Intermittent chaos in electron scattering

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The motion of an electron in a uniform magnetic field and positive point charge is not integrable. Phase space is often divided between regular regions farther from the positive charge and chaotic regions nearby. As the electron transits the chaotic region, intermittent chaotic behavior ensues. An analytic method to estimate the location of the transit parameters is also developed.

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The seemingly simple problem of a particle moving in a uniform magnetic field and the electric field of a stationary charged particle is in fact a problem of considerable complexity, since the equations of motion are in general not integrable. Electron-ion collisions in a magnetoplasma (if the collision takes place inside the Debye sphere) and the motion of charged particles in an electric discharge between a charged sphere and a distant external conductor in a uniform magnetic field [1] are examples of physical situations where this problem arises. The classical limit of a hydrogen or Rydberg atom in a strong magnetic field also falls in this category. It has been studied by several authors [2], but here a different approach is taken.

In particular, we analyze in detail a new type of chaotic behavior exhibited by this system, which we name "intermittent chaos." Typically, an electron far from the scatterer moves in a regular nonchaotic orbit, then the orbit becomes chaotic as the scatterer is approached, and later the trajectory becomes regular again as the electron exits the chaotic region. This has been noticed by Delos *et al.* (Ref. [2]), but no analysis has been given. Here this phenomenon is analyzed in detail, and the parameter regions for this behavior are found. The analysis was helped by a novel parametrization of the Hamiltonian.

The Hamiltonian is in cylindrical coordinates,

$$H=P_r^2/(2m)+P_z^2/(2m)+\frac{(P_\varphi-qBr^2/2)^2}{2mr^2}+\frac{q}{4\pi\varepsilon_0}\frac{Q}{(r^2+z^2)^{1/2}}, \tag{1}$$

where B is the magnetic field strength, q is the charge of the moving particle (in this paper an electron), and Q is the stationary charge. The total energy as well as P_φ are constants of motion. Since phase space is four-dimensional, the system is not integrable unless other conserved quantities are found. It is convenient to write the constant $P_\varphi=qBR^2/2$, where R is the radius and where $d\varphi/dt=0$ (points of zero angular momentum). One now writes

$$H=P_r^2/(2m)+P_z^2/(2m)+V(r,z), \tag{2}$$

where, introducing the dimensionless lengths $r/R\to r$, $z/R\to z$,

$$V(r,z)=\frac{q^2B^2R^2}{8m}\frac{(1-r^2)^2}{r^2}+\frac{qQ}{4\pi\epsilon_0 R(r^2+z^2)^{1/2}}. \tag{3}$$

To obtain a dimensionless Hamiltonian, one divides by $(qBR)^2/(8m)$ to get the dimensionless potential energy,

$$V(r,z)=(1/r-r)^2-\frac{c}{(r^2+z^2)^{1/2}}, \tag{4}$$

where

$$c=-\frac{2mQ}{\pi\epsilon_0 qB^2R^3}. \tag{5}$$

The minus sign was chosen since we will be concerned with the motion of an electron in an attracting electric field, so $Q/q<0$ and $c>0$. When $c\to 0$, there is no electric field and the problem is integrable, while $c\to\infty$ eliminates the magnetic field, reducing the equations to the integrable Kepler problem.

The equations of motion in these dimensionless units are

$$\ddot{r}=-\frac{\partial V}{\partial r}=2(1/r^3-r)-\frac{rc}{(r^2+z^2)^{3/2}}, \tag{6}$$

$$\ddot{z}=-\frac{\partial V}{\partial z}=-\frac{zc}{(r^2+z^2)^{3/2}}. \tag{7}$$

These two second-order equations are rewritten into four first-order equations, with the variables r, z, v_r, and v_z. Due to energy conservation $H=E=\text{const}$, only three variables are independent, so the effective phase space is three-dimensional.

