apparent. Figure 2 is a scaling plot of the square width, averaged over all trials, i.e., $W^2(L, t)/L^{2\alpha}$ versus $\tilde{t} \equiv t/L^z$, using the exponents $\alpha = 1.328$ and $z = 1.5808$ expected for DP in $1+1$ dimensions. There is a near-perfect data collapse for $\tilde{t} > 10^{-3}$. The power-law portion of the graph ($10^{-3} \leq \tilde{t} \leq 0.05$) yields the growth exponent $\beta_W = 0.839(1)$, in good agreement with the value $1 - \theta$

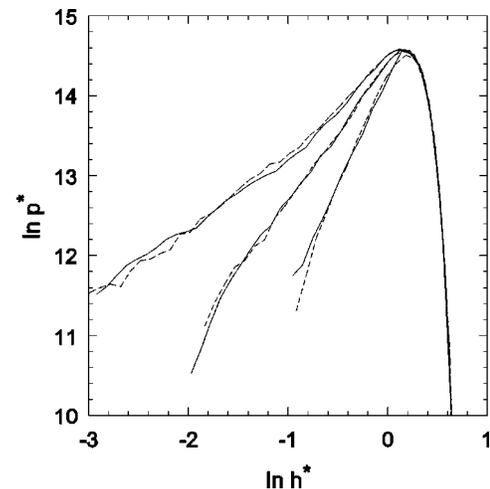


FIG. 4. Scaling plot of the height probability distributions (un-normalized) for $L = 1000$ (dashed lines) and $L = 5000$ (solid lines) at reduced times (from top to bottom on left-hand side) $\tilde{t} = 0.142, 0.71,$ and 1.42 .

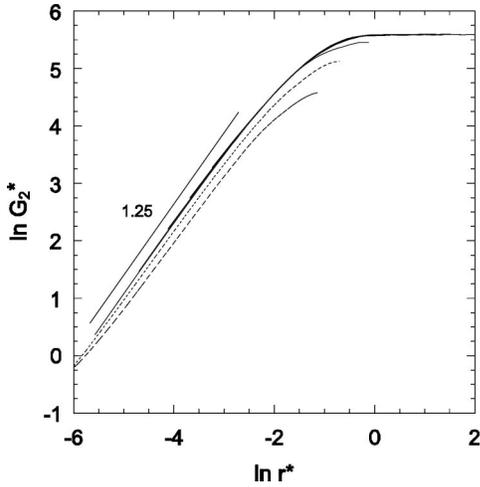


FIG. 5. Scaled height-height correlation function versus $r/t^{1/2}$ for $L=1000$. The topmost curve comprises collapsed data for times 500, 1000, 2000, 5000, and 10^4 ; below it lie results for $t=2 \times 10^4$ (solid curve), 5×10^4 (dotted curve), and 10^5 (dashed curve).

$=0.8405$ expected on the basis of the scaling argument. (Figures in parentheses denote statistical uncertainties.) Analysis of the saturation width yields $W_{sat}^2(L) \sim L^{2\alpha}$ with $\alpha = 1.325(15)$.

In Fig. 3 we test the scaling assumption for the height probability distribution $p(h)$ by plotting $t^{0.84}p(h)$ versus $h/t^{0.84}$. (Recall that \bar{h} is expected to grow proportionally to $t^{1-\theta} = t^{0.84}$.) In this system of $L=1000$ sites, there is a near-perfect data collapse, in accord with Eq. (6), for times between 500 and 10^4 . For $t=2 \times 10^4$ we begin to note a departure from scaling, which becomes more pronounced at later times. Note that $t=2 \times 10^4$ corresponds to $\ln \tilde{t} = -1$, which is where W^2 begins to depart noticeably from a power law in Fig. 2. Analysis of $p(h)$ for $L=5000$ yields similar results. The form of the scaling function for a given reduced time \tilde{t} is independent of the system size, as shown in Fig. 4, which compares height probability distributions for systems of 1000 and 5000 sites at times corresponding to the same \tilde{t} .

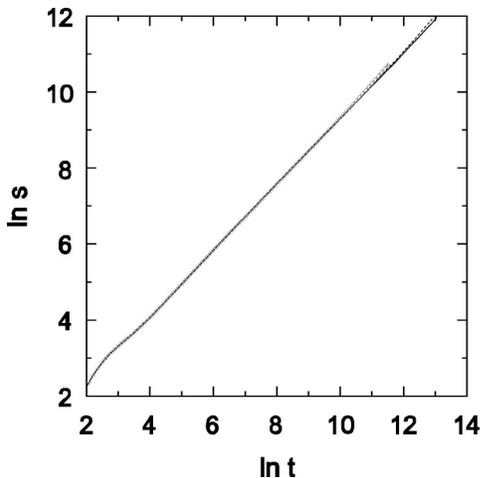


FIG. 6. Growth of the mean-square gradient $s(t) = \overline{(\nabla h)^2}$ in systems with $L=1000$ (+), 2000 (dashed line), and 5000 (solid line).