Numerical integration can be performed for given values of E and c, and surface of section plots generated. The $z=0$ plane with r and v_r plotted is convenient in generating these plots. There are two classes of parameter regions. For $E<0$, the electron is trapped in the field of the positive charge, while $E>0$ corresponds to electron scattering. Figure 1 shows surface of section plots for $E<0$ and $c=2$. As one would expect, when E is small enough to contain the electron near the bottom of the potential well, the motion is nearly regular and becomes more and more chaotic with E increased.

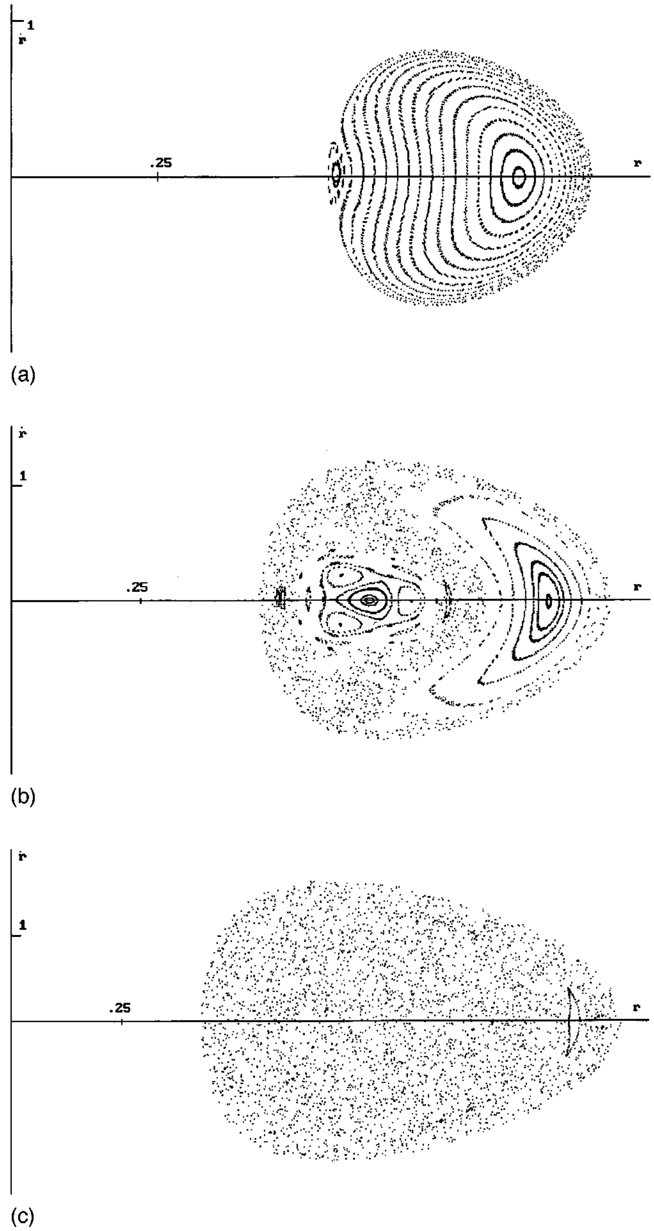

FIG. 1. Surface of section plots in the $z=0$ plane, with $c-2$. (a) $E=-2$, (b) $E=-1.6$, and (c) $E=-1$.

For Fig. 1(a), $E=-2$ close to the minimum potential. The plot looks regular, and is divided between a large island on the right and a small one on the left. Figure 1(b) with $E=-1.6$ shows a typical transition to chaos, with several islands and a chaotic region comparable in size to the regular islands. Finally, in Fig. 1(c) with $E=-1$, chaos clearly dominates. The large island in Figs. 1(a) and 1(b) has shrunk considerably.

The trajectories plotted in the r-z plane are more interesting. Figure 2 shows the plot for $c=2$ and $E=-0.2$, where the surface of section plot is completely chaotic. Figure 2(a) shows a short-time ($T=100$) trajectory. The dotted line represents $V(r,z)=E$, which limits the extent that the trajectory can move. The long-time ($T=10\,000$) behavior is shown in Fig. 2(b). Essentially all available space is visited by this trajectory, as one would expect for parameters where all KAM tori have been destroyed.

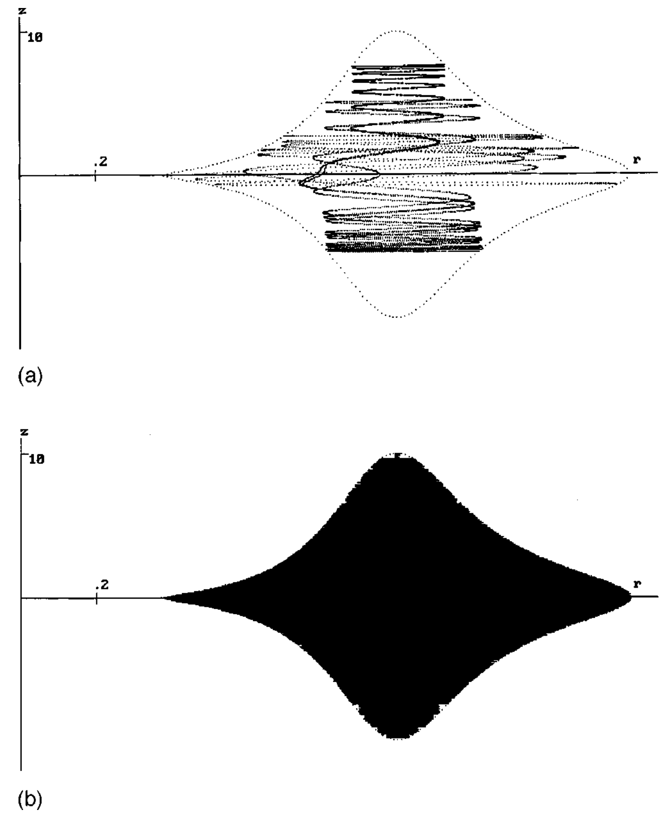

FIG. 2. Trajectory for $c=2$, $E=-0.2$. (a) $T=100$. (b) $T=10\,000$.

In Fig. 2(a), one can see that the trajectory consists of two parts: a chaotic one for small values of $|z|$ and apparently regular helical motion for larger values of $|z|$. The chaotic motion is intermittent; as soon as the particle moves farther away from the r axis, the motion becomes regular, but as it returns to the small $|z|$ region, chaotic behavior is reestablished and lasts as long as the particle stays in this region.

In order to understand this behavior, it is useful to study the scattering of the electron, launched from $|z|\gg 1$ and $E>0$. When z is sufficiently large, the electric field can be ignored. The motion is integrable with a helical path, consisting of circular motion perpendicular to B around the guiding center, which moves with constant velocity along the magnetic-field lines. The position of the guiding center is approximately at $r=1$ for electrons that do not encircle the z axis. Expanding the potential near $r=1$ gives

$$V(1+\varepsilon,\infty)=4\epsilon^2+O(\epsilon^3) \tag{8}$$

resulting in the equation of motion

$$\ddot{\epsilon}+8\epsilon=0 \tag{9}$$

from which the gyration period is $T=\pi/\sqrt{2}$.

As $|z|$ is reduced, the effect of the electric field can no longer be ignored. It gives rise to an $\mathbf{E}\times\mathbf{B}$ drift and acceleration of the guiding center along the B lines. As long as certain approximations are valid, the guiding center approximation can be used, resulting in an approximate conserved quantity, the adiabatic invariant [3] rendering the equations