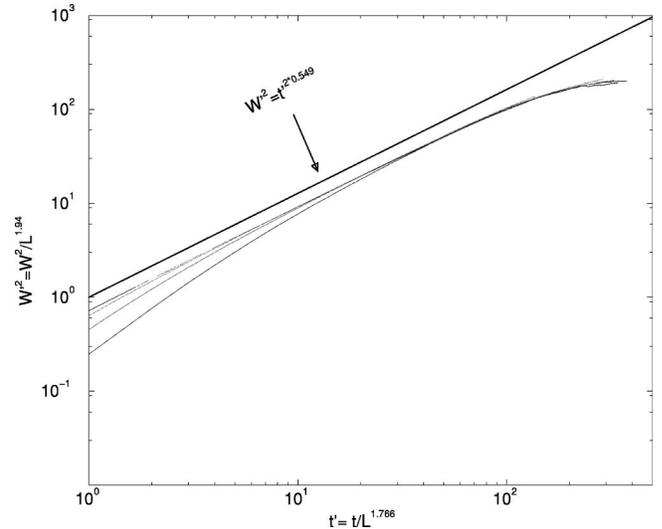


FIG. 7. Scaled mean-square width versus reduced time in two dimensions; system sizes $L=32, 64, 128,$ and 256 .

The height-height correlation function G_2 in a system of 1000 sites is shown in Fig. 5, which is a double-logarithmic plot of $G_2^* \equiv G_2/t^{2\beta_w}$ versus $r^* \equiv r/t^{1/2}$. There is a perfect data collapse for $t=500, \dots, 10^4$ (reduced times $10^{-2} \leq \tilde{t} \leq 0.2$), with a power-law portion $G_2 \sim r^{2\alpha_2}$ with $\alpha_2 \approx 0.625$. At later times G_2^* does not collapse, principally because the square width has begun to saturate (it no longer grows proportionally to $t^{2\beta_w}$). The value of α_2 is insensitive to system size. For $L=5000$ we obtain $\alpha_2=0.623(2)$ for $t=2 \times 10^5$, and 0.644 for $t=5 \times 10^5$. Thus we may, with a high degree of confidence, adopt the estimate $\alpha_2=0.63(3)$, clearly much smaller than the roughness exponent $\alpha=1.33$ found in the analysis of the square width, indicating that this one-dimensional system exhibits anomalous scaling. Recently, López argued that anomalous surface roughening is associated with a diverging height gradient [27]. In view of the growing spikes evident in the profiles shown in Fig. 1, this association seems very probable in the present instance. Indeed, the mean-square height gradient

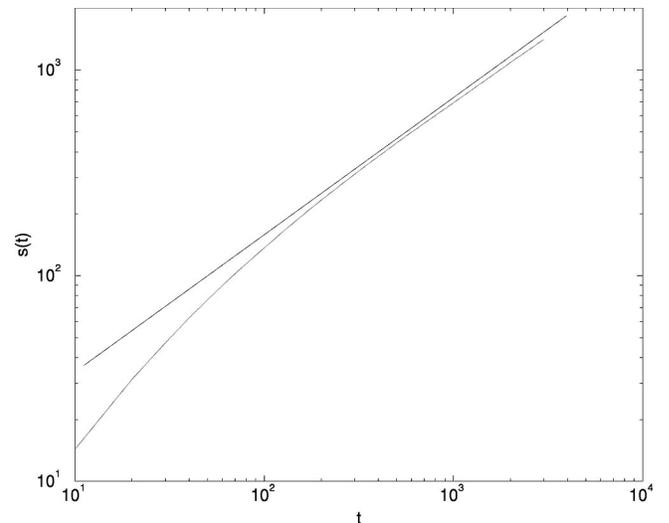


FIG. 8. Growth of the mean-square gradient $s(t) = \overline{(\nabla h)^2}$ in the two-dimensional system with $L=256$.

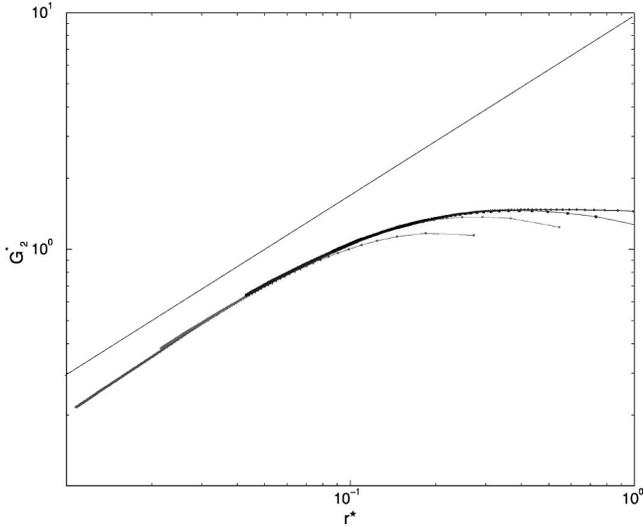


FIG. 9. Scaled height-height correlation function versus $r/t^{1/2}$ for $L=256$ in two dimensions. The curves (bottom to top) correspond to $r=2, 4, 8,$ and 16 .