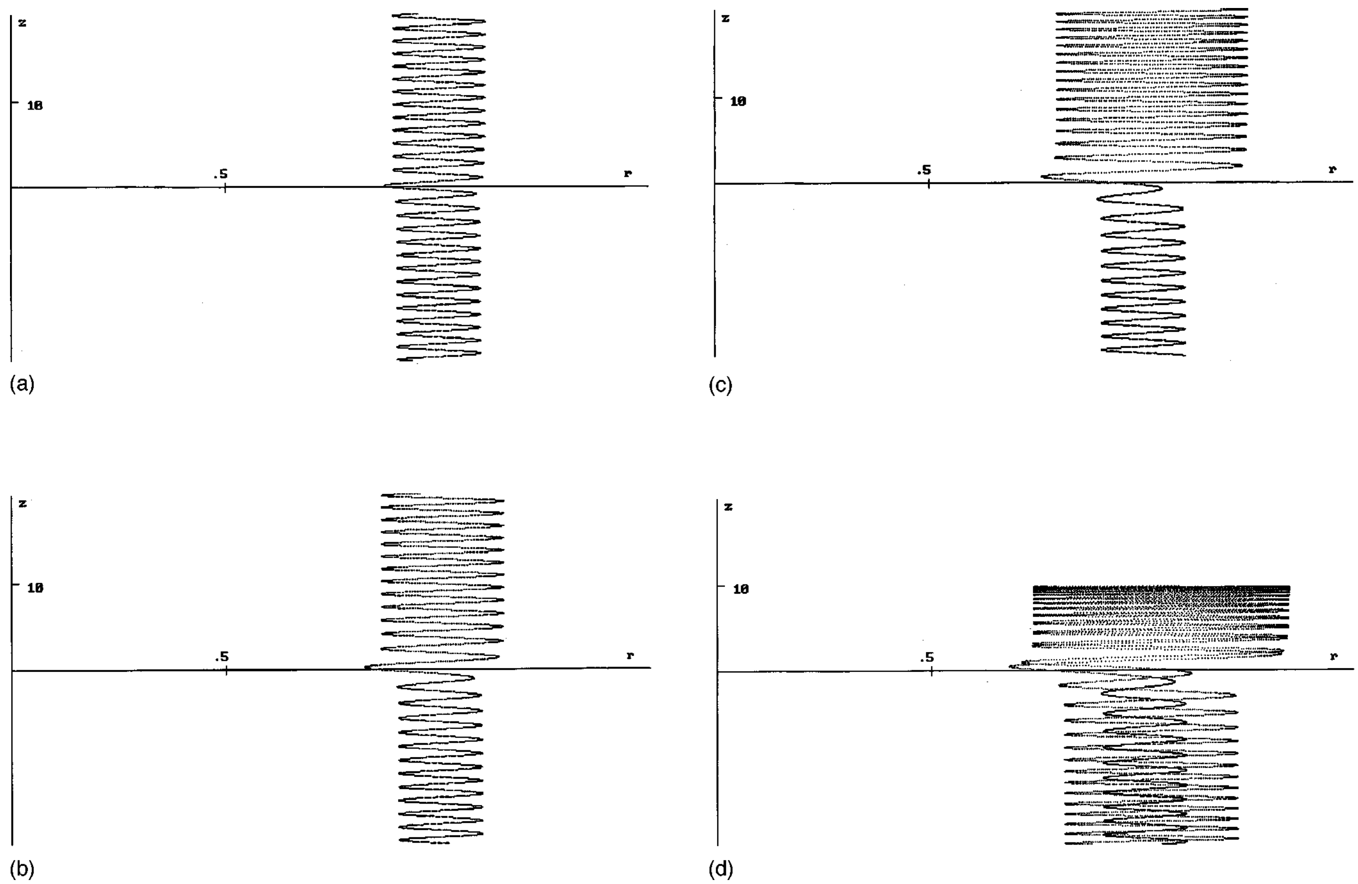

FIG. 3. Trajectories of electron scattering, with $E=0.25$. (a) $c=0.2$, (b) $c=0.5$, (c) $c=1$, and (d) $c=1.4$.

of motion integrable. In terms of the potential $V(r,z)$, there is oscillation around $r=1$, with motion in the z direction with the adiabatic invariant

$$\oint v_r dr = \text{const} \tag{10}$$

as long as there is no significant change of the potential function from one period to the next, in the frame moving with the guiding center. In the r-z plane, the trajectory is sinusoidal with slowly changing wavelength and amplitude. Such trajectories are seen in Fig. 2(a).

The adiabatic invariant equation (10) breaks down when the distance covered by the guiding center during an oscillation period $\delta = v_z T = v_z \pi/\sqrt{2}$ corresponds to a significant change of potential,

$$\delta \frac{d}{dz} V(1,z) \approx V(1,z). \tag{11}$$

This gives for z critical (z_c), where chaos sets in,

$$z_c^2 - \pi v_z z_c/\sqrt{2} + 1 = 0. \tag{12}$$

This equation has real roots only for $v_z > (2\sqrt{2}/\pi) \approx 1$ and is an approximate condition for the onset of chaos. In fact, numerical investigation of the transition region between order and chaos in Fig. 2(a) shows that this is a good approximation. Note that in this case $E<0$ and the derivation did not assume $E>0$.

If the parameters are such that during the motion $|v_z| \ll 1$ everywhere, the motion can be expected to be regular without chaotic transitions. For $r=1$, energy conservation gives

$$E = v_r^2/2 + v_z^2/2 - \frac{c}{\sqrt{1+z^2}}. \tag{13}$$

The maximum value of v_z is reached when $v_r=0$ and $z=0$ to give $v_z=\sqrt{2(E+c)}$, and yields the condition for the absence of chaos,

$$E + c \ll 1/2. \tag{14}$$

In fact, in the examples computed in Fig. 1 as well as other cases studied, one finds that $E+c \approx 1/2$ is a reasonable approximation for the dividing line between regular and chaotic behavior on surface of section plots. This becomes even more clear when scattering problems with $E>0$ and the initial $|z| \gg 1$ are computed, as shown in Fig. 3. In all four figures, $E=0.25$ and the same initial conditions ($r=1$, $z=-50$, $v_z=65$) are chosen. In Fig. 3(a), $c=0.2$ and $E+c=0.45<0.5$. The trajectory moving up from large negative z is almost unchanged as it passes the r axis. In Fig. 3(b), $c=0.5$ and $E+c=0.75>0.5$. There is now a clear change in the helical motion near $z=0$, indicating a breakdown of the adiabatic invariant close to the r axis. In Fig. 3(c), $c=1$ and $E+c=1.25$. The adiabatic invariant has changed significantly as the r axis is passed. In Fig. 3(d), $c=1.4$ and E

$+c=1.65$. The chaotic region occupies now a significant part of the r-z plane, resulting in trapping of the particle for a finite time, and after escaping the chaotic region it moves back toward $z\to-\infty$, with a changed adiabatic invariant. Changing the initial conditions can lead to the ejection after a chaotic trapping period, either in the positive or negative z direction.

It seems, based on these computations, that the approximate condition for the transition between regular and chaotic behavior, $E+c>$ or $<1/2$, is remarkably accurate. In fact it also works well for $E<0$. In Fig. 1(a), $E+c=0$ and the motion is quite regular; in Fig. 2(b), which shows the transition to chaos, $E+c=0.4$; and in Fig. 1(c), dominated by chaotic motion, $E+c=1$. Computations carried out for $c=1$ and $c=3$ have yielded similar results. Finally, the case of the electron encircling the z axis can also be treated. In this case $P_\varphi\to-P_\varphi$, and in Eq. (4) $[(1/r)-r]^2$ is replaced by $[(1/r)+r]^2$. However, $[(1/r)+r]^2=[(1/r)-r]^2+4$, so only a constant is added to the potential, leaving the equations of motion in the r-z plane unchanged and the adiabatic invariant equation (10) still valid. The motion in φ is of course different; the guiding center approximation involving flux conservation inside the particle trajectory no longer holds. Trapped and free orbits are now separated at $E=4$, and Eq.(14) becomes $E+c\ll 4.5$.

To conclude, the system studied exhibits transitions from regular to chaotic behavior. A good approximation of the parameter range where this occurs has been derived.

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[3] See, e.g., G. Schmidt, *Physics of High Temperature Plasmas*, 2nd ed. (Academic Press, New York, 1979), pp. 23–29.