$$s(t) = \overline{(\nabla h)^2} \quad (11)$$

diverges as a power law (see Fig. 6). We find $s(t) \sim t^{2\kappa}$, with $\kappa=0.439(1)$ for $L=1000$, $0.4335(4)$ for $L=2000$, and $0.4337(4)$ for $L=5000$; we adopt $\kappa=0.4336(4)$ as our best estimate. From López' analysis, one expects

$$\alpha_q = \alpha - z\kappa. \quad (12)$$

Inserting our result for κ and the DP values $\alpha = 1.32867(14)$ and $z = 1.5808(1)$ in the RHS, we obtain $\alpha_q = 0.643(1)$, in good agreement with the result found from analysis of the correlation function.

We also studied the generalized height-height correlation function G_q [Eq. (9)] for $q = 1/2, 1, 3/2, 2,$ and 3 , in a system with $L=1000$, at $t=10^4$, and found that the functions $[G_q]^{1/q}$ are identical and give $\alpha_q \approx 0.62$. We may therefore conclude that the DP interface is self-affine, not multifractal.

In two dimensions we studied systems of up to 256×256 sites at the critical point $\lambda_c \approx 1.6488$. The curves for W^2 again show a good collapse (see Fig. 7), and the derived exponents are in very good agreement with the expected values. Specifically, we find $\alpha = 0.97(1)$, $\beta_W = 0.550(5)$, and $z = 1.765(10)$. Scaling relations combined with known DP exponent values yield $\alpha = 0.970(5)$ and $\beta = 0.549(2)$, while current best estimates give $z = \nu_{\parallel}/\nu_{\perp} = 1.766(2)$ for DP in $2+1$ dimensions [28–31].

For smaller system sizes we observe a transient in W^2 and all other measured quantities, which does not appear in one dimension. Anomalous roughening is also seen in two and three dimensions. Figure 8 shows $s(t)$ growing $\sim t^{2\kappa}$ with $\kappa = 0.33(1)$. The height-height correlation function exhibits a good collapse, as shown in Fig. 9; the initial power-law growth yields $\alpha_2 = 0.385(5)$; thus the scaling relation Eq. (12) is well satisfied: $\alpha_2 - \alpha + z\kappa = 0.00(3)$.

We also determined the interface exponents in three dimensions ($\lambda_c \approx 1.3169$, system size 50^3 sites, maximum time 10^4), though to somewhat lower precision, owing to the larger computational demand. The scaling relation yields

TABLE I. Summary of interface-growth critical exponents for the contact process obtained in our simulations ($d=1-3$). Figures in parentheses denote uncertainties.

Dimension	α	α_2	β_W	κ	z
$d=1$	1.33(1)	0.63(3)	0.839(1)	0.4336(4)	1.58
$d=2$	0.97(1)	0.385(5)	0.550(5)	0.33(1)	1.765(10)
$d=3$	0.51(1)	0.09(2)	0.27(1)	0.22(1)	1.90(5)
$d=4$	0	0	0	0	2

$\beta_W = 1 - \delta = 0.274(1)$, while our simulation results for W^2 give $\beta_W = 0.27(1)$. We find $\alpha = 0.51(1)$, $z = 1.90(5)$, and $\kappa = 0.22(1)$. Together with Eq. (12), these yield $\alpha_2 = 0.09(2)$. Our results for critical exponents are collected in Table I.

IV. SUMMARY

Defining an interface representation for the contact process, we verified the expected scaling relation $\beta_W = 1 - \theta$ in dimensions 1–3, and the scaling property of the height probability distribution in one dimension. The local roughness exponent α_2 is smaller than the global value α , indicating anomalous surface growth. This anomalous scaling is attended by a diverging local slope, $s(t) = \overline{(\nabla h)^2} \sim t^{2\kappa}$. Our results for κ are consistent with the scaling relation Eq. (12) derived by López. There is, on the other hand, no evidence of multifractality in this process.

An interesting point is that the process continues to exhibit anomalous scaling for $d=2$ and 3 , even though $\alpha < 1$ in these cases. While it was pointed out some time ago that $\alpha > 1$ implies anomalous growth [32], this property appears to be an intrinsic feature of the contact process (and, by extension, of other models in the DP universality class). Finally, we note that we have introduced two critical exponents α_2 and κ , and only one scaling relation between them. We therefore have an independent exponent, κ say, that cannot be related to the standard DP exponents in any way. A very interesting theoretical task is that of computing κ in an epsilon expansion around the upper critical dimension $d_c = 4$. Our guess is that this anomalous exponent is related to the renormalization of a composite operator not considered so far in the analysis of Reggeon field theory, but this issue is beyond the scope of this paper, and will be studied elsewhere. It is interesting to note, in this connection, that Bhat-tacharjee *et al.* showed that for an interface model exhibiting super-roughening (i.e., $\alpha \geq 1$) there exist generic nonperturbative infrared singularities inaccessible to standard dynamical renormalization group analysis [33].

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